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ULTRAPRODUCTS OF VON NEUMANN ALGEBRAS AND ERGODICITY

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Abstract

An ultraproduct of any linear spaces with respect of a non-trivial ultrafilter in an index set is generalization of the non-standard expansion $^*\mathbb{R}$ of the set of real numbers $\mathbb{R}$. The non-standard mathematical analysis has the objects and methods of a research, which only to some extent depend on laws of the standard mathematical analysis.

In this work, non-standard objects – ultraproducts of von Neumann algebras – have been studied from the point of view of the standard analysis. This approach allows to receive, in particular, a criterion of contiguity of sequences of normal faithful states in terms of the equivalence of states on the corresponding ultraproducts.

We note that the classical ultraproduct of von Neumann algebras, generally speaking, is not a von Neumann algebra. Therefore, in accordance with A. Ocneanu's work, we have considered the changed construction of the ultraproduct of von Neumann algebras.

We have introduced the concept of ergodic action with respect to the normal state of group on an abelian von Neumann algebra. Its properties have been studied. In particular, we have provided the example showing that the ultraproduct of ergodic states is not, generally speaking, ergodic.

Keywords: ultraproducts, actions of group, ergodicity, states on von Neumann algebra

Introduction

Definition 1. Given a non-empty set X, a *filter* on X is a non-empty family $\mathcal{U}$ consisting of subsets of X, such that

1. the empty set is not an element of $\mathcal{U}$;

2. if A and B are subsets of X, A is a subset of B, and A is an element of $\mathcal{U}$, then B is also an element of $\mathcal{U}$;

3. if A and B are elements of $\mathcal{U}$, then the intersection of A and B is also element of $\mathcal{U}$.

A filter $\mathcal{U}$ on X is a *maximal filter* or an *ultrafilter* on X, if for every filter $\mathcal{U}'$ on X that contains $\mathcal{U}$ we have $\mathcal{U}' = \mathcal{U}$.

Note that the filter $\mathcal{U}$ on X is an ultrafilter if and only if, for any subset A of X, either A or $X \setminus A$ belongs to $\mathcal{U}$, but not both.

The family $\mathcal{U} = \{A \subseteq X : A \text{ contains an element } x_0 \in X\}$ is an example of an ultrafilter. This ultrafilter is said to be trivial. In what follows, we will use only nontrivial ultrafilters.

Definition 2. Let us consider a sequence (X_n) of nonempty sets and a nontrivial ultrafilter $\mathcal{U}$ in the set $\mathbb{N}$ of natural numbers. The *ultraproduct* $(X_n)_{\mathcal{U}}$ is the factorization of the Cartesian product $\prod\limits_{n=1}^{\infty} X_n$ by the equivalence relation:

$$(x)_n \sim (y)_n \Leftrightarrow \{n : x_n = y_n\} \in \mathcal{U}.$$

The linear structure for an ultraproduct of linear spaces is defined in the natural way:

$$(x_n)_{\mathcal{U}} + (y_n)_{\mathcal{U}} = (x_n + y_n)_{\mathcal{U}}, \quad c \cdot (x_n)_{\mathcal{U}} = (c \cdot x_n)_{\mathcal{U}}.$$

The set-theoretic ultraproduct possesses a number of extraordinary properties, for which we refer the reader to the articles by D. Mushtari and S. Haliullin [1] and S. Haliullin [2].

Definition 3. Let $\mathcal{U}$ be a nontrivial ultrafilter on the set of integers $\mathbb{N}$. Let (x_n) be a sequence of points in a metric space (X, d). A point $x \in X$ is said to be the *limit of the sequence* (x_n) *with respect to the ultrafilter* $\mathcal{U}$, denoted $x = \lim_{\mathcal{U}} x_n$, if for every $\varepsilon > 0$ we have $\{n : d(x_n, x) < \varepsilon\} \in \mathcal{U}$.

As is well-known, if $\mathbf{K}$ is a compact Hausdorff space and $\mathcal{U}$ an arbitrary ultrafilter on the set $\mathbb{N}$, then every sequence $(x_n)_{n=1}^{\infty}$, $x_n \in \mathbf{K}$, has a unique limit with respect to the ultrafilter $\mathcal{U}$.

Definition 4 (see, for example, [3]). Let $(H_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces and let $\mathcal{U}$ be a nontrivial ultrafilter in the set $\mathbb{N}$ of natural integers. Let us put $l^{\infty}(\mathbb{N}, H_n) = \{(h_n), h_n \in H_n : \sup_n \|h_n\| < \infty\}$ and $\mathcal{N}_{\mathcal{U}} = \{(h_n) \in l^{\infty}(\mathbb{N}, H_n) : \lim_{\mathcal{U}} \|h_n\| = 0\}$. The *ultraproduct* $(H_n)_{\mathcal{U}}$ of the sequence of Banach spaces is the quotient $l^{\infty}(\mathbb{N}, H_n)/\mathcal{N}_{\mathcal{U}}$; here $\mathcal{N}_{\mathcal{U}}$ is the closed subspace of $l^{\infty}(\mathbb{N}, H_n)$.

We denote an element of $(H_n)_{\mathcal{U}}$ by $(h_n)_{\mathcal{U}}$. Then, the formula

$$\|(h_n)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|h_n\|$$

defines a norm on $(H_n)_{\mathcal{U}}$. In this case, $(H_n)_{\mathcal{U}}$ is a Banach space.

It is known, (S. Heinrich, [3]), that the class of Banach algebras and the class of C^*-algebras are stable under ultraproducts.

In fact, the multiplicative and involutive structures of the ultraproduts are defined in the natural way:

$$(x_n)_{\mathcal{U}} \cdot (y_n)_{\mathcal{U}} = (x_n \cdot y_n)_{\mathcal{U}}, \ \ ((x_n)_{\mathcal{U}})^* = ((x_n)^*)_{\mathcal{U}}.$$

Let us consider the notion of the ultraproduct for a sequence of von Neumann algebras using Ocneanu's definition (see [4], [5]).

Definition 5. Let $(\mathcal{M}_n)$ be a sequence of σ-finite von Neumann algebras, and let φ_n be a normal faithful state on $\mathcal{M}_n$ for each $n \in \mathbb{N}$. Let us put

$$l^{\infty}(\mathbb{N}, \mathcal{M}_n) = \{(x_n), x_n \in \mathcal{M}_n : \sup_n \|x_n\| < \infty\},$$

$$\mathcal{N}_{\mathcal{U}}(\mathcal{M}_n, \varphi_n) = \{(x_n) \in l^{\infty}(\mathbb{N}, \mathcal{M}_n) : \lim_{\mathcal{U}} \varphi_n(x_n^* x_n + x_n x_n^*)^{1/2} = 0\}.$$

We let

$$\mathcal{M}_{\mathcal{U}}(\mathcal{M}_n, \varphi_n) = \{(x_n) \in l^{\infty}(\mathbb{N}, \mathcal{M}_n) : \\ (x_n)\mathcal{N}_{\mathcal{U}}(\mathcal{M}_n, \varphi_n) \subset \mathcal{N}_{\mathcal{U}}(\mathcal{M}_n, \varphi_n), \mathcal{N}_{\mathcal{U}}(\mathcal{M}_n, \varphi_n)(x_n) \subset \mathcal{N}_{\mathcal{U}}(\mathcal{M}_n, \varphi_n)\}.$$

Then, we define the *ultraproduct* for the sequence of von Neumann algebras with normal faithful states as the quotient

$$(\mathcal{M}_n, \varphi_n)_{\mathcal{U}} = \mathcal{M}_{\mathcal{U}}(\mathcal{M}_n, \varphi_n)/\mathcal{N}_{\mathcal{U}}(\mathcal{M}_n, \varphi_n).$$

Finally, we define a state $\varphi_{\mathcal{U}}$ on $(\mathcal{M}_n, \varphi_n)_{\mathcal{U}}$ as follows:

$$\varphi_{\mathcal{U}}\left((x_n)_{\mathcal{U}}\right) = \lim_{\mathcal{U}} \varphi_n(x_n).$$

It is known (see [5]) that if the state φ_n is normal and faithful, then $(\mathcal{M}_n, \varphi_n)_{\mathfrak{U}}$ is a von Neumann algebra with the normal faithful state $\varphi_{\mathfrak{U}}$.

There are various definitions of the absolute continuity of positive linear functionals on $*$-algebras (see, for example, the papers by S. Gudder [6] and by E. Chetcutti and J. Hamhalter [7].

Definition 6. Let $\mathcal{M}$ be a von Neumann algebra, and let φ and ψ be normal states on $\mathcal{M}$. The state φ is said to be *absolutely continuous* with respect to the state ψ if the equation $\psi(x^*x) = 0$ implies $\varphi(x^*x) = 0$, $x \in \mathcal{M}$. The states φ and ψ are called *equivalent* if they are mutually absolutely continuous. The states φ and ψ are called *singular* if there is an operator $x \in \mathcal{M}$, such that $\varphi(x^*x) = 0$ and $\psi(x^*x) = 1$.

Let us notice that it is enough to set an equivalence and singularity of states on projections in the case of von Neumann algebra. Now, we need the following concept introduced in [8].

Definition 7. Let $(\mathcal{M}_n)$ be a sequence of σ-finite von Neumann algebras, let φ_n and ψ_n be normal states on $\mathcal{M}_n$, $n \in \mathbb{N}$. The sequence (φ_n) is said to be *contigual* with respect to the sequence (ψ_n) if

$$\psi_n(x_n^* x_n) \to 0 \quad \text{implies} \quad \varphi_n(x_n^* x_n) \to 0,\ x_n \in \mathcal{M}_n, \quad (n \to \infty);$$

If the sequence (φ_n) is contigual with respect to the sequence (ψ_n) and the sequence (ψ_n) is contigual with respect to the sequence (φ_n), then the sequences (φ_n) and (ψ_n) are said to be *mutually contigual*; the sequences (φ_n) and (ψ_n) are said to be *entirely separable* if there is a subsequence (n') and operators $x_{n'} \in \mathcal{M}_{n'}$, such that

$$\varphi_{n'}(x_{n'}^* x_{n'}) \to 0, \quad \psi_{n'}(x_{n'}^* x_{n'}) \to 1, \quad (n' \to \infty).$$

These notions generalize the concepts of equivalence and singularity of states.

Theorem 1 [8]. *Let $(\mathcal{M}_n)$ be a sequence of σ-finite von Neumann algebras, let φ_n and ψ_n be normal faithful states on $\mathcal{M}_n$, $n \in \mathbb{N}$.*

i) The sequences (φ_n) and (ψ_n) are mutually contigual if and only if the states $\varphi_{\mathfrak{U}}$ and $\psi_{\mathfrak{U}}$ are equivalent for every nontrivial ultrafilter $\mathfrak{U}$ on $\mathbb{N}$.

ii) The sequences (φ_n) and (ψ_n) are entirely separable if and only if there is a non-trivial ultrafilter $\mathfrak{U}$ on $\mathbb{N}$, such that the states $\varphi_{\mathfrak{U}}$ and $\psi_{\mathfrak{U}}$ are singular.

1. Ultraproducts and ergodicity

Let G be a separable locally compact group, and let (Ω, μ) be a σ-finite standard measure space. By an action of G on (Ω, μ), we mean a Borel map $T : (s, \omega) \in G \times \Omega \to T_s(\omega) \in \Omega$, such that

1) for each fixed $s \in G$, the map $\omega \to T_s(\omega)$ is a non-singular bijection of Ω;
2) $T_s(T_t(\omega)) = T_{st}(\omega)$, $s, t \in G$, $\omega \in \Omega$;
3) $T_e(\omega) = \omega$, where e is the unit of G.

We also say that (Ω, μ) is a G-measure space and we will designate (G, Ω, μ).

Definition 8. ([9]) The action G on (Ω, μ) is said to be non-singular (with respect to the measure μ) if $E \in \mathcal{F}$ $\mu(E) = 0 \Leftrightarrow \mu(T_s(E)) = 0$ for any $s \in G$; The action G on (Ω, μ) is said to be free (with respect to the measure μ) if for any compact subset K of G, such that $e \notin K$, and any Borel subset E of Ω with $\mu(E) > 0$, there exists a Borel subset $F \subset E$, such that $\mu(F) > 0$ and $\mu(F \bigcap T_s(F)) = 0$ for every $s \in K$; The action G on (Ω, μ) is said to be ergodic (with respect to the measure μ) if $\mu(E \Delta T_s(E)) = 0$ for every $s \in G$ implies $\mu(E) = 0$ or $\mu(\Omega \setminus E) = 0$.

With (G,Ω,μ), we consider an action α of G on the abelian von Neumann algebra $\mathcal{A}=L^\infty(\Omega)$ given by the following:

$$\alpha_s(f)(\omega)=f(T_s^{-1}\omega),\quad s\in G,\quad f\in\mathcal{A},\quad \omega\in\Omega.$$

Definition 9. Let φ be a normal state on the von Neumann algebra $\mathcal{A}=L^\infty(\Omega)$. We claim that an action α of G on $\mathcal{A}$ is non-singular with respect to the state φ if $\varphi(f)=0\Leftrightarrow\varphi(\alpha_s(f))=0$, $f\in\mathcal{A}$ for any $s\in G$; we say that an action α of G on $\mathcal{A}$ is free with respect to the state φ if for any compact subset K of G, such that $e\notin K$, and for any projection $g\in\mathcal{A}, g\neq 0$, there exists a non-zero projection $f\in\mathcal{A}$, such that $f\le g$ and $\varphi(f\alpha_s(f))=0$ for every $s\in K$; we say that a non-singular action α of G on $\mathcal{A}$ is ergodic with respect to the state φ if for any projection $f\in\mathcal{A}$, $\alpha_s(f)=f$ for every $s\in G$ implies $\varphi(f)=0$ or $\varphi(\mathbf{1}-f)=0$.

At the same time, the state φ is called quasi-invariant, free, and ergodic with respect to the action α, respectively.

Theorem 2. *Let φ and ψ be normal states on the abelian von Neumann algebra $\mathcal{A}=L^\infty(\Omega)$, the action α of G on $\mathcal{A}$ is ergodic with respect to the states φ and ψ. Then, the states φ and ψ are either equivalent or singular.*

Proof. Let p and q be the supports of the states φ and ψ, respectively. It is clear that the states φ and ψ are equivalent if and only if $p=q$, and the states φ and ψ are singular if and only if $p\perp q$. Let us assume the opposite, i.e., that the states φ and ψ are non-equivalent and non-singular. We put $r=pq$. Then, $r\neq p, r\neq q, r\neq 0$, $0<\varphi(r)<1$ and, at the same time, $\alpha_s(r)=\alpha_s(pq)=pq(T_s^{-1}(\omega))=p(q(T_s^{-1}(\omega))=p(\alpha_s(q))=pq=r$ for all $s\in G$. It contradicts to an ergodicity of the state φ. □

We designate for a projection f: $\varphi_{\alpha_s}(f)=\varphi(\alpha_s(f))$.

We consider the ultraproduct $(\mathcal{A}_n,\varphi_n)_{\mathcal{U}}$ of the sequence of the abelian von Neumann algebras $(\mathcal{A}_n)_{n\in\mathbb{N}}$ with the actions α_n, on $\mathcal{A}_n$ and normal faithful states φ_n. Let $\mathcal{U}$ be a non-trivial ultrafilter on the set of natural numbers $\mathbb{N}$. Put

$$\alpha_{\mathcal{U}}((f_n)_{\mathcal{U}})=(\alpha_n(f_n))_{\mathcal{U}}.$$

Theorem 3. *If the transformation $(\alpha_n)_{s_n}$ is non-singular and free with respect to the state $\varphi_n, s_n\in G_n, n\in\mathbb{N}$, and the sequences of the states (φ_n) and $((\varphi_n)_{(\alpha_n)_{s_n}})$ are mutually contigual, then the action $\alpha_{\mathcal{U}}$ is non-singular and free on $(\mathcal{A}_n,\varphi_n)_{\mathcal{U}}$.*

Proof. It is clear that, generally, the action α can not be non-singular on the group $(G_n)_{\mathcal{U}}$. We put $G=\{s=(s_n)_{\mathcal{U}}\in(G_n)_{\mathcal{U}}:$ *the sequences (φ_n) and $((\varphi_n)_{(\alpha_n)_{s_n}})$ are mutually contigual*$\}$. Then, non-singularity of the action α_s, $s\in G$, is provided with the first statement of theorem 1. The freeness of the action α_s follows from definition 9 and the first statement of theorem 1. So, the action $\alpha_{\mathcal{U}}$ is non-singular and free. □

Let the action α_n be ergodic with respect to the normal faithful state φ_n on the algebra $\mathcal{A}_n, n\in\mathbb{N}$. Generally, $\alpha_{\mathcal{U}}$ in not ergodic with respect to the normal faithful state $\varphi_{\mathcal{U}}$. It follows from the example given below.

Example 1. Let $\Omega_n=\mathbb{R}^n$, μ_n be Gaussian measure $\mathcal{N}(0,I_n)$, G_n is the group of shifts on the elements of $\mathbb{R}^n$, $n\in\mathbb{N}$. It is known, (see [1]), that the ultraproduct $\mu_{\mathcal{U}}$ is quasi-invariant with respect to action on the group $G=\{x=(x_n)_{\mathcal{U}}:\sup_n\|x_n\|_{\ell_2}<\infty\}$. In other words, the action on the group G is non-singular with respect to measure $\mu_{\mathcal{U}}$.

Let us put $\mathcal{A}_n=L^\infty(\Omega_n,\mu_n)$ $\varphi_n(f_n)=\int f_n(x)d\mu_n$. It is clear that the state φ_n on the algebra $\mathcal{A}_n$ is normal and faithful. We consider the action α_n of the group G_n

on the algebra $\mathcal{A}_n$ as in the above. It is obvious that the action α_n is non-singular and free with respect to the state φ_n.

Now, we show that the measure $\mu_{\mathfrak{U}}$ is not ergodic with respect to action $\alpha_{\mathfrak{U}}$ on the group G. In the paper [1], it is shown that for any element $x \in G$ we have $\mu_{\mathfrak{U}}(B\Delta(B-x)) = 0$, where $B = (B_n)_{\mathfrak{U}}$, B_n is the ball on the $\mathbf{\Omega}_n$ of the radius $\sqrt{n}$ and with the center at zero. It is clear that the measure $\mu_{\mathfrak{U}}(B) = 1/2$. Therefore, the $\mu_{\mathfrak{U}}$ is not ergodic.

Further, we put $p = I_B$. Then, $\alpha_x(p) = p$ for all $x \in G$, at the same time we have $\varphi(p) \neq 0$ and $\varphi(p) \neq 1$. □

This result can be compared to Ando and Haagerup's results, see [5].

Conclusions

If to consider the results of this paper and paper [8], we can conclude that the ultraproducts of von Neumann algebras are characterized by unusual properties. We hope to obtain other interesting results in the our further investigations of the problem.

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Ультрапроизведения алгебр фон Неймана и эргодичность

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Аннотация

Ультрапроизведение произвольных линейных пространств по некоторому нетривиальному ультрафильтру в индексном множестве есть ни что иное, как обобщение нестандартного расширения ${}^*\mathbb{R}$ множества действительных чисел $\mathbb{R}$. Нестандартный математический анализ имеет свои объекты и методы исследования, которые лишь в определённой степени зависят от законов стандартного математического анализа.

В работе «нестандартные» объекты – ультрапроизведения алгебр фон Неймана – изучаются с точки зрения стандартного анализа. Такой подход позволяет, в частности, получить критерий «контигульности» последовательностей точных нормальных состояний в терминах «эквивалентности» состояний на соответствующих ультрапроизведениях.

Известно, что классическое ультрапроизведение алгебр фон Неймана, вообще говоря, не является алгеброй фон Неймана, поэтому мы рассматриваем специальную конструкцию ультрапроизведений алгебр фон Неймана, следуя работам А. Окненеу. Мы вводим понятие эргодического относительно некоторой группы преобразований состояния на алгебре фон Неймана и изучаем его свойства. Рассмотрено ультрапроизведение таких состояний и приведены их свойства. В частности, приведён пример, показывающий, что ультрапроизведение эргодических состояний не является, вообще говоря, эргодическим.

Ключевые слова: действие группы, эргодичность, состояния на алгебре фон Неймана

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