.

ON LOCAL STRUCTURE OF REARRANGEMENT INVARIANT SPACES OF FUNDAMENTAL TYPE

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1. **Global and local structure of Banach spaces.**

The simplest Banach spaces are

$$\ell^p = \{(a_n) : \|(a_n)\|_p := \Big(\sum_{n=1}^{\infty} |a_n|^p\Big)^{1/p} < \infty\}, \quad 1 \le p < \infty,$$

and $c_0 = \{(a_n) : \lim_{n\to\infty} a_n = 0\}$,

$$\|(a_n)\|_\infty := \max_{n=1,2,\dots} |a_n|.$$

Banach: Can ℓ^p and c_0 be considered as potential building blocks? Whether every (infinite–dimensional) Banach space must contain at least a copy of one of these spaces?

Tsirelson (1974), the first example of a Banach space that does not contain an isomorphic copy of (infinite–dimensional) ℓ^p, $1 \le p \le \infty$ ($\ell^\infty := c_0$).

In contrast to that, Dvoretzky (1960): any infinite–dimensional Banach space contains an arbitrarily good finite-dimensional copies of ℓ^2.

2. **Some definitions and results.**

Definition. A Banach space X contains a copy of ℓ^p, $1 \leq p \leq \infty$, if there exist a sequence $\{x_k\} \subset X$ and $C > 0$ such that for all $n \in \mathbb{N}$ and $a_k \in \mathbb{R}$

$$C^{-1}\|(a_k)\|_p \leq \Big\| \sum_{k=1}^{n} a_k x_k \Big\|_X \leq C\|(a_k)\|_p.$$

In other words: the sequence $\{x_k\}$ is equivalent to the unit vector basis of ℓ^p. We write $[x_k] \approx \ell^p$.

Definition. (James, 1967). Let X be a Banach space, $1 \leq p \leq \infty$. We say that ℓ^p is *finitely represented* in X if, for any $n \in \mathbb{N}$ and $\varepsilon > 0$, there exist $x_1, x_2, \dots, x_n \in X$ such that for all $a_1, a_2, \dots, a_n \in \mathbb{R}$

$$(1+\varepsilon)^{-1}\|(a_k)\|_p \leq \Big\| \sum_{k=1}^{n} a_k x_k \Big\|_X \leq (1+\varepsilon)\|(a_k)\|_p.$$

Theorem A (Dvoretzky (1960). *ℓ^2 is finitely represented in arbitrary Banach space.*

Theorem B (Krivine, 1976). *Let z_i, $i = 1, 2, \dots,$ do not form a relatively compact set in a Banach space X. Then ℓ^p is block finitely represented in $\{z_i\}_{i=1}^{\infty}$ for some $p \in [1, \infty]$, i.e., for every $n \in \mathbb{N}$ and $\varepsilon > 0$ there exist $0 = m_0 < m_1 < \dots < m_n$ and $\alpha_i \in \mathbb{R}$ such that the vectors*

$u_k = \sum_{i=m_{k-1}+1}^{m_k} \alpha_i z_i$, $k = 1, 2, \ldots, n$, *satisfy*

$$(1+\varepsilon)^{-1}\|(a_k)\|_p \leq \Big\| \sum_{k=1}^{n} a_k u_k \Big\|_X \leq (1+\varepsilon)\|(a_k)\|_p$$

for all $a_k \in \mathbb{R}$, $k = 1, \ldots, n$.

A problem: To identify the set of p such that ℓ^p is finitely represented in a given Banach space X.

3. **The Maurey-Pisier contribution.**

Let $r_n : [0,1] \to \mathbb{R}$ $(n \in \mathbb{N})$ be Rademacher functions, that is, $r_n(t) = sign(\sin 2^n \pi t)$. A Banach space X has *type* $p \in [1,2]$ if there is a constant $K > 0$ such that, for any $x_1, \dots, x_n \in X$, we have

$$\int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\| dt \leq K \left(\sum_{k=1}^n ||x_k||^p \right)^{1/p} .$$

A Banach space X has *cotype* $q \in [2, \infty]$ if there is a constant $K > 0$ such that, for any $x_1, \dots, x_n \in X$,

$$\left(\sum_{k=1}^n ||x_k||^q \right)^{1/q} \leq K \int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\| dt.$$

If $q = \infty$ we put $\max_{1 \leq k \leq n} ||x_k||$.

Every Banach space has type 1 and cotype ∞. Let

$$p_X := \sup\{p \in [1,2] : \ X \ \text{ has type } \ p\},$$

$$q_X := \inf\{q \in [2,\infty] : \ X \ \text{ has cotype } \ q\}.$$

Then, $p_X \leq q_X$. If ℓ^p is finitely represented in X, then $p \in [p_X, q_X]$.

Theorem C (Maurey-Pisier (1976)). *For any Banach space X l_{p_X} and l_{q_X} are finitely represented in X.*

4. **Lattice finite representability.**

Let X be a Banach lattice. Then, if ℓ^p is finitely represented in X and, moreover, elements $x_1, \dots, x_n \in X$ corresponding to the vectors of the standard basis of ℓ^p can be chosen every time disjoint, we say that ℓ^p is *lattice finitely represented* in X.

Let $1 \le p \le \infty$. A Banach lattice X satisfies an *p–upper estimate* if there exists $M > 0$ such that, for all disjoint elements $x_1, \dots, x_n$ from X,

$$\Big\| \sum_{k=1}^{n} x_k \Big\| \le M \left(\sum_{k=1}^{n} ||x_k||^p \right)^{1/p}.$$

If $1 \le q \le \infty$, then a Banach lattice X satisfies an *q–lower estimate* if there exists $M > 0$ such that, for all disjoint elements $x_1, \dots, x_n$ from X,

$$\left(\sum_{k=1}^{n} ||x_k||^q \right)^{1/q} \le M \Big\| \sum_{k=1}^{n} x_k \Big\|.$$

In the case when $p = \infty$ or $q = \infty$ we put $\max_{1 \le k \le n} ||x_k||$.

$$u_X := \sup\{p \ge 1 : \ X \ \text{ satisfies an } p\text{–upper estimate}\},$$

$$v_X := \inf\{q \ge 1 : \ X \ \text{ satisfies an } q\text{–lower estimate}\}.$$

Then, $u_X \le v_X$. If ℓ^p is lattice finitely represented in X, then $p \in [u_X, v_X]$.

Theorem D (Shepp). *Let X be a Banach lattice. Then l_{u_X} and l_{v_X} are lattice finitely represented in X.*

5. **Rearrangement invariant spaces and symmetric finite representability of ℓ^p-spaces.**

If a function $x(s)$ is measurable on $[0, \infty)$, then

$$n_x(t) := m(\{s > 0 : |x(s)| > t\}),$$

where m is the usual Lebesgue measure. Functions $x(s)$ and $y(s)$ are *equimeasurable* if $n_x(t) = n_y(t)$ for all $t > 0$.

A Banach function space X on $[0, \infty)$ is said to be a *r.i. (symmetric)* space if the conditions $x \in X$ and $n_y(t) \leq n_x(t)$ $(t > 0)$ imply that $y \in X$ and $\|y\|_X \leq \|x\|_X$.

A r.i. space X has *the Fatou property* if for any increasing sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ such that $0 \leq x_n \to x$ a.e. on $[0, \infty)$ and $\sup_n \|x_n\|_X < \infty$ we have: $x \in X$ and $\|x\|_X \leq \liminf_{n\to\infty} \|x_n\|_X$.

In what follows, a r.i. space X is either separable or it has the Fatou property.

Definition. We say that ℓ^p, $1 \leq p \leq \infty$, is *symmetrically finitely represented* in a r.i. space X $(p \in \mathcal{F}(X))$ if, for any $n \in \mathbb{N}$ and $\varepsilon > 0$, there exist equimeasurable and disjoint $x_k \in X$, $k = 1, 2, \ldots, n$, such that for any $a_k \in \mathbb{R}$

$$(1+\varepsilon)^{-1}\|(a_k)\|_p \leq \Big\| \sum_{k=1}^{n} a_k x_k \Big\|_X \leq (1+\varepsilon)\|(a_k)\|_p.$$

6. **An example: $L_{p,q}$-spaces.**

$1 < p < \infty$, $1 \le q < \infty$, $L_{p,q}$ consists of all measurable functions x on $(0,\infty)$ such that

$$\|x\|_{p,q} := \Big(\int_0^\infty x^*(t)^q \, d(t^{q/p})\Big)^{1/q} < \infty,$$

where x^* is the non-increasing rearrangement of $|x|$ (i.e., $n_{x^*}(t) = n_x(t)$, $t > 0$):

$$x^*(s) := \inf\{t > 0 : \, n_x(t) \le s\}, \quad s > 0.$$

(a) $q = p$, $L_{p,p} = L_p$: $[x_n] \approx \ell^p$ isometrically for every sequence $\{x_n\} \subset L_p$, $\|x_n\|_{L_p} = 1$, of equimeasurable and disjoint functions. For instance, for all $n \in \mathbb{N}$ and $a_k \in \mathbb{R}$

$$\Big\| \sum_{k=1}^n a_k \chi_{(k-1,k]} \Big\|_{L_p} = \Big(\sum_{k=1}^n |a_k|^p\Big)^{1/p}.$$

In particular, $\mathcal{F}(L_p) = \{p\}$.

(b) $q \ne p$: if $\{x_n\} \subset L_{p,q}$, $\|x_n\|_{p,q} = 1$, is a sequence of disjoint functions, then $[x_n]$ contains a subspace isomorphic to l_q (Carothers-Dilworth, 1988). More precisely, there exist $0 = m_0 < m_1 < \cdots < m_k < \ldots$ and $\alpha_i \in \mathbb{R}$ such that $[u_k] \approx \ell^q$, where $u_k = \sum_{i=m_{k-1}+1}^{m_k} \alpha_i x_i$, $k = 1, 2, \ldots$. Hence, if $\{x_n\} \subset L_{p,q}$ is a sequence of disjoint functions with $[x_n] \approx \ell^r$, then $r = q$.

At the same time, $\mathcal{F}(L_{p,q}) = \{p\}$.

7. **Boyd indices of r.i. spaces.**

For any $\tau > 0$, the dilation operator $\sigma_\tau x(t) := x(t/\tau)$ is bounded in any r.i. space X and $\|\sigma_\tau\|_{X\to X} \le \max(1,\tau)$. The *Boyd indices* of X:

$$\alpha_X = -\lim_{n\to\infty}\frac{1}{n}\log_2\|\sigma_{2^{-n}}\|_{X\to X},$$

$$\beta_X = \lim_{n\to\infty}\frac{1}{n}\log_2\|\sigma_{2^n}\|_{X\to X}.$$

Moreover, let $\sigma_\tau^0 x := \chi_{[0,1]}\sigma_\tau(x\chi_{[0,1]})$ and $\sigma_\tau^\infty x := \chi_{[1,\infty)}\sigma_\tau(x\chi_{[1,\infty]})$. Define the following partial Boyd indices:

$$\alpha_X^0 = -\lim_{n\to\infty}\frac{1}{n}\log_2\|\sigma_{2^{-n}}^0\|_{X\to X},$$

$$\beta_X^0 = \lim_{n\to\infty}\frac{1}{n}\log_2\|\sigma_{2^n}^0\|_{X\to X},$$

$$\alpha_X^\infty = -\lim_{n\to\infty}\frac{1}{n}\log_2\|\sigma_{2^{-n}}^\infty\|_{X\to X},$$

$$\beta_X^\infty = \lim_{n\to\infty}\frac{1}{n}\log_2\|\sigma_{2^n}^\infty\|_{X\to X}.$$

Always $0 \le \alpha_X \le \alpha_X^0 \le \beta_X^0 \le \beta_X \le 1$ and $0 \le \alpha_X \le \alpha_X^\infty \le \beta_X^\infty \le \beta_X \le 1$.

Then, $\mathcal{F}(X) \subset [1/\beta_X, 1/\alpha_X]$ for every X.

Theorem E. *For arbitrary r.i. space X: $\mathcal{F}(X) \ne \emptyset$, $\max\mathcal{F}(X) = 1/\alpha_X$, and $\min\mathcal{F}(X) = 1/\beta_X$.*

Theorem E has been stated (without proof) in Lindenstrauss-Tzafriri, Classical Banach spaces, II. Function Spaces, 1979. The proof was given in 2011 by A.

8. **A spectral characterization of the set $\mathcal{F}(X)$.**

Definition. Let X be a Banach space and let $A : X \to X$ be a bounded linear operator. A sequence $\{u_n\}_{n=1}^{\infty} \subset X$, $\|u_n\| = 1$, $n = 1, 2, \ldots$, is called *an approximate eigenvector* corresponding to an *approximate eigenvalue* $\lambda \in \mathbb{R}$ of the operator A if

$$\|Au_n - \lambda u_n\|_X \to 0$$

(equivalently: the operator $A - \lambda I$ is either not injective or not closed).

Theorem 1 (A,2011). *Let X be a separable r.i. space on $[0, \infty)$. Then $p \in \mathcal{F}(X)$ if and only if the number $\lambda := 2^{1/p}$ is an approximate eigenvalue of the doubling operator $\sigma_2 x(t) = x(t/2)$.*

The proof of the "if" part: if $\{g_k\}$ is an approximate eigenvector for λ, then for each $k \in \mathbb{N}$ we define a symmetric sequence space F_k by

$$\Big\| \sum_{j=1}^{m} a_j e_j \Big\|_{F_k} := \|a_1 g_k \oplus a_2 g_k \oplus \cdots \oplus a_m g_k\|_X, \ m \in \mathbb{R},$$

where $f \oplus g$ is the disjoint sum of functions f and g. Then, we construct (using some Rosenthal's ideas) a chain of symmetric sequence spaces, every of which is finitely represented in the previous one and the last one coincides isometrically with ℓ^p or c_0.

9. **R.i. spaces of fundamental type.**

Let ψ be a positive function on $(0,\infty)$. We introduce the *dilation functions*:

$$M_\psi(t) := \sup_{s>0} \frac{\psi(ts)}{\psi(s)}, \quad M_\psi^0(t) := \sup_{0<s\le\min(1,1/t)} \frac{\psi(ts)}{\psi(s)},$$

$$M_\psi^\infty(t) := \sup_{s\ge\max(1,1/t)} \frac{\psi(ts)}{\psi(s)}.$$

Let $\psi(t) = \phi_X(s)$, where ϕ_X is the fundamental function of X defined by $\phi_X(s) = \|\chi_{(0,s)}\|_X$. Since $\chi_{(0,st)} = \sigma_t(\chi_{(0,s)})$, then

$$M_{\phi_X}(t) = \sup_{s>0} \frac{\|\sigma_t \chi_{(0,s)}\|_X}{\|\chi_{(0,s)}\|_X} \le \|\sigma_t\|_{X\to X},$$

and similar inequalities for $M_{\phi_X}^0(t)$, $\|\sigma_t^0\|_{X\to X}$ and $M_{\phi_X}^\infty(t)$, $\|\sigma_t^\infty\|_{X\to X}$.

We say that a r.i. space X is of *fundamental type* if the opposite inequalities (up to constant) hold, i.e.,

$$\|\sigma_t\|_{X\to X} \le C M_{\phi_X}(t)\ , \ \|\sigma_t^0\|_{X\to X} \le C M_{\phi_X}^0(t),$$

$$\|\sigma_t^\infty\|_{X\to X} \le C M_{\phi_X}^\infty(t), \ \ t>0.$$

The most known and important r.i. spaces are of fundamental type (the first example of a r.i. space of non-fundamental type was constructed by Shimogaki, 1970).

The Boyd indices of a r.i. space are of fundamental type can be calculated via its fundamental function.

10. **Examples of r.i. spaces.**

(a) *Lorentz spaces.* Let $1 \leq q < \infty$, ψ be an increasing concave function on $[0, \infty)$ such that $\psi(0) = 0$. The *Lorentz space* $\Lambda_q(\psi)$ is defined by

$$\|x\|_{\Lambda_q(\psi)} := \left(\int_0^\infty x^*(t)^q d\psi(t) \right)^{1/q} < \infty.$$

$\Lambda_q(\psi)$ is a separable r.i. space with the Fatou property and $\phi_{\Lambda_q(\psi)}(t) = \psi(t)^{1/q}$. The dilation indices of $\Lambda_q(\psi)$ can be calculated by

$$\alpha_{\psi,q} = - \lim_{n \to \infty} \frac{1}{n} \log_2 \sup_{k \in \mathbb{Z}} \left(\frac{\psi(2^k)}{\psi(2^{n+k})} \right)^{1/q},$$

$$\beta_{\psi,q} = \lim_{n \to \infty} \frac{1}{n} \log_2 \sup_{k \in \mathbb{Z}} \left(\frac{\psi(2^{n+k})}{\psi(2^k)} \right)^{1/q},$$

$$\alpha^0_{\psi,q} = - \lim_{n \to \infty} \frac{1}{n} \log_2 \sup_{k \leq 0} \left(\frac{\psi(2^{k-n})}{\psi(2^k)} \right)^{1/q},$$

$$\beta^0_{\psi,q} = \lim_{n \to \infty} \frac{1}{n} \log_2 \sup_{k \leq 0} \left(\frac{\psi(2^k)}{\psi(2^{k-n})} \right)^{1/q},$$

$$\alpha^\infty_{\psi,q} = - \lim_{n \to \infty} \frac{1}{n} \log_2 \sup_{k \geq 0} \left(\frac{\psi(2^k)}{\psi(2^{k+n})} \right)^{1/q},$$

$$\beta^\infty_{\psi,q} = \lim_{n \to \infty} \frac{1}{n} \log_2 \sup_{k \geq 0} \left(\frac{\psi(2^{n+k})}{\psi(2^k)} \right)^{1/q}.$$

(b) *Orlicz spaces.* Let N be a convex continuous increasing function on $[0,\infty)$, $N(0) = 0$ and $N(\infty) = \infty$. The *Orlicz space* L_N is defined by the Luxemburg norm

$$\|x\|_{L_N} := \inf\left\{u > 0 \,:\, \int_0^\infty N(|x(t)|/u)\,dt \le 1\right\} < \infty.$$

In particular, $L^p = L_{N_p}$, $N_p(s) = s^p$, $1 \le p < \infty$.

Every Orlicz space L_N has the Fatou property and it is separable iff the function N satisfies the Δ_2-*condition*, i.e., $\sup_{u>0} \frac{N(2u)}{N(u)} < \infty$. The fundamental function $\phi_{L_N}(t) = 1/N^{-1}(1/t)$, $t > 0$, where N^{-1} is the inverse function for N. The dilation indices of L_N can be calculated by

$$\alpha_N = -\lim_{n\to\infty} \frac{1}{n} \log_2 \sup_{k\in\mathbb{Z}} \frac{N^{-1}(2^{k-n})}{N^{-1}(2^k)},$$

$$\beta_N = \lim_{n\to\infty} \frac{1}{n} \log_2 \sup_{k\in\mathbb{Z}} \frac{N^{-1}(2^k)}{N^{-1}(2^{k-n})},$$

$$\alpha_N^0 = -\lim_{n\to\infty} \frac{1}{n} \log_2 \sup_{k\ge 0} \frac{N^{-1}(2^k)}{N^{-1}(2^{k+n})},$$

$$\beta_N^0 = \lim_{n\to\infty} \frac{1}{n} \log_2 \sup_{k\ge 0} \frac{N^{-1}(2^{k+n})}{N^{-1}(2^k)},$$

$$\alpha_N^\infty = -\lim_{n\to\infty} \frac{1}{n} \log_2 \sup_{k\le 0} \frac{N^{-1}(2^{k-n})}{N^{-1}(2^k)},$$

$$\beta_N^\infty = \lim_{n\to\infty} \frac{1}{n} \log_2 \sup_{k\le 0} \frac{N^{-1}(2^k)}{N^{-1}(2^{k-n})}.$$

11. **Main results.**

Theorem 2. *Let X be a r.i. space on $(0,\infty)$ of fundamental type.*

(i) If $\alpha_X^\infty \le \beta_X^0$, then the set of approximate eigenvalues of σ_2 in X is the interval $[2^{\alpha_X}, 2^{\beta_X}]$.

(ii) If $\alpha_X^\infty > \beta_X^0$, then the set of approximate eigenvalues of σ_2 in X is the union $[2^{\alpha_X}, 2^{\beta_X^0}] \cup [2^{\alpha_X^\infty}, 2^{\beta_X}]$.

From Theorems 1 and 2 it follows a description of the set of $p \in [1,\infty]$ such that ℓ^p is symmetrically finitely represented in a given separable r.i. space of fundamental type.

Corollary 1. *Let X be a separable r.i. space of fundamental type. Then,*

(i) if $\alpha_X^\infty \le \beta_X^0$, then $\mathcal{F}(X) = [1/\beta_X, 1/\alpha_X]$;

(ii) if $\alpha_X^\infty > \beta_X^0$, then $\mathcal{F}(X) = [1/\beta_X, 1/\alpha_X^\infty] \cup [1/\beta_X^0, 1/\alpha_X]$.

Corollary 2. *If X and Y are separable r.i. spaces of fundamental type such that $\phi_X(t) \asymp \phi_Y(t)$, $t > 0$, then $\mathcal{F}(X) = \mathcal{F}(Y)$.*

In particular, $\phi_{L_{p,q}}(t) = t^{1/p}$, for all $1 \le q < \infty$. Hence, $\mathcal{F}(L_{p,q}) = \{p\}$, $1 \le q < \infty$.

12. **Reduction to the task of identification of the set of approximate eigenvalues of the shift operator in a certain Banach sequence lattice.**

Let X be a r.i. space on $(0,\infty)$. We associate to X the Banach sequence lattice E_X equipped with the norm

$$\|(a_k)_{k\in\mathbb{Z}}\|_{E_X} := \Big\| \sum_{k\in\mathbb{Z}} a_k \chi_{\Delta_k} \Big\|_X,$$

where $\Delta_k = [2^k, 2^{k+1})$, $k \in \mathbb{Z}$. The sequence of $\{\chi_{\Delta_k}\}_{k\in\mathbb{Z}}$ is equivalent in X to the unit vector basis in E_X.

Let $\sigma_2 x(t) = x(t/2)$, $t > 0$, $\tau(a_k) = (a_{k-1})$, $k \in \mathbb{Z}$, $\sigma_\lambda := \sigma - \lambda I$, $\tau_\lambda = \tau - \lambda I$, $\lambda > 0$ (I is the identity).

Proposition 1. *For every r.i. space X on $(0,\infty)$ and any $\lambda > 0$*

(i) *σ_λ is injective in X iff τ_λ is injective in E_X;*

(ii) *if σ_λ is closed in X, then τ_λ is closed in E_X;*

(iii) *if τ_λ is injective and closed in E_X, then σ_λ is closed in X.*

Corollary 3. *Let X be a r.i. space on $(0,\infty)$. Then, λ is an approximate eigenvalue of the doubling operator σ_2 in X iff λ is an approximate eigenvalue of the shift operator τ in E_X.*

13. **The shift exponents of Banach sequence lattices.**

Let E be a Banach sequence lattice modelled on $\mathbb{Z}$ such that $\tau_n a := (a_{k-n})$, where $a = (a_k)$, is bounded in E for every $n \in \mathbb{Z}$. Let $\mathbb{Z}_- = \{k \in \mathbb{Z} : \ k \le 0\}$, $\mathbb{Z}_+ = \{k \in \mathbb{Z} : \ k \ge 0\}$,

$$\tau_n^0 a := \chi_{\mathbb{Z}_-} \cdot \tau_n(a\chi_{\mathbb{Z}_-}), \quad \tau_n^\infty a := \chi_{\mathbb{Z}_+} \cdot \tau_n(a\chi_{\mathbb{Z}_+}).$$

The shift exponents of E are defined by

$$\gamma_E = -\lim_{n\to\infty} \frac{1}{n} \log_2 \|\tau_{-n}\|, \ \ \delta_E = \lim_{n\to\infty} \frac{1}{n} \log_2 \|\tau_n\|,$$
$$\gamma_E^0 = -\lim_{n\to\infty} \frac{1}{n} \log_2 \|\tau_{-n}^0\|, \ \ \delta_E^0 = \lim_{n\to\infty} \frac{1}{n} \log_2 \|\tau_n^0\|,$$
$$\gamma_E^\infty = -\lim_{n\to\infty} \frac{1}{n} \log_2 \|\tau_{-n}^\infty\|, \ \ \delta_E^\infty = \lim_{n\to\infty} \frac{1}{n} \log_2 \|\tau_n^\infty\|.$$

There are direct connections between the norms of the dilation operators in X and the shift operators in E_X.

Proposition 2. *For every r.i. space X on $(0,\infty)$*

$$\|\tau_n\|_{E_X} \asymp \|\sigma_{2^n}\|_X \ , \ \|\tau_n^0\|_{E_X} \asymp \|\sigma_{2^n}^0\|_X \ , \ \|\tau_n^\infty\|_{E_X} \asymp \|\sigma_{2^n}^\infty\|_X.$$

Hence, $\alpha_X = \gamma_{E_X}$, $\alpha_X^0 = \gamma_{E_X}^0$, $\alpha_X^\infty = \gamma_{E_X}^\infty$, $\beta_X = \delta_{E_X}$, $\beta_X^0 = \delta_{E_X}^0$ and $\beta_X^\infty = \delta_{E_X}^\infty$.

14. **A description of approximative eigenvalues of the shift operator in Banach sequence lattices.**

Denote $s_k := \|e_k\|_E$, where e_k, $k \in \mathbb{Z}$, are elements of the unit vector basis. We say that E is a lattice of fundamental type if for all $n \in \mathbb{N}$

$$\|\tau_{-n}\|_{E\to E} \asymp \sup_{k\in\mathbb{R}} \frac{s_k}{s_{n+k}}, \quad \|\tau_n\|_{E\to E} \asymp \sup_{k\in\mathbb{R}} \frac{s_{k+n}}{s_k},$$
$$\|\tau^0_{-n}\|_{E\to E} \asymp \sup_{k\geq 0} \frac{s_k}{s_{n+k}}, \quad \|\tau^0_n\|_{E\to E} \asymp \sup_{k\geq 0} \frac{s_{k+n}}{s_k},$$
$$\|\tau^\infty_{-n}\|_{E\to E} \asymp \sup_{k\leq 0} \frac{s_{k-n}}{s_k}, \quad \|\tau^\infty_n\|_{E\to E} \asymp \sup_{k\leq 0} \frac{s_k}{s_{k-n}}.$$

Then, the shift exponents of E can be calculated by using s_k, $k \in \mathbb{Z}$.

The main technical result is following.

Theorem 3. *Suppose a separable Banach sequence lattice E is of fundamental type.*

(i) if $\gamma^\infty_E \leq \delta^0_E$, then the operator τ_λ is an isomorphic embedding in E iff $\lambda \in (0, 2^{\gamma_E}) \cup (2^{\delta_E}, \infty)$;

(ii) if $\gamma^\infty_E > \delta^0_E$, then τ_λ is an isomorphic embedding in E iff $\lambda \in (0, 2^{\gamma_E}) \cup (2^{\delta^0_E}, 2^{\gamma^\infty_E}) \cup (2^{\delta_E}, \infty)$.

Moreover, if $\lambda \in (0, 2^{\gamma_E}) \cup (2^{\delta_E}, \infty)$, then $\operatorname{Im} \tau_\lambda = E$; *if $\lambda \in (2^{\delta^0_E}, 2^{\gamma^\infty_E})$, then* $\operatorname{Im} \tau_\lambda$ *is the closed subspace of codimension* 1 *in E consisting of all $(a_k) \in E$ with*

$$\sum_{k\in\mathbb{Z}} \lambda^k a_k = 0.$$

Corollary 4. *Let E be a separable Banach sequence lattice of fundamental type. Then,*

(i) if $\gamma_E^\infty \le \delta_E^0$, then the set of approximative eigenvalues of τ is $[2^{\gamma_E}, 2^{\delta_E}]$;

(ii) if $\gamma_E^\infty > \delta_E^0$, the set of approximative eigenvalues of τ is the union $[2^{\gamma_E}, 2^{\delta_E^0}] \cup [2^{\gamma_E^\infty}, 2^{\delta_E}]$.

Necessity in Theorem 3.

A key fact: Suppose τ_λ is an isomorphic mapping in E. Then, the following implication holds:

$$\lambda \notin (0, 2^{\gamma_E}] \cup [2^{\delta_E}, \infty) \implies \lambda \in [2^{\delta_E^0}, 2^{\gamma_E^\infty}].$$

Let $n \in \mathbb{N}$, $k \in \mathbb{Z}$ and $x_{n,k} = (I + \lambda^{-1}\tau + \cdots + \lambda^{-n}\tau^n)^2 e_k$. By the assumption, there exists $c > 0$ such that for all $n \in \mathbb{N}$, $k \in \mathbb{Z}$

$$\|\tau_\lambda x_{n,k}\|_E \ge c\|x_{n,k}\|_E.$$

Then the sequence $(\lambda^{k+rn}\|e_{k+rn}\|_E)_{r=0}^\infty$ is increasing for some $n \in \mathbb{N}$, $k \in \mathbb{Z}$.

Proof of Theorem 2.

Since X is a r.i. space of fundamental type and $\phi_X(2^k) = \|\chi_{\Delta_k}\|_X = \|e_k\|_{E_X} = s_k$ for all $k \in \mathbb{Z}$, by Proposition 2, E_X is also of fundamental type. It remains to combine the results of Corollaries 3 and 4 (for $E = E_X$).

15. **A description of the set $\mathcal{F}(X)$ for Lorentz and Orlicz spaces.**

Theorem 4. *(i) If* $\alpha_{\psi,q}^{\infty} \leq \beta_{\psi,q}^{0}$, *then* $\mathcal{F}(\Lambda_q(\psi)) = [1/\beta_{\psi,q}, 1/\alpha_{\psi,q}]$.

(ii) If $\alpha_{\psi,q}^{\infty} > \beta_{\psi,q}^{0}$, *then* $\mathcal{F}(\Lambda_q(\psi)) = [1/\beta_{\psi,q}, 1/\alpha_{\psi,q}^{\infty}] \cup [1/\beta_{\psi,q}^{0}, 1/\alpha_{\psi,q}]$.

An Orlicz space L_N is separable iff the function N satisfies the Δ_2-condition iff $\alpha_N > 0$.

Theorem 5. *If an Orlicz function* N *is such that* $\alpha_N > 0$, *then*

(i) if $\alpha_N^{\infty} \leq \beta_N^0$, *then* $\mathcal{F}(L_N) = [1/\beta_N, 1/\alpha_N]$;

(ii) if $\alpha_N^{\infty} > \beta_N^0$, *then* $\mathcal{F}(L_N) = [1/\beta_N, 1/\alpha_N^{\infty}] \cup [1/\beta_N^0, 1/\alpha_N]$.

16. **Concluding examples and remarks.**

Example 1. For any $1 < p < r < \infty$ we define the functions $\psi_1(t) = \max\left(t^{1/p}, t^{1/r}\right)$ and $\psi_2(t) = \min\left(t^{1/p}, t^{1/r}\right)$. Let us find the set $\mathcal{F}(X)$ for $X = \Lambda_1(\psi_i)$, $i = 1, 2$.

For ψ_1: $\alpha_{\psi_1} = \beta^0_{\psi_1} = 1/r$, $\alpha^\infty_{\psi_1} = \beta_{\psi_1} = 1/p$. Since $\alpha^\infty_{\psi_1} > \beta^0_{\psi_1}$, we get $\mathcal{F}(\Lambda_1(\psi_1)) = \{p, r\}$.

For ψ_2: $\alpha_{\psi_2} = \alpha^\infty_{\psi_2} = 1/r$, $\beta_{\psi_2} = \beta^0_{\psi_2} = 1/p$. Since $\alpha^\infty_{\psi_2} < \beta^0_{\psi_2}$, we get $\mathcal{F}(\Lambda_1(\psi_2)) = [p, r]$.

Example 2. Let $1 \le p < r \le \infty$, $X = L_p(0,\infty) \cap L_r(0,\infty)$, and $Y = L_p(0,\infty) + L_r(0,\infty)$, with usual norms:

$$\|f\|_X := \max(\|f\|_{L_p}, \|f\|_{L_r}),$$

$$\|f\|_Y := \inf\{\|g\|_{L_p} + \|h\|_{L_r} : \ f = g + h\,,\ g \in L_p\,,\ h \in L_r\}.$$

Then, X and Y are Orlicz spaces, $X = L_{N_1}$ and $Y = L_{N_2}$, where $N_1(t) = \max\left(t^p, t^r\right)$ and $N_2(t) = \min\left(t^p, t^r\right)$. By Theorem 5,

$$\mathcal{F}(X) = \{p, r\} \ \text{ and } \ \mathcal{F}(Y) = [p, r]$$

(this result has been proved earlier by Schep by using different methods).

Remark 1. For any four numbers a, b, c, and d with

$$0 < a \le \min(b,c) \le \max(b,c) \le d < 1$$

there exists a concave on $(0,\infty)$ function ψ, for which $\alpha_\psi = a$, $\alpha_\psi^\infty = b$, $\beta_\psi^0 = c$, and $\beta_\psi = d$. Then, by Theorem 4,

$$\mathcal{F}(\Lambda_1(\psi)) = [1/d, 1/a] \ \text{ if } \ b \le c,$$

$$\mathcal{F}(\Lambda_1(\psi)) = [1/d, 1/b] \cup [1/c, 1/a] \ \text{ if } \ b > c.$$

Remark 2. Let X be an arbitrary separable r.i. space X (in general, of non-fundamental type). Denote by μ_{ϕ_X}, ν_{ϕ_X}, $\mu_{\phi_X}^\infty$ and $\nu_{\phi_X}^0$ the indices corresponding to the dilation functions $M_{\phi_X}(t)$, $M_{\phi_X}^\infty(t)$ and $M_{\phi_X}^0(t)$. Then, the following result holds:

If $\mu_{\phi_X}^\infty \le \nu_{\phi_X}^0$ (resp. $\mu_{\phi_X}^\infty > \nu_{\phi_X}^0$), then the interval $[2^{\mu_{\phi_X}}, 2^{\nu_{\phi_X}}]$ (resp. the union $[2^{\mu_{\phi_X}}, 2^{\nu_{\phi_X}^0}] \cup [2^{\mu_{\phi_X}^\infty}, 2^{\nu_{\phi_X}}]$) consists of approximate eigenvalues of the operator σ_2 in X. Therefore, in the first case

$$\mathcal{F}(X) \supset [1/\nu_{\phi_X}, 1/\mu_{\phi_X}],$$

in the second

$$\mathcal{F}(X) \supset [1/\nu_{\phi_X}, 1/\mu_{\phi_X}^\infty] \cup [1/\nu_{\phi_X}^0, 1/\mu_{\phi_X}].$$

Problem 1. To characterize the set of r such that ℓ^r is lattice finitely represented in a given Banach (function) lattice X. In particular, to find the set of r such that ℓ^r is lattice finitely represented in $L_{p,q}$, $1 < p < \infty$, $1 \le q < \infty$.