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FEYNMAN CALCULUS FOR RANDOM OPERATOR-VALUED FUNCTIONS AND THEIR APPLICATIONS

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Abstract

The Feynman–Chernoff iteration of a random semigroup of bounded linear operators in the Hilbert space has been considered. The convergence of mean values of the Feynman–Chernoff iteration of a random semigroup has been studied. The estimates of the deviation of compositions of the independent identically distributed random semigroup from its mean value have been obtained as the large numbers law for the sequence of compositions of the independent random semigroup has been investigated. The relationship between the semigroup properties of the mean values of the random operator-valued function and the property of independence of the increments of the random operator-valued function has been analyzed. The property of asymptotic independence of the increments of the Feynman–Chernoff iteration of the random semigroup has been discussed. The independization of the random operator-valued function has been defined as the map of this random operator function into the sequence of random operator-valued functions, which has asymptotically independent increments. The examples of independization (which is similar to the Feynman–Chernoff iteration) of the random operator-valued function have been given.

Keywords: random operator, random semigroup, Feynman–Chernoff iteration, large numbers law

Let H be a Hilbert space and $B(H)$ be a Banach space of bounded linear operators acting in the space H. The random operator-valued function is defined as a random variable with the values in the measurable space $(C_s(R_+, B(H)), \mathcal{A}_s)$, where $C_s(R_+, B(H))$ is the topological vector space of continuous maps of the semiaxe $R_+ = [0, +\infty)$ into the space $B(H)$ endowing with a strong-operator topology and $\mathcal{A}_s$ is the corresponding σ-algebra of the Borel subsets. Two random operator-valued functions ξ, η are equivalent to each other if $P(\xi \in A) = P(\eta \in A)$ for any $A \in \mathcal{A}_s$. The random operator function is called a random semigroup if its values in the space $C_s(R_+, B(H))$ are C_0-semigroups.

For any operator-valued function $\mathbf{F} \in C_s(R_+, B(H))$, the sequence of operator-valued functions $\{(\mathbf{F}(t/n))^n,\ t \geq 0,\}$ is called the Feynman–Chernoff iteration of the function $\mathbf{F}$.

The Feynman–Chernoff iteration of the random semigroup ξ is the sequence of the random operator-valued function $\{\Phi_n \xi\}$, such that $\Phi_n\xi(t) = \xi_n(t/n) \circ \cdots \circ \xi_1(t/n)$, $t \in R_+$ for any $n \in \mathbf{N}$. Here, $\xi_k,\ k = 1, \ldots, n,$ are independent random semigroups, any of which is equivalent to the random semigroup ξ. In other words, the random operator-valued function $\Phi_n\xi(t),\ t \in R_+$ is the composition of n-independent identically distributed random semigroups $\xi_k(t/n),\ t \in R_+,\ k \in \{1, \ldots, n\}$.

The paper [1] is devoted to the investigation of the Feymnan–Chernoff iteration. The sufficient condition for convergence of the sequence of the Feynman–Chernoff iteration of operator-valued functions was obtained, and the method of generalized averaging of random semigroups and its random generators by means of the Chernoff theorem was constructed. In this approach, the methods of analysis in the topological vector spaces (see [2]) were used.

In this paper, the convergence of the mean values of the Feynman–Chernoff iteration of the random semigroup and the large numbers law for the sequence of compositions of independent identically distributed random semigroups (see [3]) is studied. The property of asymptotic independence of the Feynman–Chernoff iteration of the random semigroup is investigated. The map $\mathcal{J}$ of the linear space of the random operator-valued function into the linear space of the sequence of the random operator-valued function is called the independization of the set S of the random operator-valued function if the sequence of random functions $\mathcal{J}(\mathbf{F})$ is asymptotically independent for any $\mathbf{F} \in S$. We provide the examples of independization of the random operator-valued function.

Let $\mathcal{M}(C_s(R_+, B(H)), \mathcal{A}_s)$ be the symbol of the vector space of the random $C_s(R_+, B(H))$-valued function. Let $\mathcal{N}(C_s(R_+, B(H)), \mathcal{A}_s)$ be the symbol noting the vector space of mappings $\mathbb{N} \to \mathcal{M}(C_s(R_+, B(H)), \mathcal{A}_s)$ (the vector space of the sequences of $C_s(R_+, B(H))$-valued functions.

Definition 1. (See [1]). The functions $\mathbf{F}$, $\mathbf{G} \in C_s(R_+, B(H))$, such that $\mathbf{F}(0) = \mathbf{G}(0) = \mathbf{I}$, are equivalent in the Chernoff sense if the equality $\lim\limits_{n\to\infty} \sup\limits_{t\in[0,T]} \|((\mathbf{G}(t/n))^n - (\mathbf{F}(t/n))^n)u\| = 0$ holds for any $T > 0$ and any $u \in H$. This fact will be noted by the symbol $\mathbf{F} \sim \mathbf{G}$.

The function $\mathbf{F}$ is equivalent to the semigroup $\mathbf{U}$ if $(\mathbf{F}(t/n))^n \to \mathbf{U}(t)$ is uniform on any segment. The representation of the semigroup $\mathbf{U}$ as the limit $\lim\limits_{n\to\infty} (\mathbf{F}(t/n))^n$ is called the Feynman formula ([1]).

Definition 2. The semigroup $\mathbf{U} \in C_s(R_+, B(H))$ is the generalized mean value of the random semigroup ξ if $\mathbf{U} \sim \mathrm{M}\xi$.

Theorem 1 (See [1]**).** *Let $\xi : \Omega \to C_s(R_+, B(H))$ be a random semigroup whose values ξ_ω, $\omega \in \Omega$, is the strongly continuous contractive semigroup in space H with generators $\mathbf{L}_\omega$, $\omega \in \Omega$. Let there exists $D \subset H$, such that D be the essential domain of operators $\mathbf{L}_\omega$, $\omega \in \Omega$ and $\int\limits_\Omega \|\mathbf{L}_\omega x\| \, d\mu(\omega) < \infty$ for any $x \in D$. If the operator $\mathbf{S}u = \int\limits_\Omega \mathbf{L}_\omega u \, d\mu(\omega)$, $u \in D$, is closed and its closure is the generator of continuous semigroup, then*

$$\exp(t\mathbf{S}) \sim \mathrm{M}\xi(t),$$

where $\mathrm{M}\xi(t)$ is the usual mean value of the random semigroup.

Corollary 1. *Let $\{\mu_n\}$: $\mu_n \geq 0$, $\sum\limits_{n=1}^{\infty} \mu_n = 1$. If $\{\mathbf{L}_j\}$ is a uniformly bounded sequence in the space $B(X)$, then $\sum\limits_{j=1}^{\infty} \mu_j e^{t\mathbf{L}_j} \sim \exp\left(t \sum\limits_{j=1}^{\infty} \mu_j \mathbf{L}_j\right)$.*

Now, we investigate the properties of the sequence of compositions of independent identically distributed random semigroups of the linear operator. By using the notion of the Chernoff equivalence and the Chernoff theorem, we obtain the analog of large numbers law for the products of independent identically distributed random semigroups of the linear operator.

The random linear operator $\mathbf{U}$ is the measurable map of the set Ω of the probability space $(\Omega, \mathcal{A}, \mu)$ into the topological vector space $(B(H), \tau_s)$ endowing with the σ-algebra $\mathcal{A}_s$ of the Borel subsets. The random operators $\mathbf{U}_1, \mathbf{U}_2$ are called independent random operators if the equality $\mu(\{\mathbf{U}_1 \in A_1,\, \mathbf{U}_2 \in A_2\}) = \mu(\{\mathbf{U}_1 \in A_1\})\mu(\{\mathbf{U}_2 \in A_2\})$ holds for any $A_1, A_2 \in \mathcal{A}_s$.

Let $\{(\Omega_n, \mathcal{A}_n, \mu_n)\}$ be the sequence of probability spaces. Let $\mathbf{U}_k : \Omega_k \to C_s(R_+, B(H))$ be the random semigroup of bounded linear operators in the space H for any $k \in \mathbf{N}$. Let $\{\mathbf{U}_n\}$ be the sequence of independent random semigroup of bounded linear operators in the space H.

This sequence $\{\mathbf{U}_n\}$ can be realized in the following way. For any $n \in \mathbf{N}$, the composition of n independent random semigroups of bounded linear operators $\mathbf{U}_1, \dots, \mathbf{U}_n$ is defined as the map $\mathbf{U}_n \circ \cdots \circ \mathbf{U}_1$ of the space with the measure $(\Omega_1 \times \cdots \times \Omega_n,\, \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n,\, \mu_1 \otimes \cdots \otimes \mu_n)$ into measurable space $(C_s(R_+, B(H)), \mathcal{A}_s)$, which is defined by the equality $\mathbf{U}_n \circ \cdots \circ \mathbf{U}_1(\omega_1, \dots, \omega_n) = \mathbf{U}_n(\omega_n) \circ \cdots \circ \mathbf{U}_1(\omega_1)$, $(\omega_1, \dots, \omega_n) \in \Omega_1 \times \cdots \times \Omega_n$. Then, for any $n \in \mathbf{N}$, the composition of n independent random semigroups of linear operators is the random operator-valued function with values in the measurable space $(C_s(R_+, B(H)), \mathcal{A}_s)$.

Theorem 2. *If the random operators $\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_n$ are independent for any $n \in \mathbf{N}$, then*

$$M[\mathbf{U}_n \circ \cdots \circ \mathbf{U}_1] = M[\mathbf{U}_n]) \circ \cdots \circ M[\mathbf{U}_1] \quad \forall\, n \in \mathbf{N}.$$

This statement is the consequence of the Fubini theorem (see [4, theorem 3.4.4]) in its application to the function

$$f_{u,v,t}(\omega_1, \dots, \omega_n) = (\mathbf{U}_n(\omega_n) \circ \cdots \circ \mathbf{U}_1(\omega_1)u, v)_H, \quad (\omega_1, \dots, \omega_n) \in \Omega_1 \times \cdots \times \Omega_n,$$

for arbitrary $u,\, v \in H$.

Corollary 2. *Let $\mathbf{U}$ be some random operator. Let $n \in \mathbf{N}$ and $\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_n$ be the independent and identically distributed random operators, such that the random operator $\mathbf{U}_k$ has the same distribution as the random operator $\mathbf{U}$ for any $k \in \{1, \dots, n\}$. Then*

$$M[\mathbf{U}_n \circ \cdots \circ \mathbf{U}_1] = (M[\mathbf{U}])^n \quad \forall\, n \in \mathbf{N}. \tag{1}$$

Now, we consider the large numbers law for the compositions of linear maps.

For the sequence $\{\xi_k\}$ of independent real-valued random variables with finite mean values, the large numbers law states that

$$\lim_{n\to\infty} P\left(\left\{\left|\frac{\xi_n}{n} + \cdots + \frac{\xi_1}{n} - \mathrm{M}\left(\frac{\xi_n}{n} + \cdots + \frac{\xi_1}{n}\right)\right| > \epsilon\right\}\right) = 0.$$

We extend this definition to the sequence $\{\mathbf{U}_k\}$ of independent random semigroups of the bounded linear operators by consideration of the averaging composition $(\mathbf{U}_n)^{1/n} \circ \cdots \circ (\mathbf{U}_1)^{1/n}$ instead of the averaged sum $\frac{\xi_n}{n} + \cdots + \frac{\xi_1}{n}$. The especial property of the semigroup of linear operators in some Hilbert space is the opportunity to define the operator-valued function $(\mathbf{U}(t))^{1/n}$, $t \geq 0$, by the equality $(\mathbf{U}(t))^{1/n} = \mathbf{U}(t/n)$, $t \geq 0$. This is not a unique root of the equality $(\mathbf{X}(t))^n = \mathbf{U}(t)$, $t \geq 0$, but the choice of the solution $\mathbf{X}(t) = \mathbf{U}(t/n)$, $t \geq 0$, gives some dynamical definition of the unique root.

Therefore, the mean value of the averaged composition $(\mathbf{U}_n(t))^{1/n} \circ \cdots \circ (\mathbf{U}_1(t))^{1/n}$, $t \geq 0$, of independent random semigroups is the iteration $(\mathbf{F}(t/n))^n$, $t \geq 0$, of the operator-valued function $\mathbf{F} = \mathrm{M}[\mathbf{U}]$. Thus, the Chernoff theorem describes the asymptotic properties of the mean value of the composition of n-independent random semigroups for $n \to \infty$.

Definition 3. The sequence $\{\mathbf{U}_n\}$ of the independent random semigroups of bounded linear operators with the common mean value $\bar{\mathbf{U}}$ is said to satisfy the large numbers law in the strong operator topology if the following conditions

$$\lim_{n\to\infty} P(\{\|((\mathbf{U}_n(t))^{1/n}\circ\cdots\circ(\mathbf{U}_1(t))^{1/n} - M[(\mathbf{U}_n(t))^{1/n}\circ\cdots\circ(\mathbf{U}_1(t))^{1/n}])x\|_H > \epsilon\}) = 0 \quad (2)$$

are satisfied for any $x \in H$, any $t > 0$ and any $\epsilon > 0$.

The sequence $\{\mathbf{U}_n\}$ of the independent random semigroups of bounded linear operators with the common mean value $\bar{\mathbf{U}}$ is said to satisfy the large numbers law in the topology of operators norm if the following conditions

$$\lim_{n\to\infty} P(\{\|(\mathbf{U}_n(t))^{1/n}\circ\cdots\circ(\mathbf{U}_1(t))^{1/n} - M[(\mathbf{U}_n(t))^{1/n}\circ\cdots\circ(\mathbf{U}_1(t))^{1/n}\|_{B(H)} > \epsilon\}) = 0 \quad (3)$$

are satisfied for any $t > 0$, and any $\epsilon > 0$.

To obtain some estimates for the composition of independent random maps and to obtain the sufficient condition for the large numbers law for random maps, we introduce the notion of dispersion of random bounded linear operators and random semigroup.

The second moment of random bounded linear operator $\mathbf{A}$ is the operator $M[\mathbf{A}^*\mathbf{A}]$.

The second moment of random semigroups $\mathbf{U}$ is the operator-valued function $M[\mathbf{U}^*\mathbf{U}]$.

Thus, the second moment of random bounded linear operator is the nonnegative bounded linear operator. The dispersion of the random semigroup $\mathbf{U}$ is the nonnegative operator-valued function

$$D(\mathbf{U}) = M[\mathbf{U} - M[\mathbf{U}]] = M[\mathbf{U}^*\mathbf{U}] - M[\mathbf{U}]^* M[\mathbf{U}].$$

Now, we investigate the second moments of the values of the sequence of iterations of independent random linear operators.

Let $t \geq 0$ and $n \in \mathbf{N}$. Then, according to (1), the following equality for the composition of independent identically distributed random semigroups $\mathbf{U}_1, \dots, \mathbf{U}_n$ holds

$$D(\mathbf{U}_n(t)\circ\cdots\circ\mathbf{U}_1(t)) = \\ = M[\mathbf{U}_1^*(t)\circ\cdots\circ\mathbf{U}_n^*(t)\circ\mathbf{U}_n(t)\circ\cdots\circ\mathbf{U}_1(t)] - (M[(\mathbf{U}(t))^n])^* M[(\mathbf{U}(t))^n].$$

Let $\{\mathbf{U}_n(t),\, t \geq 0\}$ be the sequence of independent identically distributed random semigroups of unitary operators. Then

$$D(\mathbf{U}_n(t)\circ\cdots\circ\mathbf{U}_1(t)) = \mathbf{I} - ((M[\mathbf{U}(t)])^n)^*(M[\mathbf{U}(t)])^n.$$

Lemma 1 (Chebyshev inequality). *If $\mathbf{A}$ is a random variable with the values in the Banach space $B(H)$, such that its dispersion is the operator $D(\mathbf{A}) \in B(H)$, then the following Chebyshev inequality for any vector $x \in H:\ \|x\|_H = 1$ holds*

$$P(\{\|(\mathbf{A} - M\mathbf{A})x\|_H > \epsilon\}) \leq \frac{1}{\epsilon^2}\,\|D(\mathbf{A})x\|_H. \quad (4)$$

Proof. Since $D(\mathbf{A}) = \int\limits_{\Omega} ((\mathbf{A} - M\mathbf{A})^*(\mathbf{A} - M\mathbf{A}))d\mu$, then, for any vector $x \in H$, such that $\|x\|_H = 1$, the following relation holds:

$$\|D(\mathbf{A})x\|_H = \sup_{\|u\|_H=1} (u, [\int\limits_{\Omega} (\mathbf{A} - M\mathbf{A})^*(\mathbf{A} - M\mathbf{A})d\mu]x) \geq$$

$$\geq (x, [\int\limits_{\Omega} (\mathbf{A} - M\mathbf{A})^*(\mathbf{A} - M\mathbf{A})d\mu]x) = \int\limits_{\Omega} \|(\mathbf{A} - M\mathbf{A})x\|_H^2 d\mu \geq$$

$$\geq \int\limits_{\substack{\{\omega\in\Omega: \\ \|(\mathbf{A}-M\mathbf{A})x\|_H \geq \epsilon\}}} \|(\mathbf{A} - M\mathbf{A})x\|_H^2 d\mu \geq \epsilon^2 P(\{\|(\mathbf{A} - M\mathbf{A})x\|_H > \epsilon\}).$$

Therefore, the Chebyshev inequality (4) is true. □

Corollary 3. *Let $\{\mathbf{U}_n\}$ be the sequence of independent random semigroups with the common mean value $\bar{\mathbf{U}}$ and the sequence of dispersions $\{D(\mathbf{A}_n)\}$ bounded in the norm of the space $B(H)$. If the sequence $\{\mathbf{U}_n\}$ satisfies the condition*

$$\lim_{n\to\infty} \|D((\mathbf{A}_n)^{1/n} \circ \cdots \circ (\mathbf{A}_1)^{1/n})x\|_H = 0 \quad \forall\, x \in H,$$

then the large numbers law in the strong-operator topology (2) *for the sequence $\{\mathbf{U}_n\}$ is satisfied.*

Theorem 3. *Let $\xi : \Omega \to SA(H)$ be the random variable with the value from the set $SA(H)$ of self-adjoint operators in the space H (see* [1]*). Let*

$$\mathbf{U}_\omega(t) = \exp(i\xi(\omega)t), \quad t \geq 0, \quad \omega \in \Omega \tag{5}$$

be the corresponding random semigroup. Let $\mathcal{D}$ be the dense linear manifold in the space H, such that the condition $\int\limits_{\Omega} \|\xi(\omega)u\|_H \, d\mu(\omega) < \infty$ holds for any $u \in \mathcal{D}$.

Let $\{\mathbf{U}_k\}$ be the sequence of independent identically distributed random semigroups, such that any random semigroup $\mathbf{U}_k$ has the same distribution as the random semigroup (5)*.*

If the linear operator $\bar{\xi} : \mathcal{D} \to H$, which is defined by the equality

$$\bar{\xi}u = \int\limits_{\Omega} \xi(\omega)u \, d\mu(\omega), \quad u \in H,$$

is essentially self-adjoint, then the sequence $\{\mathbf{U}_n \circ \cdots \circ \mathbf{U}_1\}$ of the compositions of independent identically distributed random semigroups satisfies the condition

$$\lim_{n\to\infty} \Big[\sup_{t\in[0,T]} \Big\|D\Big(\mathbf{U}_n\Big(\frac{t}{n}\Big) \circ \cdots \circ \mathbf{U}_1\Big(\frac{t}{n}\Big)\Big)x\Big\|_H\Big] = 0 \quad \forall T > 0, \quad \forall\, x \in H. \tag{6}$$

Proof. For any $n \in \mathbf{N}$, the composition $\mathbf{U}_n \circ \cdots \circ \mathbf{U}_1$ of n-independent identically distributed semigroups is the map $\{\mathbf{U}_n(\omega_n) \circ \cdots \circ \mathbf{U}_1(\omega_1),\ \{\omega_1, \ldots, \omega_n\} \in \Omega \times \cdots \times \Omega\}$. Therefore, the mean value of the composition $\mathbf{U}_n \circ \cdots \circ \mathbf{U}_1$ satisfies (according to the theorem 2 and (1)) the equality

$$M[\mathbf{U}_n \circ \cdots \circ \mathbf{U}_1] = M[\mathbf{U}_n] \circ \cdots \circ M[\mathbf{U}_1] = (M[\mathbf{U}])^n \quad \forall\, n \in \mathbf{N}.$$

Therefore, for any $t \geq 0$ and any $n \in \mathbf{N}$ the following equality holds

$$D\Big(\mathbf{U}_n\Big(\frac{t}{n}\Big)\circ\cdots\circ\mathbf{U}_1\Big(\frac{t}{n}\Big)\Big) =$$
$$= M\Big[\mathbf{U}_1^*\Big(\frac{t}{n}\Big)\circ\cdots\circ\mathbf{U}_n^*\Big(\frac{t}{n}\Big)\circ\mathbf{U}_n\Big(\frac{t}{n}\Big)\circ\cdots\circ\mathbf{U}_1\Big(\frac{t}{n}\Big)\Big]-M\Big[\Big(\mathbf{U}^*\Big(\frac{t}{n}\Big)\Big)^n\Big]M\Big[\Big(\mathbf{U}\Big(\frac{t}{n}\Big)\Big)^n\Big].$$

Hence the equality

$$D\Big(\mathbf{U}_n\Big(\frac{t}{n}\Big)\circ\cdots\circ\mathbf{U}_1\Big(\frac{t}{n}\Big)\Big) = \mathbf{I} - \Big(\Big(M\Big[\mathbf{U}\Big(\frac{t}{n}\Big)\Big]\Big)^n\Big)^*\Big(M\Big[\mathbf{U}\Big(\frac{t}{n}\Big)\Big]\Big)^n. \tag{7}$$

is valid for any $t \geq 0$, for any $n \in \mathbf{N}$, and for any sequence of compositions of independent random unitary semigroups $\{\mathbf{U}_n(\omega_n)\circ\cdots\circ\mathbf{U}_1(\omega_1),\ \{\omega_1,\ldots,\omega_n\}\in\Omega\times\cdots\times\Omega\}$.

According to theorem 1, the operator-valued function $M[\mathbf{U}(t)]$, $t \geq 0$, is equivalent to the semigroup $\exp(i\bar{\xi}t)$, $t \geq 0$ in the Chernoff sense (See [1]). Thus, the sequence $\{(M[\mathbf{U}(t/n)])^n,\ t \geq 0\}$ converges to the semigroup $\exp(i\bar{\xi}t)$, $t \geq 0$, in the strong-operator topology uniformly on any segment of the semiaxe R_+. Therefore,

$$\lim_{n\to\infty}\Big(\sup_{t\in[0,T]}\Big\|\Big((M\Big[\mathbf{U}\Big(\frac{t}{n}\Big)\Big]\Big)^n\Big)^*\Big[\Big(M\Big[\mathbf{U}\Big(\frac{t}{n}\Big)\Big]\Big)^n - \mathbf{I}]x\Big\|_H\Big) = 0$$

for any $T > 0$ and any $x \in H$. □

Corollary 4. *Let the conditions of theorem* 3 *be fulfilled. Then*

1) *the large numbers law in the strong-operator topology is valid for the sequence* $\{\mathbf{U}_k\}$.

2) *the sequence* $\{\mathbf{U}_n^{1/n}\circ\cdots\circ\mathbf{U}_1^{1/n}\}$ *of averaging composition of random operator valued functions converges in probability sense to the one-parametric semigroup* $\exp(i\bar{\xi}t)$, $t \geq 0$ *in the strong operator topology:*

$$\forall\,\epsilon > 0,\ \forall\,x \in H,\ \forall\,T > 0\ \lim_{n\to\infty}P(\{\sup_{t\in[0,T]}\|(\mathbf{U}_n^{1/n}(t)\circ\cdots\circ\mathbf{U}_1^{1/n}(t)-\exp(i\bar{\xi}t))x\|_H \geq \epsilon\})=0.$$

Now, we give the example of the sequence $\{\mathbf{U}_n\circ\cdots\circ\mathbf{U}_1\}$ of compositions of independent identically distributed random semigroups which satisfies the large numbers law in the strong-operator topology, but it does not satisfy the large numbers law in the topology of the norm.

Example. (Violation of the large numbers law in the topology of the norm.) Let $\Omega = \{1,2\}$ and measure μ on the σ-algebra 2^Ω be given by the equality $\mu(\{1\}) = \mu(\{2\}) = 1/2$. Let $\mathbf{A}$ be the self-adjoint operator with the basis of eigenvectors $\{e_k\}$, such that $\mathbf{A}e_k = ke_k\ \ \forall\,k \in \mathbf{N}$. Let ξ be the random generator of random semigroup $\mathbf{U}(t) = \exp(it\xi)$, where $\xi(1) = -\mathbf{A}$ and $\xi(2) = \mathbf{A}$.

Then, according to theorem 3, the large numbers law in the strong- operator topology is fulfilled.

However, $M(\mathbf{U}(t)) = \cos(\mathbf{A}t)$, $t \geq 0$, $M(\mathbf{U}^*(t)\mathbf{U}(t)) = \mathbf{I}$, $t \geq 0$. Therefore, $D(\mathbf{U}(t)) = \sin^2(\mathbf{A}t)$, $t \geq 0$, and $D(\mathbf{U}_n(t/n)\circ\cdots\mathbf{U}_1(t/n)) = \mathbf{I}-\cos^{2n}(\mathbf{A}t/n)$, $t \geq 0$, for all $n \in \mathbf{N}$. Then, the large numbers law in the operator norm topology is not satisfied, because the following estimates hold for any $t > 0$:

$$P(\|\mathbf{U}_n^{1/n}(t) \circ \cdots \circ \mathbf{U}_1^{1/n}(t) - M(\mathbf{U}(t/n))^n)\|_{B(H)} > 1/2) =$$
$$= P(\|\mathbf{U}_n(t/n) \circ \cdots \circ \mathbf{U}_1(t/n) - \cos^n(\mathbf{A}t/n)\|_{B(H)} > 1/2) \geq$$
$$\geq P(\sup_{k\in\mathbb{N}} \|(\mathbf{U}_n(t/n) \circ \cdots \circ \mathbf{U}_1(t/n) - (\cos(\mathbf{A}t/n))^n)e_k\|_H > 1/2) =$$
$$= P(\sup_{k\in\mathbb{N}} |1 - (\cos(kt/n))^n| > 1/2).$$

Therefore, there is a number $n \in \mathbf{N}$ for any $t > 0$, such that

$$P(\|\mathbf{U}_n^{1/n}(t) \circ \cdots \circ \mathbf{U}_1^{1/n}(t) - M[\mathbf{U}_n^{1/n}(t) \circ \cdots \circ \mathbf{U}_1^{1/n}(t)]\|_{B(H)} > 1/2) = 1,$$

i.e., the large numbers law in the strong operator topology is violated.

The above consideration ensures the sufficient conditions for the following behavior of the Feynman–Chernoff iteration of the operator-valued function and the random operator-valued function. Under these conditions, the Feynman–Chernoff iteration of the operator-valued function is the procedure of transforming of the operator-valued function into the sequence of operator-valued functions converging to some one-parametric semigroup of operators. In addition, the Feynman–Chernoff iteration of the random operator-valued function transforming it into the sequence of random operator-valued processes, the mean values of which is a C_0-semigroup and which obeys the large numbers law, is observed.

Since the one-parametric semigroup usually can be associated with some Markovian random process with independent increment, then we investigate the property of the Feynman–Chernoff iteration procedure to create the sequence of the random operator-valued functions with an asymptotically independent increment. Let $\mathcal{B}_s(B(H))$ be the σ-algebra of the Borel subsets of the space $B(H)$ endowing with the strong operator topology.

Definition 4. The random operator-valued function ξ is called the random operator-valued function with the independent increment if the following equality satisfies for any $t_1, t_2, t_3, t_4 \in R_+ :\ t_1 < t_2 \leq t_3 < t_4$ and for any sets $B_1, B_2 \in \mathcal{B}_s(B(H))$

$$P(\{\xi(t_4) \circ (\xi(t_3))^{-1} \in B_2\})P(\{\xi(t_2) \circ (\xi(t_1))^{-1} \in B_1\}) =$$
$$= P(\{\xi(t_2) \circ (\xi(t_1))^{-1} \in B_1,\, \xi(t_4) \circ (\xi(t_3))^{-1} \in B_2\}).$$

The sequence of random operator-valued functions $\{\xi_n\}$ is called the sequence of random operator-functions with the asymptotically independent increment if the following equality satisfies for any $t_1, t_2, t_3, t_4 \in R_+ :\ t_1 < t_2 \leq t_3 < t_4$ and for any sets $B_1, B_2 \in \mathcal{B}_s(B(H))$

$$\lim_{n\to\infty}[P(\{\xi_n(t_4) \circ (\xi_n(t_3))^{-1} \in B_2\})P(\{\xi_n(t_2) \circ (\xi_n(t_1))^{-1} \in B_1\}) -$$
$$- P(\{\xi_n(t_2) \circ (\xi_n(t_1))^{-1} \in B_1,\, \xi_n(t_4) \circ (\xi_n(t_3))^{-1} \in B_2\})] = 0.$$

Definition 5. The mapping $\mathbf{J}$ of some set $\mathcal{S}$ of the linear space $\mathcal{M}(C_s(R_+, B(H)), \mathcal{A}_s)$ of the random operator function into the linear space $\mathcal{N}(C_s(R_+, B(H)), \mathcal{A}_s)$ of the sequence of random operator function $\mathbb{N} \to \mathcal{M}(C_s(R_+, B(H)), \mathcal{A}_s)$ is called the independization of the random operator functions from the set S if for any element $\xi \in \mathcal{S}$ its image $\mathbf{J}(\xi)$ is the sequence of random operator valued functions with the asymptotically independent increment.

Now, we provide the examples of the map

$$\mathcal{J}:\ \mathcal{M}(C_s(R_+,B(H)),\mathcal{A}_s)\to\mathcal{N}(C_s(R_+,B(H)),\mathcal{A}_s),$$

which is the independization (which is not independization) on some sufficiently simple subsets $\mathcal{S}$ of the set of random semigroups in the space $\mathcal{M}(C_s(R_+,B(H)),\mathcal{A}_s)$.

Let $\mathcal{S}_{th3}(H)$ be the subset of the space $\mathcal{M}(C_s(R_+,B(H)),\mathcal{A}_s)$, such that, for any $\mathbf{F}\in\mathcal{S}_{th3}(H)$, the random operator-valued function $\mathbf{F}$ is the random semigroup of type (5), which satisfies all conditions of theorem 3.

Then, for any $\mathbf{F}\in\mathcal{S}_{th3}(H)$, the sequence of the Feynman–Chernoff iteration of the random operator-valued function $\mathbf{F}$ converges in the probability sense to the C_0-semigroup $\exp(i\bar{\mathbf{F}}'(0)t),\ t\geq 0$, (where $\bar{\mathbf{F}}'(0)=\int\limits_{\Omega}\mathbf{F}'(0)\,d\mu$) according to large numbers law (see corollary 4).

Let Φ be the map $\Phi:\ \mathcal{M}(C_s(R_+,B(H)),\mathcal{A}_s)\to\mathcal{N}(C_s(R_+,B(H)),\mathcal{A}_s)$, such that, for any $\mathbf{F}\in\mathcal{M}(C_s(R_+,B(H)),\mathcal{A}_s)$, its image $\Phi(\mathbf{F})$ is the sequence of the Feynman–Chernoff iteration of the random operator-valued function $\mathbf{F}$. Then, for any $n\in\mathbf{N}$, the equality $\Phi_n(\mathbf{F})(t)=(\mathbf{F}(t/n))^n,\ t\geq 0,$ holds.

Lemma 2. *The map Φ is not independization of the set $\mathcal{S}_{th3}(H)$.*

Proof. To prove this statement, it is sufficient to obtain an example. Let $\eta(t)$ be the real-valued random process taking values $t\omega$ for any $t\geq 0$. Here, ω is the real random variable taking two values ± 1 with the probabilities $1/2$. Let $\{\eta_n\}$ be the sequence of independent identically distributed random process, such that the distribution of any process η_n is the same as the distribution of the process η. Let us consider the sequence of the process $\{\eta^n\}$, where $\eta^n(t)=\eta_n\Big(\dfrac{t}{n}\Big)+\dots+\eta_1\Big(\dfrac{t}{n}\Big)=\dfrac{t}{n}(\omega_1+\dots+\omega_n),\ t\geq 0$, $n\in\mathbf{N}$.

Let us consider one-dimensional space $H=\mathbb{C}$ and the random operator-valued functions $\xi\in\mathcal{M}(C_s(R_+,B(H)),\mathcal{A}_s)$, such that $\xi(t)u=\exp(i\eta(t))u,\ t\geq 0,\ u\in H$. Then, let $\xi\in\mathcal{S}_{th3}(H)$ and $\{\xi^n\}=\Phi(\xi)$ be the sequence of the Feynman–Chernoff iteration of the random operator-valued function ξ, where $\xi^n(t)=\exp(i\eta^n(t)),\ t\geq 0$, $n\in\mathbf{N}$. The increments of the random process ξ^n at the moments $t_1,\ t_2,\ t_1<t_2$, are functionally dependent to each other. In fact, for any $\tau\in(0,t_2-t_1)$, the following equality

$$\xi^n(t_2+\tau)\circ(\xi^n(t_2))^{-1}=\xi^n(t_1+\tau)\circ(\xi^n(t_1))^{-1}=\exp\left[i\frac{\tau}{n}(\omega_1+\dots+\omega_n)\right]$$

holds and, therefore, the increments of the process ξ^n at the different moments $t_1,\,t_2>0$ are functionally dependent. □

Let us consider another map

$$\mathcal{J}:\ (0,+\infty)\times\mathcal{M}(C_s(R_+,B(H)),\mathcal{A}_s)\to\mathcal{N}(C_s(R_+,B(H)),\mathcal{A}_s),$$

such that, for any $(T,\xi)\in(0,+\infty)\times\mathcal{M}(C_s(R_+,B(H)),\mathcal{A}_s)$, the sequence $\mathcal{J}_T(\xi)=\{\xi_T^{(n)}\}$ of the random operator-valued function $\xi_T^{(n)}$ is defined by the rule. For any $n\in\mathbf{N}$ and any $t\geq 0$, let $k=k(n,t,T)$ be the natural number $k=\Big[\dfrac{t}{n}\Big]+1$, where $\Big[\dfrac{t}{n}\Big]$ is the integer part of real number $\dfrac{t}{T}$. Then, $t\in\Big[\dfrac{(k-1)}{n}T,\ \dfrac{k}{n}T\Big)$ and the value of random linear operator $\xi_T^{(n)}(t)\in B(X)$ is defined by the equality

$$\xi_T^{(n)}(t,\omega_1,\dots,\omega_k)=\xi\Big(t-\frac{(k-1)}{n}T,\omega_k\Big)\circ\xi\Big(\frac{T}{n},\omega_{k-1}\Big)\circ\dots\circ\xi\Big(\frac{T}{n},\omega_1\Big).$$

Theorem 4. *For any* $T > 0$*, the map*

$$\mathcal{J}_T :\ \xi \to \{\xi_T^{(n)}\}$$

is the independization of the random operator-valued function on the set $\mathcal{S}_{th3}(H)$*.*

Proof. Let $T > 0$ and $\xi \in \mathcal{S}_{th3}$. Then, for any $n \in \mathbf{N}$ and any $t \geq 0$, the random linear operator $\xi_T^{(n)}(t) = ((\mathcal{J}_T)_n(\xi))(t)$ has the bounded inverse $(\xi_T^{(n)}(t))^{-1}$. Therefore, for any numbers $s, \tau, t > 0$, such that $s < s+\tau < t < t+\tau$, the random values $\xi_T^{(n)}(t+\tau)\circ(\xi_T^{(n)}(t))^{-1}$ and $\xi_T^{(n)}(s+\tau)\circ(\xi_T^{(n)}(s))^{-1}$ are independent for any sufficiently large $n \in \mathbf{N}$, such that $n > \dfrac{T}{t-s}$. □

What is the relationship between the property of independence of increments of the random operator-valued function and the property of the mean value of the random operator-function to be a one-parametric semigroup? The results on the relationship between the Markovian processes and the diffusion semigroup are well-known. Furthermore, it is commonly accepted that the mean value of functionals on the process with an independent increment generates some semigroup. However, the semigroup can be generated by functionals of different processes, which can have both independent or functionally dependent increments (see the example in lemma 2).

Note that, for any $\mathbf{F} \in \mathcal{S}_{th3}(H)$, the sequences of iteration $\Phi(\mathbf{F})$ and $\mathcal{J}_T(\mathbf{F})$ have the same limit behavior with the iteration number growth in the following sense.

Theorem 5. *Let* $T > 0$ *and* $L > 0$*. Let* $\xi \in \mathcal{S}_{th3}(H)$*. Then, for any* $u \in H$*, the following equalities hold*

$$\lim_{n\to\infty} \sup_{t\in[0,L]} \|(\mathrm{M}\Phi_n(\xi)(t) - \exp(\bar{\xi}'(0)t))u\|_H = 0;$$

$$\lim_{n\to\infty} \sup_{t\in[0,L]} \|(\mathrm{M}(\mathcal{J}_T)_n(\xi))(t) - \exp(\bar{\xi}'(0)t))u\|_H = 0,$$

where $\bar{\xi}'(0) = \int\limits_{\Omega} \xi'(0)\, d\mu$ *and* $\mathrm{M}f$ *is the mean value of the random operator-valued function* $f \in \mathcal{M}(C_s(R_+, B(H)), \mathcal{A}_s)$*.*

For both sequences of iterations $\Phi_n(\xi)$ and $(\mathcal{J}_T)_n(\xi)$, the convergence of mean values to the semigroup $\exp(\bar{\xi}'(0)t)$, $t \geq 0$, is the consequence of the Chernoff theorem.

Thus, different iterations of random operator-valued function $\xi \in \mathcal{S}_{th3}(H)$ have identical limits and different properties of asymptotic independence (see the example in lemma 2).

The convergence of the mean values of the Feynman–Chernoff iteration of the random semigroup and the large numbers law for the sequence of compositions of independent identically distributed random semigroup is studied. The property of asymptotic independence of the Feynman–Chernoff iteration of the random semigroup is investigated. The examples of independization of the random operator-valued function are given. The relationship between the properties of independence of increments of the random operator function and the semigroup properties for the mean value of this random operator-function are investigated.

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Исчисление Фейнмана случайных оператор-функций и его приложения

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Аннотация

Объектом исследования статьи являются итерации Фейнмана – Чернова случайных полугрупп ограниченных линейных операторов в гильбертовом пространстве. Изучается сходимость средних значений итераций Фейнмана – Чернова случайных полугрупп. Получены оценки отклонения композиций независимых одинаково распределенных случайных полугрупп от их математических ожиданий. Сформулирован закон больших чисел для композиций независимых случайных полугрупп, получены условия его выполнения и приведены примеры его нарушения. Исследована взаимосвязь между полугрупповым свойством среднего значения случайной оператор-функции и свойством независимости ее приращений. Исследовано свойство асимптотической независимости приращений для итераций Фейнмана – Чернова случайной полугруппы операторов. Индепендизация случайных оператор-функций определяется как отображение, сопоставляющее случайной оператор-функции последовательность случайных оператор-функций с асимптотически независимыми приращениями. Приведены примеры близких к итерациям Фейнмана – Чернова индепендизаций случайных оператор-функций.

Ключевые слова: случайные операторы, случайные полугруппы, итерации Фейнмана – Чернова, закон больших чисел

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