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The Landau–Kolmogorov Problem for the Laplace Operator on a Ball

A. A. Koshelev[1*]

[1] *Ural Federal University, ul. Mira 19, Ekaterinburg, 620002 Russia*

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Abstract—In this paper we solve the problem of maximizing the value of the Laplace operator at the origin for functions such that the second degree of the Laplace operator belongs to the space L_∞ on the unit ball of the Euclidean space. The problem is solved under restrictions on the uniform norm of a function and the L_∞-norm of the second degree of the Laplace operator of this function.

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1. STATEMENT OF THE PROBLEM AND MAIN RESULT

Let $\mathbb{R}^m$, $m \geq 2$, be Euclidean space with scalar product $(x, y) = \sum\limits_{i=1}^{m} x_i y_i$ of vectors $x = (x_1, \dots, x_m)$, $y = (y_1, \dots, y_m) \in \mathbb{R}^m$, and the norm $|x| = \sqrt{(x, x)}$, $x \in \mathbb{R}^m$. We denote by $\mathbb{S}_h^m = \{x \in \mathbb{R}^m : |x| = h\}$ the sphere of radius h with center at origin, and by $\mathbb{B}^m = \{x \in \mathbb{R}^m : |x| \leq 1\}$ the unit ball in space $\mathbb{R}^m$.

In what follows the symbol $\mathbb{G}$ stands either for the whole space $\mathbb{R}^m$, or for the unit ball $\mathbb{B}^m$. Let $C = C(\mathbb{G})$ be the space of (real-valued) functions, which are continuous and bounded on $\mathbb{G}$, with uniform norm

$$\|f\|_C = \sup\{|f(x)| : x = (x_1, \dots, x_m) \in \mathbb{G}\};$$

$L_\infty = L_\infty(\mathbb{G})$ be the space of measurable and essentially bounded on $\mathbb{G}$ functions with the norm

$$\|f\|_\infty = \operatorname{ess\,sup}\{|f(x)| : x \in \mathbb{G}\};$$

$\mathcal{D} = \mathcal{D}(\mathbb{G})$ be space of finite (i.e., with compact supports) infinitely differentiable functions on $\mathbb{R}^m$; in the case $\mathbb{G} = \mathbb{B}$ their supports belong to interior $\overset{\circ}{\mathbb{B}}$ of the ball $\mathbb{B}$.

The Laplace operator Δ is defined on twice differentiable in interior $\overset{\circ}{\mathbb{G}}$ of the set $\mathbb{G}$ functions f by the formula

$$\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \dots + \frac{\partial^2 f}{\partial x_m^2}.$$

The Laplace operator and its degrees Δ^l are extendable on classes of functions of lesser smoothness in customary way by means of theory of generalized functions, i.e., by Sobolev's scheme (see, e.g., [1], Chap. I, §7, Items 1–7). Indeed, for a pair of defined in $\overset{\circ}{\mathbb{G}}$ measurable and locally summable functions f, g and a positive integer l we write $g = \Delta^l f$, if for any function $\varphi \in \mathcal{D}(\mathbb{G})$ we have

$$\int_{\mathbb{G}} f(x)\, \Delta^l \varphi(x) dx = \int_{\mathbb{G}} g(x) \varphi(x) dx.$$

*E-mail: Anton.Koshelev@urfu.ru.