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ON THE GEOMETRY OF B-MANIFOLDS

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Abstract

The main purpose of the present paper is to study almost B-structures (Norden structures) on 8-dimensional Walker manifolds. We discuss the problem of integrability, Kähler (holomorphic) and Einstein conditions for these structures.

Key words: Walker 8-manifold, Norden metric, holomorphic metric, Einstein metric.

1. Introduction

Let M_{2n} be a Riemannian manifold with neutral metric, i. e., with pseudo-Riemannian metric g of signature (n,n). We denote by $\Im_q^p(M_{2n})$ the set of all tensor fields of type (p,q) on M_{2n}. Manifolds, tensor fields and connections are always assumed to be differentiable and of class C^∞.

Let (M_{2n},φ) be an almost complex manifold with almost complex structure φ. Such a structure is said to be integrable if the matrix $\varphi=(\varphi_j^i)$ is reduced to constant form in a certain holonomic natural frame in a neighborhood U_x of every point $x\in M_{2n}$. In order that an almost complex structure φ be integrable, it is necessary and sufficient that it be possible to introduce a torsion-free affine connection ∇ with respect to which the structure tensor φ is covariantly constant, i. e., $\nabla\varphi=0$. It is also known that the integrability of φ is equivalent to the vanishing of the Nijenhuis tensor $N_\varphi\in\Im_2^1(M_{2n})$. If φ is integrable, then φ is a complex structure and, moreover, M_{2n} is a $\mathbb{C}$-holomorphic manifold $X_n(\mathbb{C})$ whose transition functions are holomorphic mappings.

1.1. Norden metrics. A metric g is a Norden metric [1] if

$$g(\varphi X,\varphi Y)=-g(X,Y)$$

or equivalently

$$g(\varphi X,Y)=g(X,\varphi Y)$$

for any $X,Y\in\Im_0^1(M_{2n})$. Metrics of this type have also been studied under the names: B-metrics, pure metrics and anti-Hermitian metrics [2–7]). If (M_{2n},φ) is an almost complex manifold with Norden metric g, we say that (M_{2n},φ,g) is an almost Norden manifold. If φ is integrable, we say that (M_{2n},φ,g) is a Norden manifold.

1.2. Holomorphic (almost holomorphic) tensor fields. Let $\overset{*}{t}$ be a complex tensor field on $X_n(\mathbb{C})$. The real model of such a tensor field is a tensor field t on M_{2n} of the same order such that the action of the structure affinor φ on t does not depend on which vector or covector argument of t φ acts. Such tensor fields are said to be pure with respect to φ. They were studied by many authors (see, e. g., [4, 6–11]). In particular, being applied to a $(0,q)$-tensor field ω, the purity means that for any $X_1,\ldots,X_q\in\Im_0^1(M_{2n})$, the following conditions should hold:

$$\omega(\varphi X_1,X_2,\ldots,X_q)=\omega(X_1,\varphi X_2,\ldots,X_q)=\cdots=\omega(X_1,X_2,\ldots,\varphi X_q).$$

We define an operator
$$\Phi_\varphi : \Im^0_q(M_{2n}) \to \Im^0_{q+1}(M_{2n})$$
applied to a pure tensor field ω by (see [11])

$$(\Phi_\varphi \omega)(X, Y_1, Y_2, \ldots, Y_q) = (\varphi X)(\omega(Y_1, Y_2, \ldots, Y_q)) - X(\omega(\varphi Y_1, Y_2, \ldots, Y_q)) +$$
$$+ \omega((L_{Y_1}\varphi)X, Y_2, \ldots, Y_q) + \cdots + \omega(Y_1, Y_2, \ldots, (L_{Y_q}\varphi)X),$$

where L_Y denotes the Lie differentiation with respect to Y.

When φ is a complex structure on M_{2n} and the tensor field $\Phi_\varphi \omega$ vanishes, the complex tensor field $\overset{*}{\omega}$ on $X_n(\mathbb{C})$ is said to be holomorphic (see [4, 6, 11]). Thus, a holomorphic tensor field $\overset{*}{\omega}$ on $X_n(\mathbb{C})$ is realized on M_{2n} in the form of a pure tensor field ω, such that
$$(\Phi_\varphi \omega)(X, Y_1, Y_2, \ldots, Y_q) = 0$$
for any $X, Y_1, \ldots, Y_q \in \Im^1_0(M_{2n})$. Such a tensor field ω on M_{2n} is also called a holomorphic tensor field. When φ is an almost complex structure on M_{2n}, a tensor field ω satisfying $\Phi_\varphi \omega = 0$ is said to be almost holomorphic.

1.3. Holomorphic Norden (Kähler – Norden) metrics. On a Norden manifold, a Norden metric g is called *holomorphic* if

$$(\Phi_\varphi g)(X, Y, Z) = 0 \tag{1}$$

for any $X, Y, Z \in \Im^1_0(M_{2n})$.

By setting $X = \partial_k$, $Y = \partial_i$, $Z = \partial_j$ in equation (1), we see that the components $(\Phi_\varphi g)_{kij}$ of $\Phi_\varphi g$ with respect to a local coordinate system $x^1, \ldots, x^n$ can be expressed as follows:

$$(\Phi_\varphi g)_{kij} = \varphi^m_k \partial_m g_{ij} - \varphi^m_i \partial_k g_{mj} + g_{mj}(\partial_i \varphi^m_k - \partial_k \varphi^m_i) + g_{im}\partial_j \varphi^m_k .$$

If (M_{2n}, φ, g) is a Norden manifold with holomorphic Norden metric, we say that (M_{2n}, φ, g) is a *holomorphic Norden manifold.*

In some aspects, holomorphic Norden manifolds are similar to Kähler manifolds. The following theorem is an analogue to the next known result: an almost Hermitian manifold is Kähler if and only if the almost complex structure is parallel with respect to the Levi – Civita connection.

Theorem 1 [12] (For a paracomplex version see [13]). *For an almost complex manifold with Norden metric g, the condition $\Phi_\varphi g = 0$ is equivalent to $\nabla \varphi = 0$, where ∇ is the Levi – Civita connection of g.*

A *Kähler – Norden* manifold can be defined as a triple (M_{2n}, φ, g) which consists of a manifold M_{2n} endowed with an almost complex structure φ and a pseudo-Riemannian metric g such that $\nabla \varphi = 0$, where ∇ is the Levi – Civita connection of g and the metric g is assumed to be a Norden one. Therefore, there exists a one-to-one correspondence between *Kähler – Norden* manifolds and Norden manifolds with *holomorphic metric.* Recall that the Riemannian curvature tensor of such a manifold is pure and holomorphic, and the scalar curvature is a locally holomorphic function (see [5, 12]).

Remark 1. We know that the integrability of an almost complex structure φ is equivalent to the existence of a torsion-free affine connection with respect to which the equation $\nabla \varphi = 0$ holds. Since the Levi – Civita connection ∇ of g is a torsion-free affine connection, we have: if $\Phi_\varphi g = 0$, then φ is integrable. Thus, almost Norden manifolds with conditions $\Phi_\varphi g = 0$ and $N_\varphi \neq 0$, i. e., almost holomorphic Norden manifolds (analogues of almost Kähler manifolds with closed Kähler form) do not exist.

In the present paper, we shall focus our attention on Norden manifolds of dimension eight. Using a Walker metric, we construct new Norden – Walker metrics together with almost complex structures. Note that indefinite Kähler – Einstein metrics on an eight-dimensional Walker manifolds have recently been investigated in [14].

2. Norden – Walker metrics

2.1. Walker metric g. A neutral metric g on an 8-manifold M_8 is said to be a Walker metric if there exists a 4-dimensional null distribution D on M_8 which is parallel with respect to g. By Walker's theorem [15], there is a system of coordinates $(x^1,\ldots,x^8)$ with respect to which g takes the following local canonical form

$$g=(g_{ij})=\begin{pmatrix}0 & I_4\\ I_4 & B\end{pmatrix}, \tag{2}$$

where I_4 is the unit 4×4 matrix and B is a 4×4 symmetric matrix whose entries are functions of the coordinates $(x^1,\ldots,x^8)$. Note that g is of neutral signature $(++++----)$, and that the parallel null 4-plane D is spanned locally by $\{\partial_1,\partial_2,\partial_3,\partial_4\}$, where $\partial_i=\dfrac{\partial}{\partial x^i}$, $i=1,\ldots,8$.

In this paper, we consider specific Walker metrics on M_8 with B of the form

$$B=\begin{pmatrix}a & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & b & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, \tag{3}$$

where $a,\ b$ are smooth functions of the coordinates $(x^1,\ldots,x^8)$.

2.2. Almost Norden – Walker manifolds. We can construct various g-orthogonal almost complex structures φ on a Walker 8-manifold M_8 with metrics g as in (2), (3) so that (M_8,φ,g) is a (neutral) almost Norden manifold. The structure φ defined by

$$\begin{aligned}
&\varphi\partial_1=\partial_3,\quad \varphi\partial_2=\partial_4,\quad \varphi\partial_3=-\partial_1,\quad \varphi\partial_4=-\partial_2,\\
&\varphi\partial_5=\frac{1}{2}(a+b)\partial_3-\partial_7,\quad \varphi\partial_6=-\partial_8,\\
&\varphi\partial_7=-\frac{1}{2}(a+b)\partial_1+\partial_5,\quad \varphi\partial_8=\partial_6.
\end{aligned}$$

is one of the simplest examples of such an almost complex structure.

Following the terminology of [14, 16–18], we call φ a proper almost complex structure. A proper almost complex structure φ has local components

$$\varphi=(\varphi_j^i)=\begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 & -(a+b)/2 & 0\\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & (a+b)/2 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0
\end{pmatrix} \tag{4}$$

with respect to the natural frame $\{\partial_i\}$, $i=1,\ldots,8$.

Remark 2. From (4) we see that, in the case $a=-b$, φ is integrable.

2.3. Integrability of φ. We consider the general case.

The almost complex structure φ of an almost Norden – Walker manifold is integrable if and only if

$$(N_\varphi)^i_{jk} = \varphi^m_j \partial_m \varphi^i_k - \varphi^m_k \partial_m \varphi^i_j - \varphi^i_m \partial_j \varphi^m_k + \varphi^i_m \partial_k \varphi^m_j = 0. \tag{5}$$

Since $N^i_{jk} = -N^i_{kj}$, we need only consider N^i_{jk} $(j < k)$. By explicit calculations, the nonzero components of the Nijenhuis tensor are as follows:

$$\begin{aligned}
&N^1_{15} = N^1_{37} = N^3_{17} = -N^3_{35} = \frac{1}{2}(a_1 + b_1),\\
&N^3_{57} = \frac{1}{4}(a + b)(a_1 + b_1),\\
&N^1_{25} = N^1_{47} = N^3_{27} = -N^3_{45} = \frac{1}{2}(a_2 + b_2),\\
&N^1_{17} = -N^1_{35} = -N^3_{15} = -N^3_{37} = -\frac{1}{2}(a_3 + b_3),\\
&N^1_{57} = -\frac{1}{4}(a + b)(a_3 + b_3),\\
&N^1_{27} = N^1_{45} = N^3_{25} = N^3_{47} = \frac{1}{2}(a_4 + b_4),\\
&N^1_{56} = -N^1_{78} = N^3_{58} = -N^3_{67} = -\frac{1}{2}(a_6 + b_6),\\
&N^1_{58} = -N^1_{67} = -N^3_{56} = N^3_{78} = -\frac{1}{2}(a_8 + b_8).
\end{aligned} \tag{6}$$

From (6), we have

Theorem 2. *The almost complex structure φ of an almost Norden – Walker manifold is integrable if and only if the following PDEs hold:*

$$\begin{aligned}
&a_1 + b_1 = 0, \quad a_2 + b_2 = 0, \quad a_3 + b_3 = 0,\\
&a_4 + b_4 = 0, \quad a_6 + b_6 = 0, \quad a_8 + b_8 = 0.
\end{aligned} \tag{7}$$

Corollary 1. *The almost complex structure φ of an almost Norden – Walker manifolds is integrable if and only if*

$$a = -b + \xi, \tag{8}$$

where ξ is a function of x^5 and x^7 only.

Corollary 2. *A metric* (2) *with matrix*

$$B = \begin{pmatrix} -b(x^1, \dots, x^8) + \xi(x^5, x^7) & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & b(x^1, \dots, x^8) & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is always Norden – Walker.

3. Norden – Walker – Einstein metrics

We now turn our attention to the Einstein conditions for a Walker metric (2), (3) with a and b given by (8). For a and b in (8), we put $f = \frac{1}{2}(a-b) = a - \frac{1}{2}\xi = -b + \frac{1}{2}\xi$.

Since $a = f + \frac{1}{2}\xi$ and $b = -f + \frac{1}{2}\xi$, it follows that B in (2) is as follows:

$$B = \begin{pmatrix} f(x^1, \dots, x^8) + \frac{1}{2}\xi(x^5, x^7) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -f(x^1, \dots, x^8) + \frac{1}{2}\xi(x^5, x^7) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{9}$$

Let R_{ij} and S denote,respectively, the Ricci tensor and the scalar curvature of metric (2) with B given as in (9). The Einstein tensor is defined by $G_{ij} = R_{ij} - \frac{1}{8}Sg_{ij}$ and has the following nonzero components:

$$\begin{aligned}
&G_{25} = \frac{1}{2}f_{12}, \quad G_{17} = -G_{35} = -\frac{1}{2}f_{13}, \quad G_{45} = \frac{1}{2}f_{14}, \quad G_{56} = \frac{1}{2}f_{16}, \\
&G_{58} = \frac{1}{2}f_{18}, \quad G_{27} = -\frac{1}{2}f_{23}, \quad G_{47} = -\frac{1}{2}f_{34}, \quad G_{67} = -\frac{1}{2}f_{36}, \\
&G_{78} = -\frac{1}{2}f_{38}, \quad G_{15} = \frac{1}{8}(3f_{11} + f_{33}), \quad G_{26} = G_{48} = -\frac{1}{8}(f_{11} - f_{33}), \\
&G_{37} = -\frac{1}{8}(f_{11} + 3f_{33}), \quad G_{57} = \frac{1}{2}(f_{17} + f_1 f_3 - f_{35}), \\
&G_{55} = -f_{26} - f_{37} - f_{48} + \frac{3}{8}f(f_{11} - f_{33}) + \frac{1}{8}\xi(3f_{11} + 5f_{33}) - \frac{1}{2}f_3^2, \\
&G_{77} = f_{15} + f_{26} + f_{48} - \frac{3}{8}f(f_{11} - f_{33}) + \frac{1}{8}\xi(5f_{11} + 3f_{33}) - \frac{1}{2}f_1^2.
\end{aligned} \tag{10}$$

A metric g with B as in (9) is Norden – Walker – Einstein if all the above components $G_{ij} = 0$.

Theorem 3. *A Norden – Walker metric g is a Norden – Walker – Einstein one if the following PDEs hold:*

$$a_1 - b_1 = 0, \quad a_2 - b_2 = 0, \quad a_3 - b_3 = 0, \quad a_4 - b_4 = 0.$$

Proof. The assertion follows from (10) and the relation $f = \frac{1}{2}(a - b)$. □

From Theorem 2 and Theorem 3, we have

Corollary 3. *A Norden – Walker metric g is a Norden – Walker – Einstein one if the following PDEs hold:*

$$a_1 = a_2 = a_3 = a_4 = b_1 = b_2 = b_3 = b_4 = 0, \quad a_6 + b_6 = 0, \quad a_8 + b_8 = 0.$$

4. Holomorphic Norden – Walker (Kähler – Norden – Walker) metrics

Let (M_8, φ, g) be an almost Norden – Walker manifold. If

$$(\Phi_\varphi g)_{kij} = \varphi_k^m \partial_m g_{ij} - \varphi_i^m \partial_k g_{mj} + g_{mj}(\partial_i \varphi_k^m - \partial_k \varphi_i^m) + g_{im}\partial_j \varphi_k^m = 0, \tag{11}$$

then, by virtue of Theorem 1, φ is integrable and the triple (M_8, φ, g) is called a holomorphic Norden – Walker or a Kähler – Norden – Walker manifold. Taking into account Remark 1, we see that an almost Kähler – Norden – Walker manifold with conditions $\Phi_\varphi g = 0$ and $N_\varphi \neq 0$ does not exist.

Substitute (2) and (3) into (11). Since $(\Phi_\varphi g)_{ijk} = (\Phi_\varphi g)_{ikj}$, we need only consider $(\Phi_\varphi g)_{ijk}$ $(j \le k)$. By explicit calculations, the nonzero components of the tensor $\Phi_\varphi g$ are as follows:

$$\begin{aligned}
&(\Phi_\varphi g)_{155} = a_3, \quad (\Phi_\varphi g)_{157} = \frac{1}{2}(b_1 - a_1), \quad (\Phi_\varphi g)_{177} = b_3,\\
&(\Phi_\varphi g)_{255} = a_4, \quad (\Phi_\varphi g)_{257} = \frac{1}{2}(b_2 - a_2), \quad (\Phi_\varphi g)_{277} = b_4,\\
&(\Phi_\varphi g)_{355} = -a_1, \quad (\Phi_\varphi g)_{357} = \frac{1}{2}(b_3 - a_3), \quad (\Phi_\varphi g)_{377} = -b_1,\\
&(\Phi_\varphi g)_{455} = -a_2, \quad (\Phi_\varphi g)_{457} = \frac{1}{2}(b_4 - a_4), \quad (\Phi_\varphi g)_{477} = -b_2,\\
&(\Phi_\varphi g)_{517} = -(\Phi_\varphi g)_{715} = \frac{1}{2}(a_1 + b_1), \quad (\Phi_\varphi g)_{527} = -(\Phi_\varphi g)_{725} = \frac{1}{2}(a_2 + b_2),\\
&(\Phi_\varphi g)_{537} = -(\Phi_\varphi g)_{735} = \frac{1}{2}(a_3 + b_3), \quad (\Phi_\varphi g)_{547} = -(\Phi_\varphi g)_{745} = \frac{1}{2}(a_4 + b_4),\\
&(\Phi_\varphi g)_{555} = \frac{1}{2}(a+b)a_3 - a_7, \quad (\Phi_\varphi g)_{557} = -b_5,\\
&(\Phi_\varphi g)_{567} = -(\Phi_\varphi g)_{756} = \frac{1}{2}(a_6 + b_6), \quad (\Phi_\varphi g)_{577} = \frac{1}{2}(a+b)b_3 + a_7,\\
&(\Phi_\varphi g)_{578} = -(\Phi_\varphi g)_{758} = \frac{1}{2}(a_8 + b_8), \quad (\Phi_\varphi g)_{655} = -a_8,\\
&(\Phi_\varphi g)_{657} = \frac{1}{2}(b_6 - a_6), \quad (\Phi_\varphi g)_{677} = -b_8, \quad (\Phi_\varphi g)_{755} = -\frac{1}{2}(a+b)a_1 - b_5,\\
&(\Phi_\varphi g)_{757} = -a_7, \quad (\Phi_\varphi g)_{777} = -\frac{1}{2}(a+b)b_1 + b_5,\\
&(\Phi_\varphi g)_{855} = a_6, \quad (\Phi_\varphi g)_{857} = \frac{1}{2}(b_8 - a_8), \quad (\Phi_\varphi g)_{877} = b_6.
\end{aligned}$$

From the above equations, we have

Theorem 4. *A triple (M_8, φ, g) is a Kähler – Norden – Walker manifold if and only if the following PDEs hold:*

$$\begin{aligned}
a_1 = a_2 = a_3 = a_4 = a_6 = a_7 = a_8 = 0,\\
b_1 = b_2 = b_3 = b_4 = b_5 = b_6 = b_8 = 0.
\end{aligned} \tag{12}$$

Corollary 4. *A manifold (M_8, φ, g) is Kähler – Norden – Walker if and only if the matrix B in (2) is as follows:*

$$B = \begin{pmatrix} a(x^5) & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & b(x^7) & 0\\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{13}$$

5. Kähler – Norden – Walker – Einstein metrics

We now turn our attention to the Einstein conditions for a Walker metric (2), (3) with a and b given by (12). In this case, as $a = a(x^5)$ and $b = b(x^7)$, B in (3) is of the form (13).

Let R_{ij} and S denote, respectively, the Ricci tensor and the scalar curvature of metric (2) with B given as in (13). We see that all the components of the Einstein tensor defined by $G_{ij} = R_{ij} - \frac{1}{8} S g_{ij}$ are zero.

Thus, we have

Theorem 5. *A metric g with B as in* (13) *is always Kähler – Norden – Walker – Einstein.*

6. On a relation between the Goldberg conjecture of almost Norden – Walker and Kähler – Norden – Walker manifolds

Let (M_{2n}, φ, g) be an almost Norden manifold, and choose a φ-compatible 2-form Ω_φ on M_{2n}, where $\Omega_\varphi(X,Y) = h(\varphi X, Y)$, $h(X,Y) = g(X,Y) + g(\varphi X, \varphi Y)$. Then we can propose an almost Norden version of the Goldberg conjecture as follows [19]: if (G_1) M_{2n} is compact, (G_2) g is Einstein, and (G_3) the φ-compatible 2-form is closed, then φ must be integrable.

We now define two subfamilles in the set of all compact Norden – Walker 8-manifolds:

$$
\begin{aligned}
KNW &= \{(M_8, \varphi, g): \ \Phi_\varphi g = 0\}, \\
GNW &= \{(M_8, \varphi, g): \ M_8 \text{ with conditions } (G_2),\ (G_3)\}.
\end{aligned}
$$

Theorem 6. *Let $M_8 \in KNW$. Then M_8 is of type GNW, i. e., $M_8 \in GNW$.*

Proof. Suppose that $M_8 \in KNW$. Then, from Theorem 5, we see that g is Einstein. By virtue of Theorem 1 $(\nabla \varphi = 0)$, for Ω_φ we have

$$
(\nabla \Omega_\varphi)(Z; X, Y) = (\nabla_Z\, g)(\varphi X, Y) - (\nabla_Z\, g)(X, \varphi Y) + \\
+ g((\nabla_Z\, \varphi)X, Y) - g(X, (\nabla_Z\, \varphi)Y) = 0,
$$

where ∇ is the Levi – Civita connection of g. On the other hand, using the relation [20, p. 149],

$$d\Omega_\varphi = A(\nabla \Omega_\varphi),$$

where $\nabla \Omega_\varphi$ is the covariant differential of Ω_φ and A is the alternation, we have

$$d\Omega_\varphi = 0,$$

i. e., Ω_φ is closed. Thus, the proof is completed. □

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Резюме

А.А. Салимов, М. Исчан. О геометрии B-многообразий.

Изучаются почти B-структуры (структуры Нордена) на 8-мерных многообразиях Уокера. Для указанных структур исследуются вопросы интегрируемости, условия келеровости и эйнштейновости.

Ключевые слова: 8-мерное многообразие Уокера, метрика Нордена, голоморфная метрика, метрика Эйнштейна.

References

1. *Norden A.P.* On a certain class of four-dimensional A-spaces // Izvestiya VUZ. Matematika. – 1960. – No 4. – P. 145–157 [in Russian].
2. *Etayo F., Santamaria R.* $(J^2 = \pm 1)$-metric manifolds // Publ. Math. Debrecen. – 2000. – V. 57, No 3–4. – P. 435–444.
3. *Ganchev G.T., Borisov A.V.* Note on the almost complex manifolds with a Norden metric // C. R. Acad. Bulgarie Sci. – 1986. – V. 39, No 5. – P. 31–34.
4. *Kruchkovich G.I.* Hypercomplex structure on a manifold I // Trudy Seminara Vect. Tens. Anal. – M.: Moscow Univ., 1972. – No 16. – P. 174–201 [in Russian].
5. *Salimov A.A.* Almost analyticity of a Riemannian metric and integrability of a structure // Trudy Geom. Sem. Kazan. Univ. – 1983. – No 15. – P. 72–78 [in Russian].
6. *Tachibana S.* Analytic tensor and its generalization // Tôhoku Math. J. – 1960. – V. 12, No 2. – P. 208–221.
7. *Vishnevskii V.V.* Integrable affinor structures and their plural interpretations // J. Math. Sci. – 2002. – V. 08, No 2. – P. 151–187.
8. *Salimov A.A.* Generalized Yano – Ako operator and the complete lift of tensor fields // Tensor (N.S.). – 1994. – V. 55, No 2. – P. 142–146.
9. *Salimov A.A.* Lifts of poly-affinor structures on pure sections of a tensor bundle // Russian Mathematics (Iz. VUZ). – 1996. – V. 40, No 10. – P. 52–59.
10. *Vishnevskii V.V., Shirokov A.P., Shurygin V.V.* Spaces over algebras. – Kazan: Kazan Gos. University, 1985. – 263 p. (in Russian).
11. *Yano K., Ako M.* On certain operators associated with tensor fields // Kodai Math. Sem. Rep. – 1968. – V. 20. – P. 414–436.
12. *Iscan M., Salimov A.A.* On Kahler – Norden manifolds // Proc. Indian Acad. Sci. (Math. Sci.). – 2009. – V. 119, No 1. – P. 71–80.
13. *Salimov A.A., Iscan M., Etayo F.* Paraholomorphic B-manifold and its properties // Topol. and its Appl. – 2007. – V. 154. – P. 925–933.
14. *Matsushita Y., Haze S., Law P.R.* Almost Kähler-Einstein structure on 8-dimensional Walker manifolds // Monatsh. Math. – 2007. – V. 150. – P. 41–48.
15. *Walker A.G.* Canonical form for a Riemannian space with a parallel field of null planes // Quart. J. Math. Oxford. – 1950. – V. 1, No 2. – P. 69–79.
16. *Bonome A., Castro R., Hervella L. M., Matsushita Y.* Construction of Norden structures on neutral 4-manifolds // JP J. Geom. Topol. – 2005. – V. 5, No 2. – P. 121–140.
17. *Matsushita Y.* Four-dimensional Walker metrics and symplectic structure // J. Geom. Phys. – 2004. – V. 52. – P. 89–99.
18. *Matsushita Y.* Walker 4-manifolds with proper almost complex structure // J. Geom. Phys. – 2005. – V. 55. – P. 385–398.
19. *Matsushita Y.* Counterexamples of compact type to the Goldberg conjecture and various version of the conjecture, to appear // Proc. the 8th Intern. Workshop on Complex Structures and Vector Fields, Sofia, Bulgaria, August 20–26, 2004 / Ed. S. Dimiev, K. Sekigawa. – World Scientific, 2007. – P. 222–233.

20. *Kobayashi S., Nomizu K.* Foundations of Differential Geometry. V. 1. – N. Y.: Interscience Publ. (Wiley), 1963. – 329 p.

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