On definability of c.e. degrees in the 2-c.e. degree structures

Yamaleev M.M.

Kazan Federal University

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Definitions and notations

All sets are subsets of $\omega = \{0, 1, 2...\}$. Thus, let $A, B \subset \omega$.

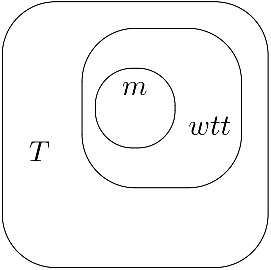
- $A \leq_T B$ if there is an algorithm that allows to answer the questions "$x \in A$?", using B as an oracle.
- $A \leq_{wtt} B$ if $A \leq_T B$ and there is a computable bound for the question.
- $A \leq_m B$ if there is a computable function f such that $x \in A \iff f(x) \in B$.
- Clearly, $A \leq_m B \implies A \leq_{wtt} B \implies A \leq_T B$.

Definitions and notations

- If $A \leq_r B$ and $B \leq_r A$ then $A \equiv_r B$.
- Let $\deg_r(A) = \{B \mid A \equiv_r B\}$.
- Here, $r \in \{m, wtt, T\}$.

Definitions and notations

Recall that in general $\deg_m(A) \subset \deg_{wtt}(A) \subset \deg_T(A)$.



Definitions and notations

Consider $\equiv_m$, $\equiv_{wtt}$, $\equiv_T$ as equivalence relations on c.e. sets.

Consider $\leq_m$, $\leq_{wtt}$, $\leq_T$ as preorders on c.e. sets.

- $\equiv_m$, $\leq_m$ are Σ^0_3-complete
- $\equiv_{wtt}$, $\leq_{wtt}$ are Σ^0_3-complete
- $\equiv_T$, $\leq_T$ are Σ^0_4-complete

The Coopers theorem

Theorem (Cooper, 1971)

There is a 2-c.e. set with proper 2-c.e. Turing degree.

Remark. In particular, this set has proper m- and wtt-degrees.

As a corollary, the universes for c.e. and 2-c.e. degree structures are different. And clearly, c.e. degrees form a substructure in the corresponding 2-c.e. degrees.

Motivations and goals

- To investigate the 2-c.e. degree structures.
- To investigate model-theoretic properties of c.e. and 2-c.e. degrees (in the different settings).
- To study relationship between c.e. and properly 2-c.e. degrees (in the different settings).

Motivations and goals

Open question (Cooper, 2002; Arslanov, 2009)

Is the class of c.e. Turing degrees definable in the partial ordering of 2-c.e. Turing degrees?

Related questions:

- The same questions for m- and wtt-degrees.
- A weaker version of the question involving parameters.
- The case of low c.e. and 2-c.e. degrees.

Definability

Let $\mathcal{A}$ be a structure, and B be a subset of $|A|$.

Definition

The class B is definable in $\mathcal{A}$ if there exists a formula $\varphi(x)$ of the first order language such that for all $a \in |A|$ it holds

$$\mathcal{A} \models \varphi(a) \Leftrightarrow a \in B$$

- As $\mathcal{A}$ we consider $(\mathbf{D}, \leq)$, where $\mathbf{D}$ is the corresponding 2-c.e. degrees, and $\leq$ is induced by the same reducibility.
- As B we consider $\mathbf{R}$, the c.e. degrees.
- In φ, there can be additional fixed variables $c_1, c_2, \dots \in |A|$ called parameters.

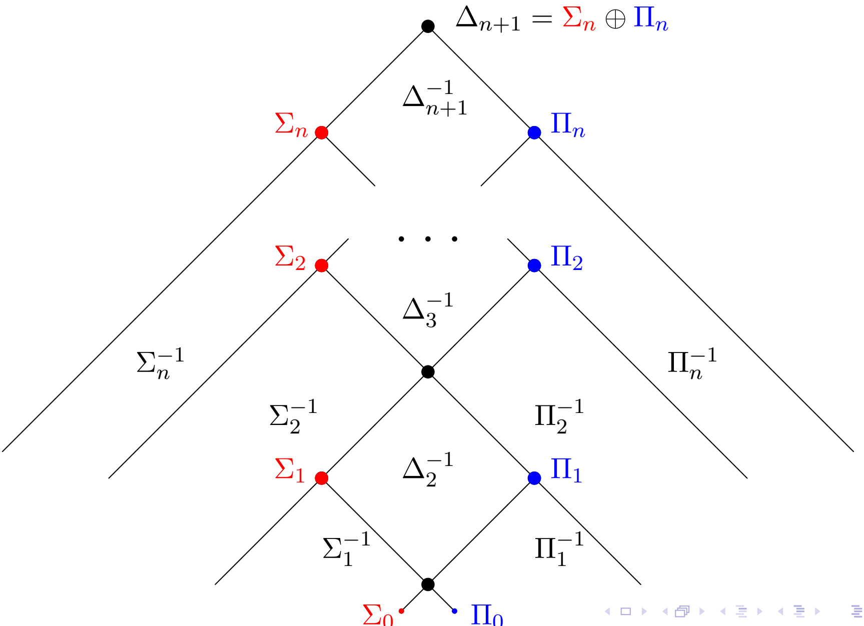
Section 1

Definability for m-degrees

A brief history

- m-degrees were actively studied since 1970 (by Degtev, Denisov, Ershov, Nies, Lachlan, Selivanov, etc.)
- The most attention was received by c.e. m-degrees and by all m-degrees.
- In general the structures of m-degrees found out to have many good properties, in particular, much better then the structures of T-degrees.
- For example, Σ_n^{-1} m-degrees have the greatest (universal) element (by Ershov), Δ_n^{-1} m-degrees have the greatest element (by Selivanov), for any fixed $n > 0$.

Picture



$\Delta_{n+1} = \Sigma_n \oplus \Pi_n$
Δ_{n+1}^{-1}
Σ_n
Π_n
$\cdots$
Σ_2
Π_2
Δ_3^{-1}
Σ_n^{-1}
Π_n^{-1}
Σ_2^{-1}
Π_2^{-1}
Σ_1
Δ_2^{-1}
Π_1
Σ_1^{-1}
Π_1^{-1}
Σ_0
Π_0

Facts and folklore

- c.e. m-degrees form an ideal in 2-c.e. m-degrees
- c.e. and co-c.e. m-degrees are isomorphic.
- c.e. and 2-c.e. m-degrees form a distributive upper semilattice (by Ershov, Lachlan, Selivanov). The same holds for c.e. wtt-degrees, but doesn't hold for 2-c.e. wtt-degrees, and for c.e. and 2-c.e. Turing degrees.
- The greatest c.e. m-degree is not splittable (by Lachlan), thus Δ- and Σ-(Π-) levels are not elementarily equivalent. The result has a direct generalization to 2-c.e. m-degrees.

Facts and folklore

- Given 2-c.e. set $A = A_0 - A_1$, let $A_0 = rng(f)$ for some computable 1-1 f, then $L(A) = f^{-1}(A_1)$ is Lachlan's set for A.
- $L(A)$ is c.e.
- $\overline{L(A)} \leq_m A$
- If $L(A)$ is c.e. then A is 2-c.e., and if $L(A)$ is computable then A is c.e.
- Then below any proper 2-c.e. m-degree there exists a noncomputable co-c.e. m-degree.

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Elementary difference

Theorem (Ershov and Lavrov, 1973)

Given noncomplete c.e. set B there exists a c.e. set $A \not\leq_m B$ which is minimal

Corollary

c.e. and 2-c.e. m-degrees are not elementarily equivalent

Note that for we can take U_{Δ_2} as B, then any set $A \not\leq_m B$ would be proper 2-c.e. and have a noncomputable element below it.

Remark. The theorem was proved for much more general case and for the c.e. setting. For details see [Erhov Yu.L., Lavrov I.A. Upper semilattice L(S), Algebra i Logica, 1973, Vol.12, No.2, P.167-189]

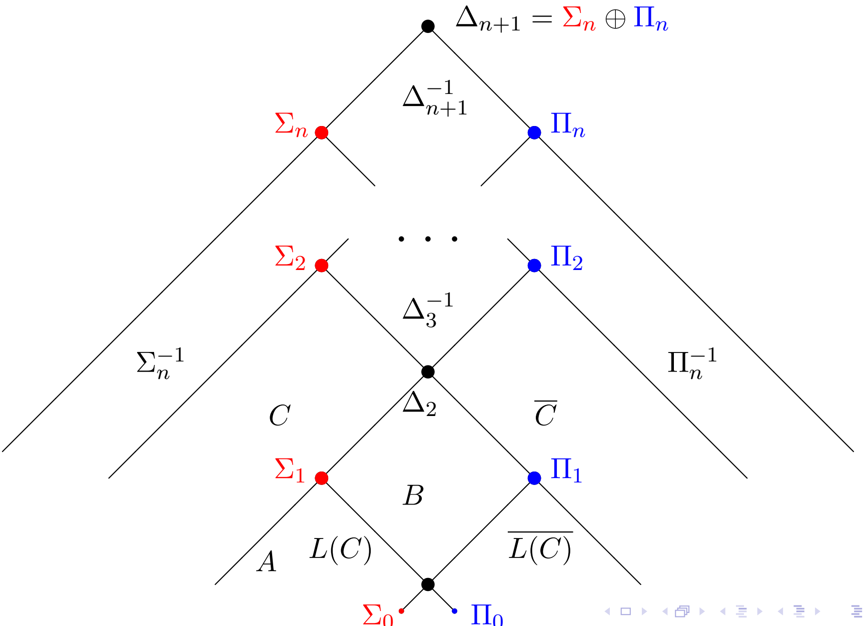
The main structural theorems

Theorem (Ng and Yamaleev, T.1)

Given $k, n > 0$, given any Σ_k^{-1} set B such that $U_{\Sigma_n} \not\leq_m B$, there exists a Σ_n^{-1} set $A \not\leq_m B$ such that for any $W <_m A$ it holds that $W \leq_m U_{\Delta_n}$.

- noncomplete $B \leftrightsquigarrow U_{\Sigma_n} \not\leq_m B$
- minimal $A \leftrightsquigarrow A$ is minimal cover for U_{Δ_n}
- c.e. $A \not\leq_m B \leftrightsquigarrow \Sigma_n^{-1}$ set $A \not\leq_m B$

Picture



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Π_n
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Σ_n^{-1}
Π_n^{-1}
C
Δ_2
$\overline{C}$
Σ_1
Π_1
B
A
$L(C)$
$\overline{L(C)}$
Σ_0
Π_0

The main structural theorems

Corollary (Ng and Yamaleev)

Given $k, n > 0$, *given any* Σ_k^{-1} *set* B *such that* $U_{\Delta_{n+1}} \not\leq_m B$, *there exists a* Δ_{n+1}^{-1} *set* $A \not\leq_m B$ *such that for any* $W <_m A$ *it holds that* $W \leq_m U_{\Delta_n}$.

Theorem (Ng and Yamaleev, T.2)

Given $n > 0$, *there exists a set* A *of properly* Σ_{n+1}^{-1} *degree such that for any* $W \in \Sigma_n^{-1}$ *if* $W \leq_m A$ *then* $W \leq_m U_{\Delta_n}$.

The main structural theorems, 2-c.e. setting

Corollary (Ershov and Lavrov, 1973)

Given noncomplete (in Δ_2^{-1} m-degrees) set B there exists a 2-c.e. set $A \not\leq_m B$ such that A has a minimal m-degree (moreover, it will be either c.e. or co-c.e.)

Corollary (Ng and Yamaleev, T.2)

There exists a set A of properly 2-c.e. m-degree such that for any c.e. W if $W \leq_m A$ then W is computable (i.e., A form minimal pair with the greatest c.e. degree).

Intuitive description

- The first part says we can build minimal m-degrees avoiding arbitrary (noncomplete) lower cones.
- The second part says that for all c.e. m-degrees we can find a half minimal pair in the 2-c.e. m-degrees.
- Note also that we cannot do it for co-c.e. m-degrees using a unique 2-c.e. degree.

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- Note also that we cannot do it for co-c.e. m-degrees using a unique 2-c.e. degree.

The corollaries

- The degree structures of c.e. and 2-c.e. m-degrees are not elementarily equivalent (and it works for all higher levels).
- The m-degree of universal Δ_2^{-1}-set is definable in 2-c.e. m-degrees.
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Definability of c.e. in 2-c.e.

- $\theta(x) := \forall b\ [x \not\leq b \Rightarrow \exists a\ (a \not\leq b \wedge \forall w\ [w < a \Rightarrow w \leq 0])]$
- $\psi(x) := \theta(x) \wedge \forall z[x < z \Rightarrow \neg\theta(z)]$,
- Thus, $\psi(x)$ is true in Σ_2^{-1} iff $x = U_{\Delta_2}$

- $\varphi(x, y) := \exists u\ \psi(u) \wedge x \cup y = u \wedge [\forall x_1\ \forall y_1(x_1 < x \Rightarrow x_1 \cup y < u) \wedge (y_1 < y \Rightarrow x \cup y_1 < u)]$
- Thus, $\varphi(x, y)$ defines the pair of U_{Σ_1} and U_{Π_1} but cannot distinguish them.

- $\varphi^{\Sigma}(x) := \exists y\ (\varphi(x, y) \wedge \exists z\ \forall w[z \not\leq x \cup y \wedge w < z \wedge w \leq x \Rightarrow w \leq 0])$

Complexity of the formulas

- Elementarily difference of c.e. and 2-c.e.: Σ^0_2
- Definability of c.e. in 2-c.e.: Σ^0_4
- For higher level the complexity depends on the level of Ershov's hierarchy

Questions

- Is Σ_1^{-1} level definable in the structure of Σ_ω^{-1}-level?
- Could the same approach work for infinite levels? (probably with parameters)
- What to do with the limit levels?

Acknowledgement

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- A.G. Melnikov (and the workshop in Almaty in 2018)

Definability (with parameters) for Turing degrees

In this section, we talk about Turing degrees.

Approaches

to the problem of definability of c.e. Turing degrees in partial ordering of 2-c.e. Turing degrees.

Proposed by Arslanov and Yamaleev (2018)

1. Density of double bubbles
2. Nonspilliting pairs
3. Lachlan sets and degrees
4. Isolation from side

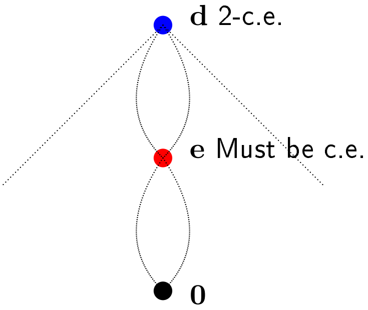
Approach 1

Definition (Arslanov, Kalimullin, and Lempp, 2010)

Let $\mathbf{e}, \mathbf{d}$ be 2-c.e. degrees such that $\mathbf{0} < \mathbf{e} < \mathbf{d}$. We say that these degrees form a *double bubble* (also, *a bubble pair*, *2-bubble*, *bubble*) in 2-c.e. degrees if any 2-c.e. degree $\mathbf{u} < \mathbf{d}$ is comparable with $\mathbf{e}$. Also, we say that $\mathbf{d}$ is the top of bubble, and $\mathbf{e}$ is the middle of bubble. By default, we consider bubbles in 2-c.e. degrees.

- The degree $\mathbf{e}$ must be c.e.
- The degree $\mathbf{d}$ is an exact 2-c.e. degree.
- The degree $\mathbf{d}$ is not splittable avoiding upper cone of $\mathbf{e}$.

Approach 1. The picture.



Approach 1. The idea.

- To show that between any two c.e. degrees we can find a degree $\mathbf{e}$.
- Then any c.e. degree has a splitting where the both parts are middles of bubbles.
- Such splitting doesn't exists for properly 2-c.e. degrees.

Approach 1. The results.

- [Liu, Wu, Yamaleev, 2015]The exact 2-c.e. degrees are downward dense.
- [Andrews, Kuyper, Lempp, Soskova, Yamaleev, 2017] There exists a nonzero c.e. degree such that no double bubble can be found below it.
- Conjecture [Arslanov, Yamaleev, 2018] The middles of double bubbles can be found below any nonzero c.e. degree, moreover it can be combined with lower cone avoidance.

Approach 1. Conclusion.

- Definable middle of bubbles with fairly “easy” construction.
- Even if we cannot prove the density. The middles of bubbles is still a reliable class of c.e. degrees. And can be combined with downward density and cone avoidance.
- Can a middle of bubble be constructed above any low or superlow c.e. degree?

Approach 2. Idea.

- So far, the middles of bubbles is the only known definable class in the 2-c.e. degrees.
- [Cooper, Li, 2002] There exists a nonzero c.e. degree $\mathbf{e}$, 2-c.e. $\mathbf{d} > \mathbf{e}$ such that $\mathbf{d}$ is not splittable avoiding the upper cone of $\mathbf{e}$.
- [Yamaleev, 2009] Given properly 2-c.e. degrees $\mathbf{d} > \mathbf{b}$. If there is no a c.e. degree between them then $\mathbf{d}$ is splittable avoiding upper cone of $\mathbf{b}$.

Approach 2. Idea.

- Change definable singletons to the intervals containing c.e. degrees.
- Every splitting of a properly 2-c.e. degree has a half whose interval doesn't contain a c.e. degree.
- However, there were many difficulties as in the first approach (namely, we met similar difficulties when tried to split a c.e. degree into two middles of bubbles)

Approach 2. Idea.

- Other definable singleton is a degree $\mathbf{e}$ with the following property. For a given pair $\mathbf{e} < \mathbf{d}$ it satisfies: any degree $\mathbf{f}$ such that $\mathbf{e} < \mathbf{f} \leq \mathbf{d}$ is not splittable avoiding the upper cone of $\mathbf{e}$.
- However, it coincides with the middles of bubbles because of the following result.
- [Liu, Ng, Yamaleev, 2018] Given 2-c.e. sets $E \leq_T D$. If $E \not\leq_T L(D)$ then D is splittable avoiding the upper cone of B.
- Thus, if $\mathbf{d} > \mathbf{e}$ satisfies the mentioned property then clearly $L(D) \in \mathbf{e}$. Also, it holds for any $\mathbf{f}$ between $\mathbf{e}$ and $\mathbf{f}$. Now, if there was 2-c.e. $\mathbf{b}$ incomparable with $\mathbf{e}$ then either $L(B) <_T E$ or $L(B)|_T E$. Then we take join of B and E.

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Approach 2. Conclusion.

- Other definable c.e. classes coincide with the middles of bubbles.
- An alternative is to use the whole intervals which contain a c.e. degree. For instance, if $\mathbf{d} > \mathbf{e}$ is not splittable avoiding the upper cone of $\mathbf{e}$ then there is a c.e. degree in $[\mathbf{e}, \mathbf{d})$. However, here we stick to a pair of variables (in particular, it is important for the downward density).

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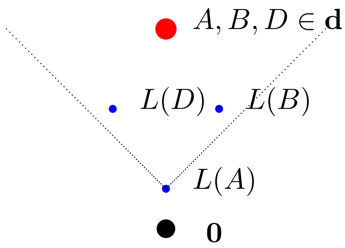
Approach 3. Idea.

- To use $L(D)$ which reflects to enumerability properties of a 2-c.e. set D. Then consider a collection of $L(B)$ such that $B \equiv_T D$.
- Make a connection between the associated degrees $L(B)$ and the degree of D.
- The good case is when for each properly 2-c.e. degree of B the collection of the degrees of $L(B)$ is bounded from below by some nonzero c.e. degree.

Approach 3. Results.

- Series results by Ishmukhametov [1999,2000] and by Fang, Liu, Wu, Yamaleev [2013-2019] showed that different distributions for $L(B)$ are possible.
- In particular, there is a properly 2-c.e. degree with unbounded collection of its associated degrees of $L(B)$.
- Nice to mention: if $D \equiv_T B$ and have a proper 2-c.e. degree then $L(D)$ and $L(B)$ cannot form a minimal pair.

Approach 3. Picture.



Approach 3. Conclusion.

- [Fang, Liu, Wu, Yamaleev, 2018] Unfortunately, for the most viable way there is a negative result, and some properly 2-c.e. degree $\mathbf{d}$ has unbounded $L[\mathbf{d}]$.
- [Yamaleev] Though, there are still some special properties for 2-c.e. degrees, namely, for any properly 2-c.e. degree $\mathbf{d}$ there is a c.e. degree $\mathbf{c}$ such that $\mathbf{c} \notin L[\mathbf{d}]$.

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Approach 4. Isolation from side.

- [Yang and Yu, 2006] Inapparently used isolation from side to show that c.e. degrees doesn't form a Σ_1-substructure of 2-c.e. degrees.
- [Cai, Slaman, and Shore, 2012] Inapparently used isolation from side to show that k-c.e. degrees doesn't form a Σ_1-substructure of n-c.e. degrees for all $k < n$
- [Wu and Yamaleev, 2012] A 2-c.e. degree $\mathbf{d}$ is *isolated from side nontrivially* if $\mathbf{d}$ is nonisolated and there exists a c.e. degree $\mathbf{a}|\mathbf{d}$ such that for all c.e. degrees $\mathbf{w}$ if $\mathbf{w} \leq \mathbf{d}$ then $\mathbf{w} \leq \mathbf{a}$.

Approach 4. The result.

Any low properly 2-c.e. Turing degree $\mathbf{d}$ is isolated from side

Theorem (Yamaleev, 2019)

For any low 2-c.e. set D with a properly 2-c.e. Turing degree there exists a c.e. set A such that $D \not\leq_T A$ and for any c.e. set $W \leq_T D$ it holds $W \leq_T A$.

- The set A can be made low
- If $D \leq_T C$ then the set A can be made below C.

Approach 4. The consequences.

- Recall from Approach 1,
  Conjecture [Arslanov, Yamaleev, 2018] The middles of double bubbles can be found below any nonzero c.e. degree, moreover it can be combined with lower cone avoidance.
- In particular, for isolation from side we bound all middles of double bubbles.

Corollary

The low c.e. degrees are definable in the partial ordering of low 2-c.e. Turing degrees

- For any low c.e. degree we can construct a definable c.e. degree below it, avoiding any lower cone
- Due to isolation from side we cannot do it for any properly low 2-c.e. degrees.

Approach 4. The definability (with parameters).

- [Welch, 1980] There exists low c.e. degrees $\mathbf{c}_1$ and $\mathbf{c}_2$ such that for any c.e. degree $\mathbf{a}$ there exists its splitting $\mathbf{a}_1 \cup \mathbf{a}_2 = \mathbf{a}$ such that $\mathbf{a}_i \leq \mathbf{c}_i$ for $i = 1, 2$.
- The parameters $\mathbf{c}_1$ and $\mathbf{c}_2$ are the desired ones. Lets fix them.
- Consider a c.e. degree $\mathbf{a}$. It has the mentioned above splitting such that the both parts are below the parameters, also those parts are not isolated from side (i.e. we always can find a definable c.e. degree below them).
- Consider a properly 2-c.e. degree $\mathbf{d}$. If it doesn't have a splitting below the parameters then it is clearly proper 2-c.e. Assume it has such splitting. Then at least one part must be properly 2-c.e. Then at least one part must be isolated from side (recall that $\mathbf{c}_1$ and $\mathbf{c}_2$ are low).

Approach 4. Misc.

- Assume that for a given c.e. degrees $\mathbf{a} \not\leq \mathbf{c}$ there is a middle of bubble $\mathbf{e} < \mathbf{a}$ such that $\mathbf{e} \not\leq \mathbf{c}$. How to avoid the case when $\mathbf{c}$ could be 2-c.e.?
- Then we should update isolation from side as follows: given 2-c.e. degree $\mathbf{d}$ and c.e. degree $\mathbf{a}$ such that $\mathbf{a} \not\leq \mathbf{d}$. Then there is a c.e. degree $\mathbf{c}$ such that it covers the c.e. degrees below $\mathbf{d}$ (can include $\mathbf{d}$ as well) and $\mathbf{a} \not\leq \mathbf{c}$.

Approach 4. Backup plans.

- For a given properly 2-c.e. degree $\mathbf{d}$ do there exists c.e. degrees $\mathbf{c}$ and $\mathbf{g}$ such that one of them isolates $\mathbf{d}$ from side?
- For a given properly 2-c.e. degree $\mathbf{d}$ do there exists c.e. degrees $\mathbf{c}$ and $\mathbf{g}$ such that any c.e. degree below $\mathbf{d}$ is either below $\mathbf{c}$ or $\mathbf{g}$?
- Note that then we obtain definable degrees which are join of two middles of bubbles? Does this class coincide wth the middles of bubbles?

Open questions

- Is any properly 2-c.e. degree isolated from side?
- Can c.e. degree be definable with 1 parameter in the partial ordering of 2-c.e. degrees?

Section 3

The wtt-degrees setting

wtt-degrees, problems for transferring

- [Yamaleev, 2019]. Any properly 2-c.e. wtt-degree is isolated from side.
- However, there are difficulties to realize the conjecture about middle of the double bubble.

- How far can we go with the lower cone avoidance in 2-c.e. wtt-degrees?
- Is it possible to use the nonsplitting pair?

Question

Are c.e. wtt-degrees definable in the partial ordering of 2-c.e. wtt-degrees?

Conclusion

Chances for success (in the nearest future).

- Definability of c.e. m-degrees in 2-c.e. m-degrees (very high).
- Definability of c.e. low Turing degrees in low 2-c.e. Turing degrees (high).
- Definability with two parameters of c.e. Turing degrees in 2-c.e. Turing degrees (high).
- Definability of c.e. wtt-degrees in 2-c.e. wtt-degrees (low, there are difficulties in avoiding lower cones).
- Definability of c.e. Turing degrees in 2-c.e. Turing degrees (very low, there are difficulties in isolation from side).

Conclusion

- Often each structure requires its own approach.
- In good cases an elementary difference can be considered as an intermediate step to definability.
- Among tools usually works well cone avoidance and minimal covering.
- It could be that for m-degrees there are other (and simpler) formulas for definability and elementarily difference. Though they could require much more subtle structural properties.
- There is some progress towards different directions

Thank you for your attention!