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On Positivity Conditions for the Cauchy Function of Functional-Differential Equations

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Abstract—We study how the statements on estimates of solutions to linear functional-differential equations, analogous to the Chaplygin differential inequality theorem, are connected with the positivity of the Cauchy function and the fundamental solution. We prove a comparison theorem for the Cauchy functions and the fundamental solutions to two functional-differential equations. In the theorem, it is assumed that the difference of the operators corresponding to the equations (and acting from the space of absolutely continuous functions to the space of summable ones) is a monotone totally continuous Volterra operator. We also obtain the positivity conditions for the Cauchy function and the fundamental solution to some equations with delay as long as those of neutral type.

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Introduction. Let $X = (X, \succeq)$ and $\Lambda = (\Lambda, \succeq)$ be partially ordered spaces, and Y be a set. Let a mapping $\Phi : X \times \Lambda \to Y$ and an element $\theta \in Y$ be given. Consider the equation $\Phi(x, \lambda) = \theta$ with respect to $x \in X$ with a parameter $\lambda \in \Lambda$. To obtain estimates of solutions, some statements on monotone dependence of its solution x on λ can be effective. For differential equations, some functions involved into them (e.g., the known right-hand sides of the equations), as long as values of initial or boundary conditions, are often selected as parameters. The Chaplygin theorem [1] states that, under the assumption of continuity of the function φ, if some function x_0 satisfies the condition $\dot{x}_0(t) > \varphi\big(t, x_0(t)\big)$, $t \geq 0$, then for the solution to the equation $\dot{x} = \varphi(t, x)$, $t \geq 0$, with initial condition $x(0) = x_0(0)$ the strong inequality $x(t) < x_0(t)$, $t > 0$, is valid. This statement can be considered (with appropriate ordering of functions) as a theorem on strongly monotone dependence of the solution x to the Cauchy problem for the equation $\dot{x} - \varphi(t, x) = \lambda(t)$ on λ.

Many investigations are devoted to dissemination and generalization the statements on inequalities; fundamental results on differential and integral inequalities are obtained by Azbelev (see [2], section 1). Functional-differential inequalities were investigated by the participants of the Perm seminar ([3–5]), conditions for dissemination of the Chaplygin theorem to linear boundary value problems we obtained in [6, 7]. In [8] we investigated non-linear functional inequalities. In the recent works [9, 10] investigation of implicit differential ad functional-differential inequalities commenced.

An analog of the Chaplygin theorem on non-strong inequality for a linear evolution functional-differential equation takes place if and only if its Cauchy function $\mathcal{C}(t, s)$ is nonnegative. To be fair, the corresponding statement on strong inequality additionally requires that for every $t > 0$ the measure of the set $\{s : \mathcal{C}(t, s) > 0\}$ be positive. Monotone dependence of the solution on the initial value is equivalent to nonnegativeness of the normal fundamental solution $X(t)$; under increasing of the initial value, the solution strongly increases if and only if $X(t) > 0$.

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