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SOME DEVELOPMENTS OF PETROV'S WORK ON CONFORMAL AND PROJECTIVE STRUCTURE

G. Hall

Abstract

This paper discusses the achievements of A.Z. Petrov in the area of conformal and projective structure of space-times. In fact, it will be mostly concerned with the latter topic but points out uses of the former in developing the projective theory of space-times. Some new developments in this area of Petrov's research will be given.

Key words: projective structure, conformal structure, Petrov classification, curvature map.

Introduction

Petrov's work on the algebraic classification of the Weyl tensor of the gravitational field in Einstein's general theory of relativity was a major breakthrough in achieving a fuller understanding of this theory through its applications to the generation of much-needed exact solutions of Einstein's field equations. The final form of Petrov's work first appeared in [1] (and in English translation in [2]) and also in his book [3]. The essence of this classification, for Petrov, was an algebraic classification of the Riemann tensor at a point of a space-time which was itself an Einstein space. However, in the important special case when the space-time is a vacuum space-time, the associated Riemann tensor has identical algebraic properties to the Weyl tensor of *any* space-time and so Petrov's algebraic classification is usually taken to apply, quite generally, to the Weyl tensor. Petrov showed that there were essentially three distinct algebraic types for such a Riemann tensor (and hence for the Weyl tensor of any space-time) at a particular space-time point. Within a decade of his original paper, many other workers had realised its usefulness and extended his ideas to a comprehensive theory of what is now referred to as the *Petrov Classification*. Two of Petrov's types have been divided by eigenvalue degeneracy, and so one now speaks of the Petrov types **I**, **D**, **II**, **N** and **III**, with type **O** reserved for the trivial case when the Weyl tensor vanishes at the point in question. Further details can be found in the above mentioned works of Petrov and also in many other places, for example, [4–6].

In his book [3], Petrov also discussed the idea of "geodesic mappings" of gravitational fields, that is, roughly speaking, "when two gravitational fields have the same (unparametrised) geodesics". This problem has been revived recently [7–13] and has interested both geometers (for obvious reasons) and relativists (because of the application to the Newton-Einstein principle of equivalence in general relativity theory). The main part of this paper will be concerned with this problem. In fact, Petrov's classification of the Weyl tensor is rather useful in some of the ways of attacking it. In addition, Petrov's algebraic ideas suggested to the present author the idea of the curvature map [6] and this also turns out to be useful in the study of projective structure.

1. The Petrov classification

Let M be a smooth, connected, Hausdorff 4-dimensional manifold with smooth Lorentz metric g of signature $(-,+,+,+)$ so that (M,g) is a *space- time*. Let ∇ denote the Levi – Civita connection arising from g and $Riem$ the associated curvature tensor. The components of $Riem$ are $R^a{}_{bcd}$; the associated Ricci tensor, $Ricc$, has components $R_{ab} \equiv R^c{}_{acb}$ and the Ricci scalar is $R \equiv R_{ab}\,g^{ab}$. The Weyl tensor is denoted by C and has components $C^a{}_{bcd}$. Using the usual algebraic symmetries of the Riemann tensor components R_{abcd} at some point $m \in M$, Petrov [1–3] introduced the well-known 6×6 notation for this tensor to turn it into a 6×6 symmetric matrix, $R_{\alpha\beta}$ at m where Greek letters take the values 1,. . . ,6 and represent a skew-symmetric pair of indices according to Petrov's scheme $1 \leftrightarrow (14), 2 \leftrightarrow (24), 3 \leftrightarrow (34), 4 \leftrightarrow (23), 5 \leftrightarrow (31)$, and $6 \leftrightarrow (12)$. Thus, $R_{1234} \rightarrow R_{(12)(34)} = R_{63}$, etc. Petrov also noticed that the symmetric matrix $R_{\alpha\beta}$ possessed certain other convenient symmetries because of the fact that (M,g) was assumed to be an Einstein space ($Ricc = \frac{R}{4}\,g$). Nowadays, these are usually expressed, using the duality operator $*$, in terms of the left and right "duals" of $Riem$. Since Petrov's ideas apply to the Weyl tensor of any space-time, the remainder of the argument will be given in terms of it. The dual relation referred to above is then given by

$$ {}^*C_{abcd} = C^*_{abcd} \tag{1} $$

One then considers the matrix $C_{\alpha\beta}$ and studies its possible Jordan canonical forms with respect to the (6×6) form of the *bivector metric* $G_{abcd} = g_{ac}g_{bd} - g_{ad}g_{bc}$ (which is permitted by the algebraic symmetries of G_{abcd}) and is non-degenerate with signature $(-,-,-,+,+,+)$. This Petrov did by first transferring attention to what is essentially the complex tensor $\overset{+}{C}$ derived from C and with components $\overset{+}{C}_{abcd} = C_{abcd} + i\overset{*}{C}_{abcd}$ where $\overset{*}{C}_{abcd}$ denotes either the left or right dual of C, these being equal, by (1), and then showing how this led to a study of a certain 3×3 trace-free complex matrix derived from and containing all the information in the original one. (It is remarked that the trace-free condition arises from the condition $C^{ab}{}_{ab} = 0$ and is stronger that the corresponding condition on $Riem$ (in an Einstein space) which is $R^{ab}{}_{ab} = R$.)

Thus the possible Segre types (over $\mathbb{C}$) of the original (6×6 Weyl) matrix $C_{\alpha\beta}$ are those of a complex 3×3 matrix and are $\{111\}$, $\{21\}$ and $\{3\}$. These are Petrov's three types of gravitational field at $m \in M$. By refining the classification on the basis of eigenvalue degeneracy, they are usually given in six types as $\{111\}$ (type **I**), $\{(11)1\}$ (type **D**), $\{21\}$ (type **II**) $\{(21)\}$ (type **N**) and $\{3\}$ (type **III**) together with type **O** if $C(m) = 0$. The symbol **D** here refers to the term "degenerate" whilst **N** stands for "null" (this latter term arising from certain algebraic similarities between this type and the Maxwell-Minkowski tensor in "pure radiation" electromagnetic fields). The trace-free condition referred to earlier shows that all the eigenvalues of the original 6×6 matrix (or the 3×3 one derived from it) in the types **N**, **III** (and **O**) cases are zero. If the Petrov type at $m \in M$ is **D**, **II**, **N**, **III** or **O**, the Weyl tensor is said to be *algebraically special* at m and if it is **I** it is said to be *algebraically general* at m.

It is remarked that Petrov's classification is pointwise and can vary from point to point over M subject to continuity requirements. In fact, one may decompose M disjointly in the forms $M = \mathbf{I} \cup \mathbf{D} \cup \mathbf{II} \cup \mathbf{N} \cup \mathbf{III} \cup \mathbf{O} = \mathbf{I} \cup \operatorname{int}\mathbf{D} \cup \operatorname{int}\mathbf{II} \cup \operatorname{int}\mathbf{N} \cup \operatorname{int}\mathbf{III} \cup \operatorname{int}\mathbf{O} \cup \mathbf{F}$ where a Petrov symbol now refers to all those points of M which are of that Petrov type, int denotes the interior operator in the manifold topology of M and the subset $\mathbf{F}$ is determined by the disjointness of the decomposition and is thus closed and can be shown to have empty interior [6].

Of the main developments of Petrov's work since its announcement in Kazan one should mention the algebraic work of Bel [14], Geheniau [15], and Debever [16], who discovered the beautiful reformulation of Petrov's classification using the idea of principal (or repeated principal) null directions (the *Bel criteria*) and the resultant canonical forms for C based on them (see also Sachs [17] and Ehlers and Kundt [5] for a rather elegant treatment of this problem). A comprehensive spinor treatment of similar matters was also given by Penrose [18] and a discussion of the physical interpretation of the Petrov classification was given by Pirani [19].

2. Projective relatedness

Now let M be a manifold of dimension $n \geq 2$ and g a smooth metric on M of any signature with Levi–Civita connection ∇ and with curvature and Ricci tensors, etc. denoted as before. Suppose g' is another smooth metric on M of arbitrary signature whose associated structures are denoted by adding a prime to the corresponding ones for g. Call ∇ and ∇' (or g and g', or (M, g) and (M, g')) *projectively related* if the unparamatrised geodesics of ∇ and ∇' coincide. If such is the case then M admits an exact global 1-form ψ such that, in any coordinate domain of M, the Christoffel symbols from ∇ and ∇' satisfy

$$\Gamma'^{a}_{bc} - \Gamma^{a}_{bc} = \delta^{a}{}_{b}\psi_c + \delta^{a}{}_{c}\psi_b \tag{2}$$

and, conversely, if (2) holds in any coordinate domain, (M, g) and (M, g') are projectively related. Equation (2) can, by using the identity $\nabla' g' = 0$, be written in the equivalent form

$$g'_{ab;c} = 2g'_{ab}\psi_c + g'_{ac}\psi_b + g'_{bc}\psi_a \tag{3}$$

where a semi-colon denotes a covariant derivative with respect to ∇. Equation (2) reveals a relation between the type $(1,3)$ curvature tensors $Riem$ and $Riem'$ of ∇ and ∇', respectively, given by

$$R'^{a}{}_{bcd} = R^{a}{}_{bcd} + \delta^{a}{}_{d}\psi_{bc} - \delta^{a}{}_{c}\psi_{bd} \quad (\Rightarrow R'_{ab} = R_{ab} - 3\psi_{ab}) \tag{4}$$

where $\psi_{ab} \equiv \psi_{a;b} - \psi_a\psi_b$. Since ψ is exact, $\psi = d\chi$ for some smooth function χ on M and then $\psi_{ab} = \psi_{ba}$. The problem thus becomes that of solving (3) for g' and ψ.

Petrov studied this problem in some detail (see [3]). He approached it as a problem regarding two quadratic forms g and g' and considered the associated Jordan forms of g' with respect to g. In Petrov's work, (M, g) was a space-time and g' also had Lorentz signature. In this case the only possible Segre types for g' are $\{1,111\}$, $\{211\}$ $\{31\}$ and $\{z\bar{z}11\}$ together with their degeneracies. Here, $\{1,111\}$ means that g' is diagonalisable over $\mathbb{R}$ (with a comma separating the "timelike eigenvalue" from the "spacelike" ones), and $\{z\bar{z}11\}$ necessarily occurs when g' is diagonalisable over $\mathbb{C}$ but not $\mathbb{R}$. Petrov then proceeds to solve (an equivalent form of) (3) for each of these Segre types using a method based on the Ricci rotation coefficients.

An alternative approach was suggested by the Russian mathematician Sinjukov [20] (and for the remainder of this section the manifold M is of any dimension $n \geq 2$ and the metrics g and g' are of arbitrary signature). His idea is essential to modify the approach contained in (2) and (3) by drawing attention away from the pair (g', ψ) to the pair (a, λ) where a is a (necessarily) non-degenerate, smooth, type $(0,2)$ symmetric tensor and λ a smooth (necessarily) exact 1-form, on M given by

$$a_{ab} = e^{2\chi} g'^{cd} g_{ac} g_{bd}, \quad \lambda_a = -e^{2\chi}\psi_b g'^{bc} g_{ac} \; (\Rightarrow \lambda_a = -a_{ab}\psi^b) \tag{5}$$

(where g'^{ab} are the contravariant components of g' and not g'_{ab} with indices raised using g). Then (5) can be inverted to give

$$g'^{ab} = e^{-2\chi} a_{cd}\, g^{ac}\, g^{bd}, \quad \psi_a = -e^{-2\chi} \lambda_b\, g^{bc}\, g'_{ac}. \tag{6}$$

The condition (3) for projective relatedness is now, from (5) and (6), equivalent to the more convenient (Sinjukov) equation for the (Sinjukov) tensor a [20]:

$$a_{ab;c} = g_{ac}\, \lambda_b + g_{bc}\, \lambda_a. \tag{7}$$

The idea then is to solve (7) for a and λ and convert back, using (6), to find g' and ψ. With a and λ thus found, one first defines a type $(2,0)$ tensor a^{-1} on M which is, at each $m \in M$, the inverse matrix of a $(a_{ac}\, a^{-1cb} = \delta^b_a)$. Then one defines a related type $(0,2)$ tensor on M by $a^{-1}_{ab} = g_{ac}\, g_{bd}\, a^{-1cd}$. Finally, one defines a global function $\chi = \dfrac{1}{2} \log \left| \left(\dfrac{\det g}{\det a} \right) \right|$ and a global exact 1-form $\psi \equiv d\chi$ on M. Then $g'_{ab} = e^{2\chi} a^{-1}_{ab}$, which is a global metric on M, and ψ, together satisfy (3) and constitute the required solution on M (see, e.g., [13, 21]; the expression here for g'_{ab} corrects a typographical error in [13]).

It is remarked at this point that one may associate with (M, g) its type $(1,3)$ *Weyl projective tensor* W with components given by

$$W^a{}_{bcd} = R^a{}_{bcd} - \frac{1}{n-1} (\delta^a{}_c R_{bd} - \delta^a{}_d R_{bc}). \tag{8}$$

This tensor was discovered by Weyl [22] and has the property that if (M, g) and (M, g') are projectively related, the Weyl projective tensors associated with g and g' are equal on M.

In the remaining sections, the aim is to survey and extend some of Petrov's results on projective relatedness using the Sinjukov transformation, Petrov's classification of the Weyl conformal tensor C, the *curvature map* [6] and holonomy theory.

3. First order systems, curvature maps and holonomy

For the remainder of this paper, let (M, g) be a space-time and let g' be any other (not necessarily Lorentz) metric on M so that (M, g) and (M, g') are projectively related. Then (2)–(7) hold (and $n = 4$ in (8)). On applying the Ricci identity to the Sinjukov tensor a and using (7) one finds

$$(a_{ab;cd} - a_{ab:dc} =) a_{ae} R^e{}_{bcd} + a_{be} R^e{}_{acd} = g_{ac}\, \lambda_{bd} + g_{bc}\, \lambda_{ad} - g_{ad}\, \lambda_{bc} - g_{bd}\, \lambda_{ac}, \tag{9}$$

where $\lambda_{ab} \equiv \lambda_{a;b} (= \lambda_{ba})$. On applying certain standard procedures to this equation and with repeated use of (7) one can show [12, 21] that if Φ is any of the components of a, the components of λ or the scalar $\lambda^a_{;a}$, then, in any coordinate domain of M, Φ satisfies a first-order differential equation of the form $\Phi_{,a} = F_a$ where a comma denotes a partial derivative and the quantities F_a depend only on the various quantities that Φ can represent and those describing the geometry of M. Thus, any global solution for a and λ of (7) is uniquely determined by giving the quantities a, λ and $\lambda^a_{;a}$ at any point $m \in M$. From this it follows [21] that if the pairs $(a, \lambda), (b, \mu)$ are global solution pairs of (7) and if there exists a non-empty open subset $U \subset M$ such that $b = a + \alpha g$ $(\alpha \in \mathbf{R})$ on U then $b = a + \alpha g$ and $\lambda = \mu$, on M. In particular, if $a = b$ on U, then $a = b$ and $\lambda = \mu$ on M and so $(a, \lambda) = (b, \mu)$. Thus, if M admits a non-empty, open subset U such that the only solution of (7) on U is $\lambda = 0$ and $a = \alpha g$ $(0 \neq \alpha \in \mathbf{R})$, the only solution of (7) on M is $\lambda = 0$ and $a = \alpha g$ (and so $\nabla = \nabla'$ on M).

This last result is useful in getting global solutions for (7) from local ones. To attack the local problem, the following construction is helpful. Let Λ_m denote the set of all tensor type $(2,0)$ 2-forms (bivectors) at $m \in M$ and consider the linear map $f : \Lambda_m \to \Lambda_m$ constructed from the curvature tensor $Riem$ of (M,g) where

$$f : F^{ab} \to R^{ab}{}_{cd}F^{cd} \quad (F^{ab} = -F^{ba}). \tag{10}$$

(The similarity with Petrov's 6×6 matrix and its associated linear map is clear.) Let $\ker f$ and $\mathrm{rg} f$ denote the kernel and range space of f. The rank of f (the dimension of $\mathrm{rg} f$) at m is called the *curvature rank* (of (M,g)) at m [6] and this curvature rank together with the nature of $\mathrm{rg} f$ contain much useful information. Clearly, if $F \in \ker f$ and $G \in \mathrm{rg} f$, $F^{ab}G_{ab} = 0$ and so F and G are orthogonal with respect to the bivector metric and $\dim(\ker f) + \dim(\mathrm{rg} f) = 6$. However, $(\ker f) \cap (\mathrm{rg} f)$ need not consist only of the zero bivector (as it would if the bivector metric were positive definite). In fact, for *vacuum* metrics of Petrov type **I**, **D**, **II**, **N**, **III** and **O**, one has, for the pair $(\dim(\mathrm{rg} f),\ \dim(\ker f))$, the respective values $(6,0)$ (or $(4,2)$), $(6,0)$, $(6,0)$, $(2,4)$, $(4,2)$ and $(0,6)$ and for the types **N** and **III**, $(\mathrm{rg} f)\cap(\ker f)$ is 2-dimensional (and 0-dimensional for all other types). It is remarked that the evenness of $\dim(\mathrm{rg} f)$ follows, since g has Lorentz signature, from (1) because in this case $Riem = C$. For the case of vacuum metrics, the Petrov canonical types trivially give a complete classification of the map f but for general space-times, a more detailed classification is required. To achieve this, it is convenient to introduce five *curvature classes* in the following way. In these definitions if $F \in \Lambda_m$ is of matrix rank 2 it is called *simple*. In this case it may be written as $F^{ab} = p^a q^b - q^a p^b = p \wedge q$ for tangent vectors p, q at m and the 2-space spanned by p and q is called the *blade* of F. Otherwise F is *non-simple* and gives rise to a canonical pair of blades at m which are orthogonal [6, 17].

Class A

This covers all possibilities not covered by classes **B**, **C**, **D** and **O** below. For this class, the curvature rank at m is 2, 3, 4, 5 or 6.

Class B

This occurs when $\dim(\mathrm{rg} f) = 2$ and when $\mathrm{rg} f$ is spanned by a timelike-spacelike pair of simple bivectors with orthogonal blades (chosen so that one is the dual of the other). In this case, one can choose a null tetrad $l, n, x, y \in T_mM$ such that these bivectors are $F = l \wedge n$ and $\overset{*}{F} = x \wedge y$, so that F is timelike and $\overset{*}{F}$ is spacelike and then (using the algebraic identity $R_{a[bcd]} = 0$ to remove cross terms) one has, at m,

$$R_{abcd} = \alpha F_{ab}F_{cd} + \beta \overset{*}{F}_{ab}\overset{*}{F}_{cd} \tag{11}$$

for $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0 \neq \beta$.

Class C

In this case $\dim(\mathrm{rg} f) = 2$ or 3 and $\mathrm{rg} f$ may be spanned by independent simple bivectors F and G (or F, G and H) with the property that there exists $0 \neq r \in T_mM$ such that r lies in the blades of $\overset{*}{F}$ and $\overset{*}{G}$ (or $\overset{*}{F}$, $\overset{*}{G}$ and $\overset{*}{H}$). Thus, $F_{ab}r^b = G_{ab}r^b (= H_{ab}r^b) = 0$ and r is then unique up to a multiplicative non-zero real number.

Class D

In this case $\dim(\mathrm{rg} f) = 1$. If $\mathrm{rg} f$ is spanned by the bivector F then, at m,

$$R_{abcd} = \alpha F_{ab}F_{cd} \tag{12}$$

for $0 \neq \alpha \in \mathbb{R}$ and $R_{a[bcd]} = 0$ implies that $F_{a[b}F_{cd]} = 0$ from which it may be checked that F is necessarily simple.

Class O
In this case $Riem$ vanishes at m.

This curvature classification is, like the Petrov classification, pointwise. One may topologically decompose M into its curvature classes in a similar way to that done for Petrov's types [6].

There is a particularly useful result regarding the curvature rank of f. If F is a simple bivector in $\ker f$ (for (M,g)), then the blade of F is a 2-dimensional eigenspace of the symmetric type $(0,2)$ tensor $\nabla\lambda$, whilst if F is a non-simple bivector in $\ker f$ the canonical blade pair of F give two g-orthogonal 2-dimensional eigenspaces of $\nabla\lambda$. It can then be shown [7, 13] that if $\ker f$ is such that the tangent space T_mM to M at $m\in M$ is an eigenspace of $\nabla\lambda$ at each point of M then, on M,

$$(i)\ \ \lambda_{a;b}=cg_{ab};\quad (ii)\ \ \lambda_d R^d{}_{abc}=0;\quad (iii)\ \ a_{ae}R^e{}_{bcd}+a_{be}R^e{}_{acd}=0, \tag{13}$$

where c is constant in (i). Then λ is a homothetic (co)vector field on M and if λ vanishes over some non-empty open subset of M it vanishes on M. Further, the equations in parts (ii) and (iii) may be solved algebraically for λ and a at any $m\in M$ if the curvature class is known at m [6]. Part (iii) is also usefully related to the curvature class at the appropriate point.

The final technique required in the study of the projective problem is the holonomy algebra of (M,g). The details here are a little lengthy and are given in [6, 13, 21, 23]. Briefly, the holonomy algebra of (M,g) can be shown to be a subalgebra of the Lie algebra of the Lorentz group and can be classified conveniently into fifteen types [24] which are labeled $R_1,\dots,R_{15}$. Of course, the holonomy group of (M,g) depends not only on its holonomy algebra but also on the topological properties of M. But the holonomy algebra is all that will be required here. The type R_1 is the trivial flat case, R_5 is impossible for a space-time, R_{15} is the general case and the holonomy types R_2-R_4 and R_6-R_{14} reflect the number of independent (locally) covariantly constant and recurrent vector fields admitted by ∇ on M. The holonomy classification of (M,g) is, unlike the Petrov and curvature classifications, a *global* statement about (M,g). Taken together with the *infinitesimal holonomy algebra* [6, 23], it combines with the various curvature classes over M described above to provide a powerful tool in solving the projective problem.

4. Main results

First consider the case when (M,g) is a space-time which is an Einstein space (and which includes the special case when (M,g) is vacuum). For this situation one has, in the notation established above, the following result [7–9, 11–13].

Theorem 1. *Let (M,g) be a space-time which is an Einstein space and let g' be another smooth metric on M of arbitrary signature, which is projectively related to g. Then either $\nabla=\nabla'$ or (M,g) and (M,g') each have constant curvature. If (M,g) is vacuum and not flat, then (M,g') is also vacuum (and not flat) and, further, $g'=cg$ for constant c except possibly when (M,g) is a pp-wave space-time (when the simple relation between g and g' can easily be found).*

Proof. A very brief sketch of several proofs will be given (Petrov's approach has already been mentioned). The first approach, given in [7], was actually given only for vacuum space-times but is easily extended to Einstein spaces. This approach relies on (3) and (8) and makes no use of the Sinjukov transformation. First, one disjointly decomposes M into its regions of constant Petrov type, as described in Section 1. A relationship between the curvature map f and the Petrov types of $Riem$ is then established and canonical forms for each Petrov type are written down for each region in the above decomposition with use being made of the equality of the tensors W in (8) for g and g'. In an improved proof given in [8], use is made of the Sinjukov equations (5)–(7). Here, one is able to show that either $\nabla' = \nabla$ or the Weyl tensor and any solution pair (a, λ) of (7) satisfy

$$a_{ae}C^{e}{}_{bcd} + a_{be}C^{e}{}_{acd} = 0, \quad C_{abcd}\,\lambda^{d} = 0 \tag{14}$$

on M. One then decomposes M as $M = A \cup B$ where $A = \{m \in M : C(m) \neq 0\}$ and $B = \{m \in M : C(m) = 0\}$ so that A is open and B is closed in M. (In [8] this argument was given rather clumsily and will be clarified here). It was then shown that λ^a was a *projective* vector field on M and so if it vanishes on some non-empty open subset of M it vanishes on M (and then from (6) and (2) $\psi = 0$ and so $\nabla = \nabla'$, on M). (In fact, this result is essentially a consequence of the first order system described in Section 3.) If $m \in A$ and $\lambda(m) \neq 0$, it follows from the second equation in (14) together with the Bel criteria (Section 1) that, at m and in some open neighbourhood W of m, the Petrov type of (M, g) is **N** and λ spans a (repeated) principal *null* direction at m. Then one derives the contradiction that λ vanishes on W and so λ vanishes on A. Since A is open it follows that, if $A \neq \emptyset$, λ again vanishes on M. Finally, if $A = \emptyset$, $M = B$ and so (M, g) is of constant curvature (and so also is (M, g) [25]). It can also be shown that if g and g' are not of constant curvature they have the same signatures (something which was assumed by Petrov) but that (M, g') may not be an Einstein space. Other proofs have also been given, but the *pp*-wave possibility described in the statement of the theorem only seems to have been pointed out clearly in [7]. (In theorem 1, if g is of signature $(+,+,+,+)$ or $(+,+,-,-)$ again one gets the result that either each of (M, g) and (M, g') is of constant curvature or $\nabla = \nabla'$, that g' need not represent an Einstein space nor have the same signature as g but that (excluding the constant curvature case) Ricci flatness is preserved [8, 9]). □

For a general space-time, it is convenient to proceed by considering the holonomy type of (M, g). All types except the most general one can be solved and the following theorem summarizes part of the situation.

Theorem 2. *Let (M, g) and (M, g') be projectively related space-times.*

(i) If g and g' are (locally) conformally related on M, then they are globally conformally related on M and further $\nabla = \nabla'$ and $g' = cg$ on M for c constant.

(ii) If (M, g) has holonomy type R_2, R_3, R_4, R_6, R_7, R_8 or R_{12}, then $\nabla = \nabla'$ and the relation between g and g' can be calculated easily using holonomy theory.

(iii) If (M, g) has holonomy type R_{10}, R_{11} or R_{13} and *with curvature rank > 1 at some $m \in M$, then $\nabla = \nabla'$ and the relation between g and g' can be calculated easily using holonomy theory.*

Proof. Again a brief sketch only will be given. The result in part (i) was partly noticed by Thomas [26]. The full result is proved by choosing $m \in M$ and a connected open neighbourhood U of M on which $g' = \phi g$ for $\phi : U \to \mathbb{R}$ and substituting into (3), contracting with g^{ab} and then substituting back and contracting with g^{ac} to get $\psi = 0$ and $\chi = \text{const}$ on U. The result follows from a topological argument using the connectedness of M. For parts (ii) and (iii), one first uses a result mentioned just before (13) regarding the kernel, $\ker f$, of the curvature map f and shows that for each of the holonomy types in parts (ii) and (iii) T_mM is an eigenspace of $\nabla\lambda$ and hence that λ is a homothetic vector field on M. Then (13) can be used in conjunction with the remarks following it to choose $m \in M$ and an open connected neighbourhood U of m and to write a canonical form for the tensor a in U, which is determined by part (iii)

of (13), in terms of a null tetrad chosen to “fit” the holonomy invariant distributions and/or vector fields. One then substitutes into (7) and performs certain contractions to see that λ vanishes on U and hence, since it is homothetic, on M.

This solves the problem for all holonomy types except types R_{10}, R_{11} or R_{13} (and with curvature rank ≤ 1 at each point of M), and for types R_9, R_{14} and R_{15}. The solution in these cases is more complicated and can be found in [13, 21] (and further holonomy details are available in [27]). For these cases, one does not necessarily achieve $\nabla = \nabla'$ but the relationship between g and g' can still be found. The general case R_{15} is not completely solved (although some progress can be made [21]). In particular, in the important case for relativistic cosmology when (M, g) is a “generic” FRWL cosmological model (necessarily of type R_{15}), it has been completely solved [10] and (M, g') must also be an FRWL metric.

□

Another problem which has been essentially solved is the following one [21]. If (M, g) and (M, g') are projectively related, how are their holonomy groups related? Clearly, from the above results, there is a close link between such holonomy groups (often equality), as Theorem 2 shows, but it does not follow that they are the same and examples of non-equality have been given. Further, the Petrov type and the curvature class at $m \in M$ and the holonomy type of (M, g) are, as may be expected, closely related. In fact, it can be shown [6] that if the holonomy type of (M, g) is R_2 or R_4, the Petrov type at any point is either **O** or **D** and similarly for holonomy type R_3 it is **O** or **N**, for R_7 it is **O** or **D**, for R_{13} it is **O**, **I** or **D** and for all other holonomy types except R_{10} and R_{15} it is algebraically special. In addition, for the Petrov types R_2, R_3 and R_4, it is Petrov type **O** at m if and only if $Riem(m) = 0$. Similarly, if the holonomy type is R_2, R_3 or R_4, the *curvature class* at any $m \in M$ is O or D, for holonomy types R_6, R_8, R_{10}, R_{11} and R_{13} it is O, C or D, for R_7 it is O, B or D, for R_9 and R_{12}, it is O, C, D or A and for R_{14} or R_{15}, it could be any curvature class (but, if R_{14}, it cannot be curvature class B everywhere). Another question, perhaps less interesting for physicists, is the problem when the original (M, g) satisfies $\dim M = 4$ and with g *positive definite*. Techniques similar to those above also lead to a solution for this problem in all but the most general holonomy case [28]. Similarly, the case when $\dim M = 4$ and g has *neutral* signature $(+,+,-,-)$ has been considered[1]. Further details for space-times can be found in [29].

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Резюме

Г. Холл. Развитие исследований А.З. Петрова по конформной и проективной структурам.

В статье обсуждаются достижения А.З. Петрова в области исследования конформной и проективной структур пространства-времени. Основное внимание уделено последней, однако показано и значение первой в развитии проективной теории пространства-времени. Приведены некоторые новые результаты в данной области исследования А.З. Петрова.

Ключевые слова: проективная структура, конформная структура, классификация А.З. Петрова, карта кривизны.

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Hall, Graham – PhD, Professor Emeritus of Mathematics, Institute of Mathematics, University of Aberdeen, Scotland, UK.

Холл, Грэм – доктор наук, почетный профессор математики Института математики Абердинского университета, Шотландия, Великобритания.

E-mail: *g.hall@abdn.ac.uk*