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On Mappings of Plane Domains by Solutions of Second-Order Elliptic Equations

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Abstract—In this paper, we study sufficient conditions for the one-to-one solvability of second-order partial differential equations in a plane Jordan domain. For a continuous one-to-one and orientation-keeping map of the boundary of a Jordan domain to the rectifiable boundary of some other Jordan domain, we prove the following property: If the Cauchy integral whose measure is generated by this map is bounded by some constant in the exterior domain, then the solution to the corresponding Dirichlet problem in the domain with this boundary function maps these domains one-to-one. In the proof of the main result we use integral representations of equation solutions, particularly, properties of Fredholm-type integral equations on the domain boundary.

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INTRODUCTION

Let $L = \overline{\partial}\partial_\beta$; here $\overline{\partial}$ is the Cauchy–Riemann operator, while $\partial_\beta = \frac{\partial}{\partial x} + \beta i \frac{\partial}{\partial y}$, $-1 < \beta < 0$. In this paper, we study properties of solutions to the equation

$$Lu = 0. \tag{1}$$

Consider the following problem.

Problem. *Let a function Φ be continuous on the boundary Γ of a Jordan domain D and let it one-to-one map Γ on some closed contour Γ_1, preserving the orientation. We are going to establish conditions for the function Φ and the contour Γ which guarantee that the solution to the Dirichlet problem for Eq.* (1) *in D with the boundary function Φ one-to-one maps D on the domain D_1 bounded by the contour Γ_1.*

Note that the Dirichlet problem for Eq. (1) is solvable with any function Φ which is continuous on Γ ([1], theorem 7.4).

For harmonic functions the following Rado–Kneser–Choquet theorem ([2], P. 33) is valid.

Let a function Φ be continuous on the boundary ∂B of the unit disk B and let it one-to-one map ∂B on some convex closed contour Γ_1, preserving the orientation. Then the harmonic extension of the function Φ in B one-to-one maps B on the domain D_1 bounded by the contour Γ_1.

The study of harmonic maps of D on D_1 is reduced to studying harmonic maps of the disk B on D_1 with the help of a conformal map of B on D.

According to [3] (theorem 1.4), with $\beta \neq -1$ for any closed contour Γ there exists a continuous on ∂B function Φ which one-to-one maps ∂B on some closed contour Γ, preserving the orientation, for which the solution to the Dirichlet problem for Eq. (1) is not biunique in B.

In [4] (theorem 1) we establish sufficient conditions for the function Φ, which is continuous on ∂B, guaranteeing that the solution to the Dirichlet problem for Eq. (1) in B with the boundary function Φ is biunique.

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