УЧЕНЫЕ ЗАПИСКИ КАЗАНСКОГО УНИВЕРСИТЕТА.
СЕРИЯ ФИЗИКО-МАТЕМАТИЧЕСКИЕ НАУКИ

2018, Т. 160, кн. 2 ISSN 2541-7746 (Print)
С. 384–391 ISSN 2500-2198 (Online)

UDK 519.2

SHIFT-INVARIANT MEASURES ON INFINITE-DIMENSIONAL SPACES: INTEGRABLE FUNCTIONS AND RANDOM WALKS

V.Zh. Sakbaev, D.V. Zavadsky

Moscow Institute of Physics and Technology, Dolgoprudny, 141701 Russia

Abstract

Averaging of random shift operators on a space of the square integrable by shift-invariant measure complex-valued functions on linear topological spaces has been studied. The case of the l_∞ space has been considered as an example.

A shift-invariant measure on the l_∞ space, which was constructed by Caratheodory's scheme, is σ-additive, but not σ-finite. Furthermore, various approximations of measurable sets have been investigated. One-parameter groups of shifts along constant vector fields in the l_∞ space and semigroups of shifts to a random vector, the distribution of which is given by a collection of the Gaussian measures, have been discussed. A criterion of strong continuity for a semigroup of shifts along a constant vector field has been established.

Conditions for collection of the Gaussian measures, which guarantee the semigroup property and strong continuity of averaged one-parameter collection of linear operators, have been defined.

Keywords: strongly continuous semigroups, averaging of operator semigroups, shift-invariant measures, square integrable functions

Introduction

In this paper, we investigate semigroups of shift operators on a Hilbert space $L_2(l_\infty, \mathfrak{B}(l_\infty), \lambda)$ of complex square integrable functions by a shift-invariant and σ-additive measure λ on a space l_∞. Data on various constructions of shift-invariant measures on linear topological spaces and possible connections between the measures are presented in [1–4].

We consider semigroups indexed by parameter $t \in \mathbb{R}_+ \equiv [0, +\infty)$ of bounded operators A_h^t, which are defined in the following way: $A_h^t f(x) = f(x + th)$, where $h \in l_\infty$, $f \in L_2(l_\infty, \mathfrak{B}(l_\infty), \lambda)$. We establish a criterion of strong continuity for operator semigroups A_h^t. We prove that an operator semigroup A_h^t is strongly continuous if and only if $h \in l_1$.

Then, we construct a class of Gaussian measures on a space l_∞, which are concentrated on a space l_1 – a domain of strong continuity for operator semigroups A_h^t.

Our next step is to average operator semigroups A_h^t by a random vector $h \in l_\infty$, whose distribution is given by the previously constructed Gaussian measures. As a result, we obtain an operator set H_t, $t \in \mathbb{R}_+$, which is a strongly continuous contraction operator semigroup. The obtained results show the applicability of the approach to study random shift operators in the Hilbert spaces developed in [3, 4], as well as to the study of random shift operators in linear topological spaces.

1. Supporting results

In this section, we introduce supporting constructions and results in order to define and operate with strongly continuous operator semigroups on the spaces of integrable functions.

Let us denote Borel σ-algebras corresponding to the topology of pointwise convergence on spaces R^∞ and l_∞ as $\mathfrak{B}(R^\infty)$ and $\mathfrak{B}(l_\infty)$, respectively, and a Borel σ-algebra corresponding to the standard topology on an arbitrary subspace $K \subset R^k$ – as $\mathfrak{B}(K)$, where $k \in N$.

Lemma 1. *Let X_1, and X_2 be the Frechet spaces. Then $\mathfrak{B}(X_1 \times X_2) = \mathfrak{B}(X_1) \times \mathfrak{B}(X_2)$, where $\mathfrak{B}(X_1 \times X_2)$, $\mathfrak{B}(X_1)$, and $\mathfrak{B}(X_2)$ denote Borel σ-algebras on X_1, X_2, and X_3, respectively.*

Proof. $X_1 \times X_2$ is a Frechet space. Let us define a subset $L \subset (X_1 \times X_2)^*$ in the following way: $h \in L$ if and only if for all $x_1, x_2 \in X_1 \times X_2 : h(x_1; x_2) = l_1(x_1)$, where $l_1 \in X_1^*$; or for all $x_1, x_2 \in X_1 \times X_2 : h(x_1; x_2) = l_2(x_2)$, where $l_2 \in X_2^*$. A set L separates points of a space $X_1 \times X_2$. That is why, according to the paragraph A.3.7. from [5], we have that $\mathfrak{B}(X_1 \times X_2) \subset \mathfrak{B}(X_1) \times \mathfrak{B}(X_2)$. Furthermore, $\mathfrak{B}(X_1) \times \mathfrak{B}(X_2) \subset \mathfrak{B}(X_1 \times X_2)$. Hence, $\mathfrak{B}(X_1 \times X_2) = \mathfrak{B}(X_1) \times \mathfrak{B}(X_2)$. $\square$

Let us denote a shift-invariant and σ-additive measure on S as λ, where S is a σ-algebra on a space R^∞ (the precise definition of S and λ may be found in [2]). Then, the following statements hold.

Lemma 2. *For all $B \in \mathfrak{B}(R), n \in N : R^n \times B \times R^\infty \in S$.*

Lemma 3. *For all $A \in S$ and for all $\epsilon > 0$ there exist $n, k \in N : \lambda(A \triangle \bigcup_{i=1}^{n} h_i + Q_i \times [0;1] \times [0;1] \times \cdots) < \epsilon$, where $Q_i \in \mathfrak{B}([0;1]^k)$, and $h_i + Q_i \bigcap h_j + Q_j = \varnothing$, when $i \neq j$.*

Lemma 4. $\mathfrak{B}(l_\infty) \subset S|l_\infty$.

Proof. It is enough to prove that $\mathfrak{B}(R^\infty) \subset S$. A space R^∞ is a separable Frechet space. Let us define linear continuous mappings $P_i : R^\infty \to R$, where $i \in N$ in the following way: $P_i(x_1, x_2, \ldots) = x_i$. Mappings P_i separate points of a space R^∞. That is why, according to the paragraph A.3.7. from [5], it is enough to prove that for all $B \in \mathfrak{B}(R), n \in N : R^n \times B \times R^\infty \in S$. So, according to lemma 2, we have that $\mathfrak{B}(l_\infty) \subset S|l_\infty$. $\square$

Lemma 5. *A set $\{I_{h+Q\times[0;1]\times[0;1]\times\cdots} | h \in R^\infty, Q \in \bigcup_{i=1}^{\infty} \mathfrak{B}([0;1]^i)\}$ is a total set in a space $L_2(l_\infty, \mathfrak{B}(l_\infty), \lambda|l_\infty)$.*

Lemma 6. *$\mathfrak{B}_s(l_2) = \mathfrak{B}(R^\infty) \bigcap l_2$, where $\mathfrak{B}_s(l_2)$ is a Borel σ-algebra corresponding to the standard topology on a space l_2.*

Hereafter, we consider λ as a measure on a space $(l_\infty, \mathfrak{B}(l_\infty))$.

2. Operator semigroups of shifts

Let us define operators $A_h^t : L_2(l_\infty, \mathfrak{B}(l_\infty), \lambda) \to L_2(l_\infty, \mathfrak{B}(l_\infty), \lambda)$, where $t \geq 0$, $h \in l_\infty$, in the following way:

$$A_h^t u(x) = u(x + th).$$

For each $h \in l_\infty$, operators A_h^t form a semigroup, besides for each $t \geq 0,\ h \in l_\infty$: $\|A_h^t\| = 1$. Let us explore conditions, under which operator semigroups A_h^t should satisfy in order to be strongly continuous.

Lemma 7. *Let for all* $n \in N$: $t_n \geq 0,\ h_n \geq 0$; $t_n \to 0$; $(h_1, h_2, \ldots) \in l_1$; $Q \in \mathfrak{B}([0;1]^k)$; $b \in h_\infty$. *Then*

$$\lim_{n\to\infty} \sup_{|x_i|\leq h_i, i\in N} \|A^{t_n}_{(x_1,x_2,\ldots)} I_{b+Q\times[0;1]\times[0;1]\times\cdots} - I_{b+Q\times[0;1]\times[0;1]\times\cdots}\|_{L_2} = 0.$$

Proof. It is enough to prove the statement in the case $b = 0$. We have that for all $n \in N$:

$$\sup_{|x_i|\leq h_i, i\in N} \|A^{t_n}_{(x_1,x_2,\ldots)} I_{b+Q\times[0;1]\times[0;1]\times\cdots} - I_{b+Q\times[0;1]\times[0;1]\times\ldots}\|_{L_2} =$$

$$= \sup_{|x_i|\leq h_i, i\in N} \left(\int\limits_{l_\infty} |I_{Q\times[0;1]\times[0;1]\times\cdots}(x+t_n(x_1,x_2,\ldots)) - I_{Q\times[0;1]\times[0;1]\times\cdots}(x)|^2 d\lambda(x) \right)^{1/2} =$$

$$= \sup_{|x_i|\leq h_i, i\in N} (\lambda(t_n(-x_1,-x_2,\ldots) + Q\times[0;1]\times[0;1]\times\cdots \triangle\, Q\times[0;1]\times[0;1]\times\ldots)^{1/2} \leq$$

$$\leq 2 \sup_{|x_i|\leq h_i, i\in N} (\lambda(t_n(x_1,x_2,\ldots) + Q\times[0;1]\times[0;1]\times\cdots \setminus Q\times[0;1]\times[0;1]\times\cdots)^{1/2} \leq$$

$$\leq 2 \sup_{|x_i|\leq h_i, i\in N} \lambda^k(t_n(x_1,x_2,\ldots x_k) + Q\setminus Q) +$$

$$+C \sup_{|x_i|\leq h_i, i\in N} (t_n|x_{k+1}| + (1-t_n|x_{k+1}|)t_n|x_{k+2}| + (1-t_n|x_{k+1}|)(1-t_n|x_{k+2}|)t_n|x_{k+3}| + \cdots),$$

where λ^k is the Lebesgue measure on a space R^k and $C = 2\lambda^k(Q)$. We know that $(h_1, h_2, \ldots) \in l_1$. That is why

$$\sup_{|x_i|\leq h_i, i\in N} (t_n|x_{k+1}| + (1-t_n|x_{k+1}|)t_n|x_{k+2}| + (1-t_n|x_{k+1}|)(1-t_n|x_{k+2}|)t_n|x_{k+3}| + \cdots) \to 0.$$

As a result, we have

$$\lim_{n\to\infty} \sup_{|x_i|\leq h_i, i\in N} \|A^{t_n}_{(x_1,x_2,\ldots)} I_{b+Q\times[0;1]\times[0;1]\times\cdots} - I_{b+Q\times[0;1]\times[0;1]\times\cdots}\|_{L_2} = 0.$$

$\square$

Theorem 1. *An operator semigroup* $A_h^t,\ t \in \mathbb{R}_+$ *is strongly continuous if and only if* $h \in l_1$.

Proof. If $h \in l_1$, then, according to lemma 7, a semigroup A_h^t is a strongly continuous operator semigroup, because a set $\{I_{h+Q\times[0;1]\times[0;1]\times\cdots} | h \in l_\infty,\ Q \in \bigcup\limits_{i=1}^{\infty} \mathfrak{B}([0;1]^i)\}$ is total in a space $L_2(l_\infty, \mathfrak{B}(l_\infty), \lambda)$.

Suppose that $h = (h_1, h_2, \ldots) \notin l_1$. Let us calculate a value of the following measure:

$$\lambda(-th + [0;1]\times[0;1]\times\cdots \bigcap [0;1]\times[0;1]\times\cdots),$$

where $t \geq 0$. We have that

$$\lambda(-th + [0;1]\times[0;1]\times\cdots \bigcap [0;1]\times[0;1]\times\cdots) = \lim_{n\to\infty} \prod_{k=1}^{n} (1 - t|h_k|) =$$

$$= \lim_{n\to\infty} \exp\left(\sum_{k=1}^{n} ln(1 - t|h_k|) \right) \leq \lim_{n\to\infty} \exp\left(-\sum_{k=1}^{n} t|h_k| \right) = 0,$$

when t is close enough to 0. Hence,

$$\lim_{t\to 0}\|A^t_{(h_1,h_2,\ldots)}I_{[0;1]\times[0;1]\times\cdots} - I_{[0;1]\times[0;1]\times\cdots}\|_{L_2} = 2^{1/2}.$$

That is why an operator semigroup A^t_h is not strongly continuous when $h \notin l_1$. □

3. Gaussian measures concentrated on a space l_1

Our goal in this paragraph is to construct a Gaussian measure on a space l_2, which is concentrated on a space l_1.

Let us take a sequence of numbers a_n, which satisfies the following properties: for all $n \in N : a_n \in (0,1)$,

$$0 < \prod_{n=1}^{\infty} a_n < 1.$$

Let us take $h = (h_1, h_2, \ldots) \in l_1$, where $\forall n \in N : h_1 > 0$. The next step is to choose a sequence σ_n, where for all $n \in N : \sigma_n > 0$ and for all $n \in N$:

$$\frac{1}{\sqrt{2\pi}\sigma_n}\int_{-h_n}^{h_n} \exp\left(-\frac{x^2}{2\sigma_n^2}\right)dx = a_n.$$

We can rewrite the equation in the following way:

$$\frac{1}{\sqrt{2\pi}\sigma_n}\int_{-h_n}^{h_n} \exp\left(-\frac{x^2}{2\sigma_n^2}\right)dx = \frac{1}{\sqrt{2\pi}}\int_{-h_n/\sigma_n}^{h_n/\sigma_n} \exp\left(-\frac{x^2}{2}\right)dx = a_n.$$

As a result, we obtain that $\sigma_n = \dfrac{h_n}{\sqrt{2}erf^{-1}(a_n)}$. So, let us denote a Gaussian measure on R, corresponding to σ_n, as γ_n and construct a measure μ in the following way: $\mu = \bigotimes_{n=1}^{\infty} \gamma_n$. According to the paragraph 2.2 from [5], a measure μ is well defined on $\mathfrak{B}(R^\infty)$, and $\mu|l_2$ is a centered Gaussian measure on a space l_2. A measure μ depends on a sequence a_n and a vector h. Let us denote a class of measures which can be constructed by the suggested scheme as M.

Lemma 8. *For all $\mu \in M : \mu(l_1) = 1$.*

Proof. According to the definition of a sequence a_n, there exists $n \in N$ for all $\epsilon > 0$:

$$\prod_{k=n+1}^{\infty} a_n > 1 - \epsilon.$$

For this reason, we can conclude that $\mu(R^n \times [-h_{n+1}; h_{n+1}] \times [-h_{n+2}; h_{n+2}] \times \cdots) > 1 - \epsilon$. Hence, we obtain that

$$\mu(\bigcup_{n=1}^{\infty} R^n \times [-h_{n+1}; h_{n+1}] \times [-h_{n+2}; h_{n+2}] \times \cdots) = 1.$$

As a result, we have that $\mu(l_1) = 1$. □

Let us define measures μ_t, where $t > 0$, by the formula:

$$\forall A \in \mathfrak{B}_s(l_2) : \ \mu_t(A) = \int_{l_\infty} I_A(\sqrt{t}h)\, d\mu(h),$$

where $\mu \in M$. Measures μ_t are Gaussian on a space l_2, and we have that for all $t,\, s > 0 : \mu_{t+s} = \mu_t * \mu_s$.

4. Averaging of operator semigroups

Let us define an operator set $H_t : L_2(l_\infty, \mathfrak{B}(l_\infty), \lambda) \to L_2(l_\infty, \mathfrak{B}(l_\infty), \lambda)$, where $t \geq 0$, in the following way:

$$H_t u(x) = \int\limits_{l_\infty} u(x + \sqrt{t}h)\, d\mu(h).$$

The vector $H_t u$ should be understood as the Pettis integral:

$$\forall\, v \in L_2(l_\infty, \mathfrak{B}(l_\infty), \lambda) :\ (H_t u; v)_{L_2} = \int\limits_{l_\infty} \left(\int\limits_{l_\infty} u(x + \sqrt{t}h)\bar{v}(x)\, d\lambda(x) \right) d\mu(h),$$

where $\mu \in M$. Let us note that

$$\int\limits_{l_\infty} \left(\int\limits_{l_\infty} u(x + \sqrt{t}h)\bar{v}(x)\, d\lambda(x) \right) d\mu(h) = \int\limits_{l_\infty} \left(\int\limits_{\mathrm{supp}(v)} u(x + \sqrt{t}h)\bar{v}(x) d\lambda(x) \right) d\mu(h),$$

$$\int\limits_{\mathrm{supp}(v)} |u(x + \sqrt{t}h)\bar{v}(x)|\, d\lambda(x) \leq \|u\|_{L_2} \|v\|_{L_2}.$$

That is why, according to Fubini's theorem, a function

$$h \to \int\limits_{l_\infty} u(x + \sqrt{t}h)\bar{v}(x)\, d\lambda(x)$$

is defined everywhere and measurable. Hence, the definition on an operator set H_t is correct. Moreover, for all $t \geq 0 : \|H_t\| \leq 1$.

Theorem 2. *For an arbitrary measure $\mu \in M$, a set $H_t,\ t \in \mathbb{R}_+$ is an operator semigroup.*

Proof. Let us fix elements $u, v \in L_2(l_\infty, \mathfrak{B}(l_\infty), \lambda)$ and introduce a function $f : l_\infty \to R$ as:

$$f(h) = \int\limits_{l_\infty} u(x + h)\bar{v}(x)\, d\lambda(x).$$

Let us fix $t, s > 0$. Then, we have that

$$(H_{t+s}u; v)_{L_2} = \int\limits_{l_\infty} \left(\int\limits_{l_\infty} u(x + \sqrt{t+s}h)\bar{v}(x) d\lambda(x) \right) d\mu(h) =$$

$$= \int\limits_{l_\infty} f(\sqrt{t+s}h)\, d\mu(h) = \int\limits_{l_2} f(\sqrt{t+s}h)\, d\mu(h) = \int\limits_{l_2} f(h)\, d\mu_{t+s}(h).$$

According to paragraph 3, we obtain that

$$(H_{t+s}u; v)_{L_2} = \int\limits_{l_2 \times l_2} f(x + y)\, d(\mu_t \otimes \mu_s)(x, y) =$$

$$= \int\limits_{l_2} \left(\int\limits_{l_2} f(x + y) d\mu_t(x) \right) d\mu_s(y) = \int\limits_{l_2} \left(\int\limits_{l_2} f(x + y) d\mu_s(y) \right) d\mu_t(x).$$

On the other hand,

$$(H_tH_su;v)_{L_2}=\int\limits_{l_\infty}\left(\int\limits_{l_\infty}(H_su)(x+\sqrt{t}h_1)\bar{v}(x)\,d\lambda(x)\right)d\mu(h_1)=$$

$$=\int\limits_{l_\infty}\left[\int\limits_{l_\infty}\left(\int\limits_{l_\infty}u(x+\sqrt{t}h_1+\sqrt{s}h_2)\bar{v}(x)\,d\lambda(x)\right)d\mu(h_2)\right]d\mu(h_1)=$$

$$=\int\limits_{l_2}\left(\int\limits_{l_2}f(x+y)\,d\mu_s(y)\right)d\mu_t(x).$$

That is why $H_{t+s}=H_tH_s$ and, as a result, H_t is an operator semigroup. $\square$

Theorem 3. *For an arbitrary measure $\mu\in M$, a set H_t, $t\in\mathbb{R}_+$ is an operator continuous operator semigroup.*

Proof. Let us fix arbitrary functions $u,v\in L_2(l_\infty,\mathfrak{B}(l_\infty),\lambda)$. Let us consider a sequence $t_n\geq 0$ and let $\lim\limits_{n\to\infty}t_n=0$. Then

$$|(H_{t_n}u-u;v)_{L_2}|\leq$$

$$\leq\int\limits_{l_\infty}\left(\int\limits_{l_\infty}|(u(x+\sqrt{t_n}h)-u(x))\bar{v}(x)|\,d\lambda(x)\right)d\mu(h)\leq$$

$$\leq\|v\|_{L_2}\int\limits_{l_\infty}\sqrt{\int\limits_{l_\infty}|u(x+\sqrt{t_n}h)-u(x)|^2\,d\lambda(x)}\,d\mu(h)=$$

$$=\|v\|_{L_2}\int\limits_{l_1}\sqrt{\int\limits_{l_\infty}|u(x+\sqrt{t_n}h)-u(x)|^2\,d\lambda(x)}\,d\mu(h).$$

According to Lebesgue's dominated convergence theorem, we obtain that

$$\int\limits_{l_1}\sqrt{\int\limits_{l_\infty}|u(x+\sqrt{t_n}h)-u(x)|^2\,d\lambda(x)}\,d\mu(h)\to 0.$$

So, we have that $\sup\limits_{\|v\|_{L_2}=1}|(H_{t_n}u-u)_{L_2}|\to 0$. That is why $H_{t_n}u-u\to 0$. Therefore, an operator semigroup H_t is strongly continuous. $\square$

Conclusions

In this paper, we proved the criterion of strong continuity for shift operator semigroups, along a constant vector field in a space l_∞, on a space of complex square integrable by shift-invariant measure on a space l_∞ functions. In addition, it was shown that averaging of shift-invariant operator semigroups by Gaussian measures concentrated on a space l_1 is a strongly continuous contraction operator semigroup.

Acknowledgements. The work was performed according to the Russian Government Program of Competitive Growth of Moscow Institute of Physics and Technology (project 5-100).

References

1. Baker R. Lebesgue measure on R^∞. *Proc. Am. Math. Soc.*, 1991, vol. 113, no. 4, pp. 1023–1029. doi: 10.2307/2048779.
2. Zavadsky D.V. Shift-invariant measures on sequence spaces. *Tr. Fiz.-Tekh. Inst.*, 2017, vol. 9, no. 4, pp. 142–148. (In Russian)
3. Sakbaev V.Zh. Averaging of random walks and shift-invariant measures on a Hilbert space. *Theor. Math. Phys.*, 2017, vol. 191, no. 3, pp. 886–909. doi: 10.1134/S0040577917060083.
4. Sakbaev V.Z. Random walks and measures on Hilbert space that are invariant with respect to shifts and rotations. *Itogi Nauki Tekh., Ser.: Sovrem. Mat. Prilozh. Temat. Obz.*, 2017, vol. 140, pp. 88–118. (In Russian)
5. Bogachev V.I. *Gaussovskie mery* [Gaussian Measures]. Moscow, Fizmatlit, 1997. 352 p. (In Russian).

Recieved
October 17, 2017

Sakbaev Vsevolod Zhanovich, Doctor of Physics and Mathematics, Professor of the Higher Mathematics Department

Moscow Institute of Physics and Technology
Institutskiy per., 9, Dolgoprudny, Moscow Region, 141701 Russia
E-mail: *fumi2003@mail.ru*

Zavadsky Dmitrii Viktorovich, Student of the Department of Control and Applied Mathematics

Moscow Institute of Physics and Technology
Institutskiy per., 9, Dolgoprudny, Moscow Region, 141701 Russia
E-mail: *Dmitriyx2013@yandex.ru*

УДК 517.98+519.2

Трансляционно-инвариантные меры на бесконечномерных пространствах, интегрируемые функции и случайные блуждания

В.Ж. Сакбаев, Д.В. Завадский

Московский физико-технический институт, г. Долгопрудный, 141701, Россия

Аннотация

В работе изучается усреднение случайных операторов сдвига аргумента в пространстве квадратично интегрируемых по трансляционно-инвариантной мере комплекснозначных функций на линейных топологических пространствах. В качестве примера рассмотрен случай пространства l_∞. Трансляционно-инвариантная мера на пространстве l_∞, построенная при помощи схемы Каратеодори, обладает свойством счетной аддитивности, но не обладает свойством σ-конечности. Также рассматриваются различные приближения измеримых множеств. Рассматриваются однопараметрические группы сдвигов вдоль постоянного векторного поля в пространстве l_∞ и полугруппы сдвигов на случайный вектор, распределение которого задается семейством гауссовских мер. Получен критерий сильной непрерывности группы сдвигов вдоль постоянного векторного поля. Установлены условия на семейство гауссовских мер, достаточные для сохранения полугруппового свойства усредненного однопараметрического семейства линейных операторов и его сильной непрерывности.

Ключевые слова: сильно непрерывные полугруппы, усреднение операторных полугрупп, трансляционно-инвариантные меры, квадратично интегрируемые функции

Поступила в редакцию
17.10.17

Сакбаев Всеволод Жанович, доктор физико-математических наук, профессор кафедры высшей математики

Московский физико-технический институт
Институтский пер., д. 9, г. Долгопрудный, Московская обл., 141701, Россия
E-mail: *fumi2003@mail.ru*

Завадский Дмитрий Викторович, студент факультета управления и прикладной математики

Московский физико-технический институт
Институтский пер., д. 9, г. Долгопрудный, Московская обл., 141701, Россия
E-mail: *Dmitriyx2013@yandex.ru*

For citation: Sakbaev V.Zh., Zavadsky D.V. Shift-invariant measures on infinite-dimensional spaces: Integrable functions and random walks. *Uchenye Zapiski Kazanskogo Universiteta. Seriya Fiziko-Matematicheskie Nauki*, 2018, vol. 160, no. 2, pp. 384–391.

Для цитирования: *Sakbaev V.Zh., Zavadsky D.V.* Shift-invariant measures on infinite-dimensional spaces: Integrable functions and random walks // Учен. зап. Казан. ун-та. Сер. Физ.-матем. науки. – 2018. – Т. 160, кн. 2. – С. 384–391.