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SPECTRAL ORDER ON UNBOUNDED OPERATORS AND THEIR SYMMETRIES

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Abstract

The spectral order on positive unbounded operators affiliated with von Neumann algebras have been considered. The spectral order has physical meaning of comparing distribution functions of quantum observables and organizes the structure of unbounded positive operators into a complete lattice. In the previous investigation, we clarified the structure of canonical preservers of the spectral order relation in the bounded case. In the present paper, we have discussed new results on preservers of the spectral order for unbounded positive operators affiliated with von Neumann algebras. We proved earlier that any spectral automorphisms (bijection preserving the order in both directions) of the set of all positive unbounded operators acting on a Hilbert space is a composition of function calculus with a natural extension of projection lattice automorphism. Our investigation starts with observation that this does not hold if the underlying von Neumann algebras have a non-trivial center. However, we have shown that for any von Neumann algebra the following holds. The spectral automorphism preserves positive multiples of projections if and only if it is a composition of the function calculus given by a strictly increasing bijection of the positive part of the real line and an extension of projection lattice automorphism.

Keywords: spectral order, unbounded operators

Introduction

In this paper, we summarize and deepen the results concerning the spectral order on bounded and unbounded operators and their algebras.

Let H be a Hilbert space. Then, there is a standard order on the structure of self-adjoint bounded operators acting on H defined as

$$T \leq S \quad \text{if} \quad \langle T\xi, \xi \rangle \leq \langle S\xi, \xi \rangle$$

for all $\xi \in H$. This order is widely used in the operator theory and its application. The surprising result obtained by R.V. Kadison [1] shows us that the structure of bounded self-adjoint operators acting on a Hilbert space endowed with this standard operator order forms an anti-lattice. It means that two elements in this poset have supremum if and only if they are comparable. In contrast to this, M.P. Olson introduced and studied another order on self-adjoint operators, called the spectral order, that organizes self-adjoint operators into a conditionally complete lattice [2]. In this order, an operator T is majorized by an operator S if each spectral projection of T corresponding to an interval $(-\infty, \lambda]$ majorizes the same spectral projection for S. Viewed from the perspective of quantum mechanics, where self-adjoint operators correspond to observables, the spectral order means that the distribution functions of one observable are less than the distribution function of the second observable in each state of the system.

(For application of the operator algebraic approach to the quantum theory, we refer the reader to [3–5].) The spectral order, first mentioned probably by W. Arveson [6], has been studied in the context of bounded operators and matrices so far [7–12]. We have analyzed the structure of preserves of the spectral order on von Neumann and AW^*-algebras in a series of papers [9, 10, 13–16]. Here, we would like to bring their new aspects to investigation by putting them in a new perspective of unbounded positive operators affiliated to von Neumann algebras.

1. Spectral automorphisms of unbounded operators

Unlike the standard operator order, the spectral order can be naturally defined for unbounded operators as well. Moreover, from the point of view of the quantum theory, where important observables are given by unbounded operators, it is natural to study the spectral order and its automorphisms in the context of unbounded operators affiliated to certain von Neumann algebras. Let us fix notation and introduce all needed concepts. For basic facts on unbounded operators, we refer the reader to [17] and for the theory of von Neumann algebras to [18]. Throughout the paper, let H be a complex Hilbert space. The symbol $\mathcal{E}(H)$ will stand for the set of all positive self-adjoint operators (bounded or not) acting on H. If M is a von Neumann algabra acting on H, then by $\mathcal{E}(M)$ we shall denote the set of all operators in $\mathcal{E}(H)$ that are affiliated to M. Note that $\mathcal{E}(M) = \mathcal{E}(H)$ if $M = B(H)$. By $E(M)$ we shall denote the unit interval in M, that is the set of all (bounded) operators $T \in M$ with $0 \leq T \leq I$, where I is the identity operator. Set $E(H) = E(B(H))$. Let $P(H)$ and $P(M)$ denote the sets of all projections, i.e., self-adjoint idempotent elements, in $B(H)$ and M, respectively. By spectral resolution we mean a map $E : \mathbb{R} \to P(H) : \lambda \to E_\lambda$, such that the following conditions are satisfied:

(i) E is monotone, meaning that $E_\lambda \leq E_\mu$ whenever $\lambda \leq \mu$;

(ii) $\lim\limits_{\lambda\to\infty} E_\lambda = 0$, $\lim\limits_{\lambda\to\infty} E_\lambda = I$;

(iii) E is continuous from the right, meaning that $E_\lambda = \lim\limits_{\mu\to\lambda+} E_\mu$ for all $\lambda \in \mathbb{R}$.

All limits above are taken in the strong operator topology. If $V : \mathbb{R} \to P(H)$ is a map satisfying (i) and (ii) above, then we can define a so-called regularization $\widetilde{V}$ of V by putting

$$\widetilde{V}_\lambda = \lim_{\mu\to\lambda+} V_\mu.$$

Regularization is always a resolution of identity.

As it is well known, there is a one-to-one map between self-adjoint operators and spectral resolutions of identity that restricts to a correspondence between elements of $\mathcal{E}(H)$ and spectral resolutions of identity vanishing for negative numbers. Given a self-adjoint operator A we shall denote by $E^A(\cdot)$ its spectral measure and by (E^A_λ) its spectral resolution. Let us recall that $E^A_\lambda = E^A((-\infty, \lambda])$ for all $\lambda \in \mathbb{R}$. Let us note that a self-adjoint operator A is affiliated to a von Neumann algabra if and only if all values of its spectral measure belong to M. We are now in a position to define the spectral order, $\preceq$, on $\mathcal{E}(H)$. Let A and B be in $\mathcal{E}(H)$. We say that $A \preceq B$ if

$$E^A_\lambda \geq E^B_\lambda \quad \text{for all} \quad \lambda \in \mathbb{R}.$$

We endow the sets $\mathcal{E}(M)$ and $E(M)$ by the spectral order. By this way, we obtain complete lattices. Indeed, let us have a family (T_α) in $\mathcal{E}(H)$. Then, supremum, T, of this family in the spectral order is the self-adjoint operator T with the spectral resolution

$$E^T_\lambda = \bigwedge_\alpha E^{T_\alpha}_\lambda, \quad \lambda \in \mathbb{R}.$$

(here, infimum is taken in the projection lattice). On the other hand, infimum of this family, with respect to the spectral order, is the self-adjoint operator S, the spectral resolution of which is the regularization of the projection-valued map

$$\lambda \to \bigvee_{\alpha} E_{\lambda}^{T_{\alpha}}.$$

This can be checked directly and was shown in the case of bounded operators by M.P. Olson [2]. Let us remark that projection lattice $P(M)$ is a sublattice of $P(H)$. Therefore, $\mathcal{E}(M)$ is a sublattice of $\mathcal{E}(H)$.

We shall summarize a few important properties of the spectral order. More details can be found in the papers [15, 16]. For bounded operators, the spectral order implies the standard order and both orders coincide for bounded commuting operators. On $P(M)$, the spectral order is equivalent to the standard operator order, and supremum and infimum are the same whether computed in $P(H)$ or in $E(M)$.

Let $(P, \preceq)$ be a poset. A bijection $\varphi : P \to P$ is called an order automorphism if it preserves the order in both directions:

$$a \preceq b \Longleftrightarrow \varphi(a) \leq \varphi(b) \quad a,\, b \in P.$$

In [16], we described completely ordered automorphisms of $\mathcal{E}(H)$. We showed that any such automorphism has two components — one is induced by order automorphism of the lattice of projections and the another is given by the function calculus. The former transformation is not changing the spectrum, but rather the set of spectral projections. The latter transformation is changing the spectrum, but leaving the set of spectral projections invariant. Spectral order automorphisms of the set of bounded positive operators were described in the remarkable paper by L. Molnar and P. Semrl [12]. We generalized this result to an unbounded context. Moreover, our approach showed clearly how the spectral order automorphism is connected with the order automorphism of the lattice of projections that form the logic of quantum mechanics. We shall need the following notation: Let $\tau : P(M) \to P(M)$ be a spectral order automorphism. Having a positive operator T affiliated with M, we can define another operator, $\varphi_\tau(T)$, the spectral resolution of which is $\lambda \to \tau(E_\lambda^T)$. It can be shown that the map $T \to \varphi_\tau(T)$ is an order automorphism of $\mathcal{E}(M)$. This automorphism leaves the projection lattice $P(M)$ invariant and restricts it to another automorphism of $P(M)$. In fact, it can be checked quite easily that

$$\varphi_\tau(P) = I - \tau(I - P), \quad P \in P(M).$$

Another important example of spectral automorphism is given by function calculus. The following proposition can be proved in the same way as proposition 1.4 in [15].

Proposition 1. *Let $f : [0, \infty) \to [0, \infty)$ be a strictly increasing bijection. Then, the map $T \to f(T)$ is a spectral order automorphism.*

In case of the structure of unbounded positive operators, the main result characterizing automorphisms of $\mathcal{E}(H)$ was proved in [16].

Theorem 1. *Let $\varphi : \mathcal{E}(H) \to \mathcal{E}(H)$ be a spectral order automorphism. Then, there is $c > 0$, a projection lattice isomorphism $\tau : P(H) \to P(H)$, and strictly increasing bijection $f : [0, \infty) \to [0, \infty)$, such that*

$$\varphi(T) = c\varphi_\tau(f(T)), \quad T \in \mathcal{E}(H). \tag{1}$$

In the proof of the foregoing theorem, an important role is played by the presence of atomic projections in $P(H)$. These projections are not available for general von Neumann algebras. Furthermore, the result above is not true for non-factorial von Neumann algebras. For example, any canonical spectral order automorphism given by φ is sending the unit I to

$$\varphi(\lambda I) = \varphi_\tau(f(\lambda I)) = \varphi_\tau(f(\lambda)I) = f(\lambda)I\,.$$

Therefore, any canonical automorphism preserves multiples of the unit. Now, let us consider von Neumann algebra $M = B(H) \oplus B(H)$ acting on $H \oplus H$. Let $\varphi : \mathcal{E}(M) \to \mathcal{E}(M)$ be defined as

$$\varphi : (T_1, T_2) \to (T_1^2, T_2).$$

Then, it can be easily verified that φ is a spectral order automorphism of $\mathcal{E}(M)$. However, $\varphi(1/2\,I, 1/2\,I) = (1/4\,I, 1/2\,I)$, which is not a multiple of the unit. Therefore, φ cannot be canonical.

It turns out that, in case of bounded operators, a spectral order automorphism is canonical if and only if it preservers multiples of projections in the sense specified in the next theorem. This theorem was proved in [16].

Theorem 2. *Let $\varphi : E(M) \to E(M)$ be a spectral order automorphism. Then, the following statements are equivalent:*

(i) For each projection $P \in P(M)$, the image $\varphi(P)$ is a projection and there is a bijection $f_P : [0,1] \to [0,1]$, such that

$$\varphi(\lambda P) = f_P(\lambda)\varphi(P), \quad P \in P(M).$$

(ii) There is a projection lattice isomorphism $\tau : P(H) \to P(H)$ and strictly increasing bijection $f : [0,\infty) \to [0,\infty)$, such that

$$\varphi(T) = \varphi_\tau(f(A)), \quad T \in \mathcal{E}(H). \tag{2}$$

In order to generalize the foregoing result for unbounded operators, we need the following definition:

Definition 1. Let $\varphi : \mathcal{E}(M) \to \mathcal{E}(M)$ be a spectral order automorphism. We say that φ preserves multiples of projections if there is a projection P' in M and strictly increasing bijection $f_P : [0,\infty) \to [0,\infty)$ for each projection P in M, such that

$$\varphi(\lambda P) = f_P(\lambda)P', \quad \lambda \in \mathbb{R}.$$

We have the following properties of maps specified above.

Proposition 2. *Let $\varphi : \mathcal{E}(M) \to \mathcal{E}(M)$ be a a spectral order automorphism preserving multiples of projections. Then, the following assertions hold true:*

(i) There is $c > 0$, such that $\varphi(I) = cI$.

(ii) If $\varphi(I) = I$, then φ preserves projections.

Proof. (i) We know that there is a projection Q in M and $c > 0$, such that $\varphi(P) = cQ$. For each projection P in M, we have that $P \preceq I$, and so $f_P(1)P' \le cI$. Since the standard order for bounded positive operators is preserved by range projections, we have that $P' \le Q$. However, the set of all possible P' is all set $P(M)$. It implies that $Q = I$.

(ii) Let us suppose that $\varphi(I) = I$. Let us take an arbitrary projection P in M. Then, $P \le I$, and so $f_P(1)P' \le I$. It means that $f_P(1) \le 1$. Let us suppose for a contradiction that there is a $c > 0$ with $1 > c > f_P(1)$. Then, $I = \varphi^{-1}(I) \ge \varphi^{-1}(cP') = f_P^{-1}(c)P$. It follows that $f_P^{-1}(c) \le 1$. Therefore, $c \le f_P(1)$ is a contradiction. We showed that $\varphi(P) = P'$. $\square$

Theorem 3. *Spectral order automorphism $\varphi : \mathcal{E}(M) \to \mathcal{E}(M)$ preserves multiples of projections if and only if there is a positive number $c > 0$, an order automorphism $\tau : P(M) \to P(M)$, and a strictly increasing function $f : [0,\infty) \to [0,\infty)$, such that*

$$\varphi(T) = c\varphi_\tau(f(T)), \quad T \in \mathcal{E}(M).$$

Proof. Let us suppose that φ preserves multiples of projections. In the light of proposition 2, by multiplying φ by a suitable positive constant we can suppose that $\varphi(I) = I$. In this case, φ preserves projections. Therefore, for each projection P in M, there is a strictly increasing bijection $f_P : [0,\infty) \to [0,\infty)$ with $\varphi(\lambda P) = f_P(\lambda)\varphi(P)$.

Let us take now two non-zero projections P and Q in M and consider their supremum $R = P \vee Q$. Then, for $\lambda \geq 0$ we have

$$f_R(\lambda)\varphi(R) = \varphi(\lambda R) = \varphi(\lambda(P \vee Q)) = \varphi(\lambda P) \vee \varphi(\lambda Q) = f_P(\lambda)\varphi(P) \vee f_Q(\lambda)\varphi(Q). \quad (3)$$

It can be verified easily that for $0 \leq s \leq t$ and projections E and F we have

$$sE \vee tF = s(E \vee F - F) + tF$$

(see, e.g., [15, lemma 1.2]). Therefore, comparing the spectra of operators on both sides of (3) we obtain

$$\{f_R(\lambda), 0\} = \{f_P(\lambda), f_Q(\lambda), 0\}.$$

It implies that $f_P(\lambda) = \varphi_Q(\lambda)$. In other words, we have that $\varphi(\lambda P) = f(\lambda)\varphi(P)$, for all projections $P \in M$ and $\lambda \geq 0$. By composing φ with the spectral automorphism $a \to f^{-1}(a)$, we can suppose that $\varphi(\lambda p) = \lambda\varphi(p)$. for all projections $P \in M$ and $\lambda \geq 0$. As we know, *varphi* restricts to a projection lattice automorphism on $P(M)$. Let us denote this restriction by τ'. Let us consider another projection lattice automorphism

$$\tau(P) = I - \tau'(I - P), \quad P \in P(M).$$

Then, for $\lambda \geq 0$, we have

$$\varphi_\tau(\lambda P) = \lambda(I - \tau(I - P)) = \lambda(I - (I - \tau'(P)) = \lambda\tau'(P) = \lambda\varphi(P) = \varphi(\lambda P).$$

Therefore, φ and φ_τ coincide on all elements of the form λP, where P is a projection and $\lambda \geq 0$. However, by applying lemma 1.2 in [15], we have that each element in $\mathcal{E}(M)$ is a supremum (in the spectral order) of elements that are multiples of projections. Therefore, any automorphism is uniquely determined by its values on multiples of projections. It means that $\varphi = \varphi_\tau$ and the proof is completed. □

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Спектральный порядок на неограниченных операторах и их симметрии

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Аннотация

В работе рассмотрен спектральный порядок на положительных неограниченных операторах, присоединенных к алгебре фон Неймана. Физический смысл спектрального порядка состоит в сравнении функций распределения квантовых наблюдаемых. С математической точки зрения спектральный порядок интересен, кроме всего прочего, тем, что организует множество положительных неограниченных операторов, присоединенных к алгебре фон Неймана, в полную решетку. В предыдущих исследованиях авторами было получено описание преобразований, сохраняющих спектральный порядок в случае ограниченных операторов. В настоящей работе приводятся новые результаты по описанию преобразований, сохраняющих спектральный порядок, в случае положительных неограниченных операторов, присоединенных к алгебре фон Неймана. Ранее было показано, что любой спектральный автоморфизм (биекция, сохраняющая спектральный порядок в обоих направлениях) на множестве положительных неограниченных операторов, действующих в гильбертовом пространстве, представим в виде композиции функционального исчисления с естественным продолжением автоморфизма на решетке ортопроекторов. В работе показано, что это утверждение неверно для алгебры фон Неймана, имеющей нетривиальный центр. Но для произвольной алгебры фон Неймана получено описание спектральных автоморфизмов, сохраняющих операторы, кратные ортопроекторам.

Ключевые слова: спектральный порядок, неограниченные операторы

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