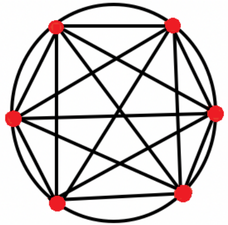
International Conference on Algebra, Analysis and Geometry
Kazan, Russia, August 22 – 28, 2021

Optimal Green energy points on the circles

Vladimir N. Dubinin

Institute for Applied Mathematics,
Far Eastern Branch of the Russian Academy of Sciences
Vladivostok
e-mail: dubinin@iam.dvo.ru

Polya-Schur inequality



"Auf einen weiteren Satz... hat mich Herr G. Polya aufmerksam gemacht":

$$\prod_{k=1}^{n}\prod_{\substack{l=1\\ l\neq k}}^{n}|z_k - z_l| \leq \prod_{k=1}^{n}\prod_{\substack{l=1\\ l\neq k}}^{n}|z_k^* - z_l^*| = n^n.$$

I. Schur, Mathemetische Z., 1(4), 1918, p. 385.

Discrete energy

$$\sum_{k=1}^{n}\sum_{\substack{l=1\\ l\neq k}}^{n} K(\mathbf{x}_k,\mathbf{x}_l)\nu(\mathbf{x}_k)\nu(\mathbf{x}_l), \qquad \mathbf{x}_k \in \mathbb{R}^d, \quad k=1,\dots,n.$$

$$K(z_k, z_l) = -\log|z_k - z_l|,$$

$$\sum_{k=1}^{n}\sum_{\substack{l=1\\ l\neq k}}^{n} -\log|z_k - z_l| \geq \sum_{k=1}^{n}\sum_{\substack{l=1\\ l\neq k}}^{n} -\log|z_k^* - z_l^*|.$$

The Riesz $s-$energy ($s \neq 0$) of n points $z_1, ..., z_n$ of the complex plane is defined by

$$\sum_{k=1}^{n} \sum_{\substack{l=1 \\ l \neq k}}^{n} |z_k - z_l|^{-s}.$$

It can be shown using the classical Tóth's result and a convexity argument that for $s \geq -1$ and each $n \geq 2$, the n-th roots of unity

$$z_k^* = \exp \frac{2\pi i(k-1)}{n}, \quad k = 1, ..., n,$$

form minimal $n-$point $s-$energy configuration for the unit circle $|z| = 1$:

$$\sum_{k=1}^{n} \sum_{\substack{l=1 \\ l \neq k}}^{n} |z_k - z_l|^{-s} \geq \sum_{k=1}^{n} \sum_{\substack{l=1 \\ l \neq k}}^{n} |z_k^* - z_l^*|^{-s}.$$

L. Fejes Tóth, Regular Figures, A Pergamon Press Book, The Macmillan Co., New York, 1964.

Motivation

Various sophisticated problems related to the optimality of the Riesz $s-$energy for different values of s and for the points z_k lying in the plane sets or in $\mathbb{R}^d$ have been treated in a number of papers.

1. J.S. Brauchart, D.P. Hardin, E.B. Saff, The Riesz energy of the Nth roots of unity: an asymptotic expansion for large N, Bulletin of the London Mathematical Society, Volume 41, Part 4, August 2009, 621-633.

2. J.S. Brauchart, D.P. Hardin, and E.B. Saff. The next-order term for optimal Riesz and logarithmic energy asymptotics on the sphere. Contemp. Math, 2012. V. 578. P.31-61.

3.S. Borodachov, D. Hardin, E. Saff, Discrete energy on rectifiable sets. Springer, 2019.

Green energy

$$\sum_{k=1}^{n}\sum_{\substack{l=1\\l\neq k}}^{n} g_B(\mathbf{x}_k,\mathbf{x}_l)\nu(\mathbf{x}_k)\nu(\mathbf{x}_l), \quad \mathbf{x}_k\in B\subset\mathbb{R}^d, \quad k=1,\dots,n.$$

For $d=2,\ B=\{z=re^{i\theta}:\ \mu_1<r<\mu_2\}$ and a real ζ we have

$$g_B(z,\zeta)=\frac{\log\frac{r}{\mu_1}\log\frac{\zeta}{\mu_2}}{\log\frac{\mu_1}{\mu_2}}-\log\left|\frac{\mu_2^2(1-\zeta^{-1}re^{i\theta})}{\mu_2^2-\zeta re^{i\theta}}\right|+$$

$$+\sum_{n=1}^{\infty}\frac{\left(\frac{\zeta}{\mu_2}\right)^n-\left(\frac{\mu_2}{\zeta}\right)^n}{1-\left(\frac{\mu_2}{\mu_1}\right)^{2n}}\left[\left(\frac{r}{\mu_2}\right)^n-\left(\frac{\mu_2}{r}\right)^n\right]\frac{\cos n\theta}{n}.$$

Lower bound

Let $\theta_k,\ k = 1, ..., 2n,\ n \geq 2$, be real numbers, such that

$$\theta_1 < \theta_2 < ... < \theta_{2n} < \theta_1 + 2\pi,$$

and let $Z = \{z_k\}_{k=1}^{2n},\ z_k = e^{i\theta_k},\ k = 1, ..., 2n$. Denote by

$$E(Z, B) = \sum_{k=1}^{2n} \sum_{\substack{l=1 \\ l \neq k}}^{2n} g_B(z_k, z_l)$$

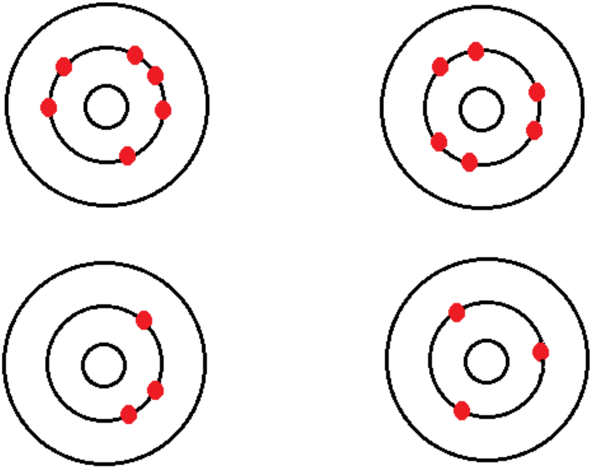
the Green energy of the collection Z with respect to the ring $B := \{z :\ \mu_1 < |z| < \mu_2\}$, $0 < \mu_1 < 1 < \mu_2 < \infty$.

Theorem 1.

$$E(Z, B) \geq E(Z^*, B),$$

where $Z^* = \{z_k^*\}_{k=1}^{2n},\ |z_k^*| = 1$, $\arg z_{2j-1}^* = -\frac{\eta}{2n} + \frac{2\pi j}{n}$, $\arg z_{2j}^* = \frac{\eta}{2n} + \frac{2\pi j}{n}$, $j = 1, ..., n$, $\eta = \sum_{j=1}^{n}(\theta_{2j} - \theta_{2j-1})$.

Lower bounds



Condenser capacities

$$\mathrm{cap}\; C^*(r) = -\frac{4\pi n}{\log r} -$$

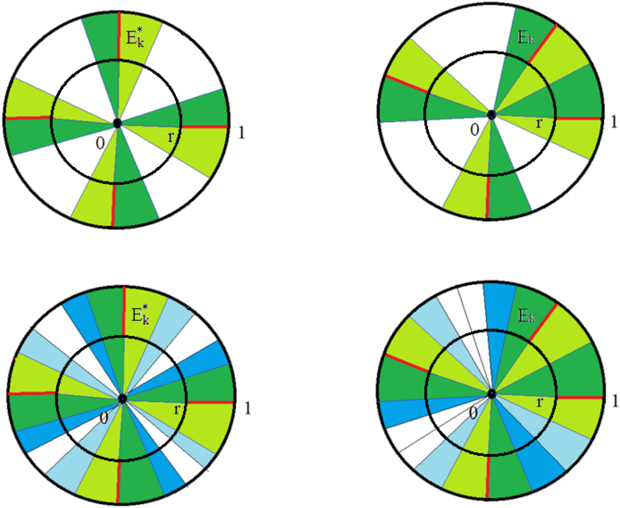
$$-2\pi \left\{ \sum_{k=1}^{2n} \log r(B, z_k^*) + E(Z^*, B) \right\} \left(\frac{1}{\log r} \right)^2 + o\left(\left(\frac{1}{\log r} \right)^2 \right), \;\; r \to 0;$$

$$\mathrm{cap}\,\mathrm{Dis}\; C^*(r) = -\frac{4\pi n}{\log r} -$$

$$-2\pi \left\{ \sum_{k=1}^{2n} \log r(B, z_k) + E(Z, B) \right\} \left(\frac{1}{\log r} \right)^2 + o\left(\left(\frac{1}{\log r} \right)^2 \right), \;\; r \to 0,$$

where $r(B, z_k)$ is the inner radius of B with respect to the point $z_k, \;\; k = 1, ..., n$.

Dissymmetrization



Theorem 2. *Let Z and B be as above. Then, for the discrete Green energy*

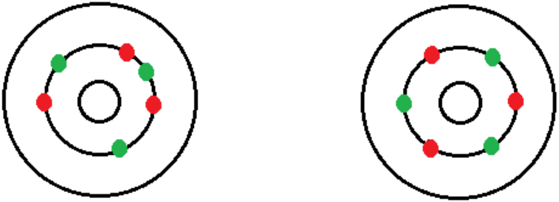
$$\mathcal{E}(Z,B) := \sum_{k=1}^{2n} \sum_{\substack{l=1 \\ l \neq k}}^{2n} (-1)^{k+l} g_B(z_k, z_l)$$

we have

$$\mathcal{E}(Z,B) \leq \mathcal{E}(\mathcal{Z}^*, B)$$

where $\mathcal{Z}^* = \{\exp\left(\frac{\pi i k}{n}\right)\}_{k=1}^{2n}$.

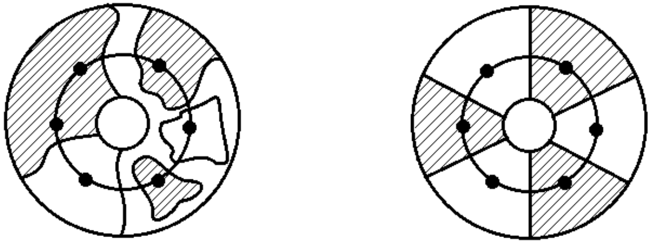
The proof of the inequality is carried out by reduction to the extremal decomposition problems, which have a long history and many applications.



By taking the limit $\mu_1 \to 0,\ \mu_2 \to \infty$ we have

$$\prod_{k=1}^{n} \prod_{\substack{l=1 \\ l \neq k}}^{n} |z_k - z_l|^{(-1)^{k+l}} \geq \left(\frac{n}{2}\right)^{2n}.$$

Extremal decomposition



$$\prod_{k=1}^{n} r(B_k, z_k) \leq \prod_{k=1}^{n} r(B_k^*, z_k^*).$$

Notation and definitions

$\mathbb{R}^d$ is a $d-$dimensional Euclidean space with the usual norm $||\cdot||$, with points $\mathbf{x} = (x_1, ..., x_d)$, $d \geq 3$. A domain B in $\mathbb{R}^d$ is admissible if it has the Green function for the Laplace operator vanishing at the points of the boundary ∂B of the domain B. This Green function with pole at the point $\mathbf{x}_0 \in B$ will be denoted by $g_B(\mathbf{x}, \mathbf{x}_0)$. In the neighborhood of $\mathbf{x}_0$ the following expansion holds

$$g_B(\mathbf{x}, \mathbf{x}_0) = \lambda_d ||\mathbf{x} - \mathbf{x}_0||^{2-d} + O(1), \quad \mathbf{x} \to \mathbf{x}_0,$$

where $\lambda_d = ((d-2)\omega_{d-1})^{-1}$, $\omega_{d-1} = 2\pi^{d/2}/\Gamma(d/2)$ is the surface measure of the unit hyper-sphere. In all points of B different from the pole $\mathbf{x}_0$, the Green function is harmonic.

Denote by J the $(d-2)-$dimensional plane

$$\{\mathbf{x} \in \mathbb{R}^d : \ \mathbf{x} = (0, 0, x_3, ..., x_d)\}.$$

We will need the cylindrical coordinates $(r, \theta, \mathbf{x}')$ of the point $\mathbf{x} = \{x_1, ..., x_d\}$ in $\mathbb{R}^d$, related to the Cartesian coordinates by $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, $\mathbf{x}' \in J$. A domain $B \subset \mathbb{R}^d$ will be called the *rotation domain* (with respect to the axis J), if for any point $(r, \theta, \mathbf{x}') \in B$ and any φ the point $(r, \varphi, \mathbf{x}')$ belongs to B.

Green energy

Suppose that B is an admissible rotation domain and let $\Omega = \{S\}$ be the collection comprising a finite number of distinct circles S of the form $S = \{(r_0, \theta, \mathbf{x}'_0) : \ 0 \leq \theta \leq 2\pi\}$ lying in the domain B (here $r_0 > 0$ and $\mathbf{x}'_0 \in J$ are assumed to be fixed). For arbitrary real numbers $\theta_j, \ j = 0, ..., m-1$,

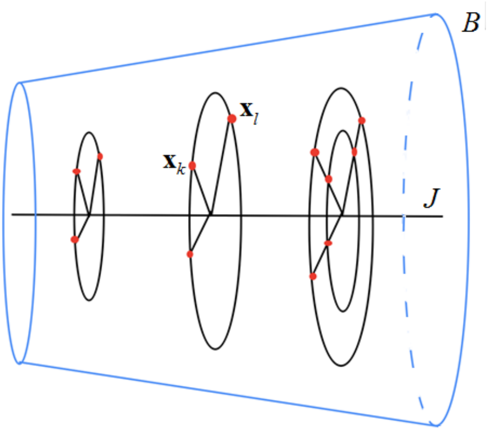
$$0 \leq \theta_0 < \theta_1 < ... < \theta_{m-1} < 2\pi,$$

denote by $X = \{\mathbf{x}_k\}_{k=1}^n$ the collection of all distinct points of B at which the circles from Ω intersect the half-planes

$$L_j = \{(r, \theta, \mathbf{x}') : \ \theta = \theta_j\}, \ j = 0, ..., m-1.$$

Let $\Delta = \{\nu(\mathbf{x}_k)\}_{k=1}^n$ be an arbitrary discrete charge. The *Green energy* of this charge with respect of the domain B is defined by

$$E(X, \Delta, B) = \sum_{k=1}^{n} \sum_{\substack{l=1 \\ l \neq k}}^{n} g_B(\mathbf{x}_k, \mathbf{x}_l)\nu(\mathbf{x}_k)\nu(\mathbf{x}_l).$$



$$\nu(\mathbf{x}_k) = \nu(\mathbf{x}_l)$$

Theorem 3. *Let $X^* = \{\mathbf{x}_k^*\}_{k=1}^n$ be the collection of points at which the circles from Ω intersect the half-planes*

$$L_j^* = \{(r, \theta, \mathbf{x}') : \ \theta = 2\pi j/m\}, \ j = 0, ..., m-1.$$

Suppose that the charge $\Delta = \{\nu(\mathbf{x}_k)\}_{k=1}^n$ takes equal values $\nu(\mathbf{x}_k) = \nu(\mathbf{x}_l)$ at the points $\mathbf{x}_k$ and $\mathbf{x}_l$ from the collection X that lie on the same circle from Ω and, furthermore, that the points $\mathbf{x}_k \in X$ and $\mathbf{x}_k^* \in X^*$ lie on the same circle from Ω, $k = 1, ..., n$. Then

$$E(X, \Delta, B) \geq E(X^*, \Delta, B).$$

Upper bound

For any $d \geq 3$

$$\sum_{k=1}^{2n} \sum_{\substack{l=1 \\ l \neq k}}^{2n} \frac{(-1)^{k+l}}{|z_k - z_l|^{d-2}} \leq \sum_{k=1}^{2n} \sum_{\substack{l=1 \\ l \neq k}}^{2n} \frac{(-1)^{k+l}}{|z_k^* - z_l^*|^{d-2}}$$

where $z_k, \ k = 1, ..., 2n$, are located on the circle $|z| = 1$ in the ascending order of the index k and $z_k^* = \exp(\pi i(k-1)/n)$, $k = 1, ..., 2n$.
V.N. Dubinin, E.G. Prilepkina, "Optimal Green energy points on the circles in d-space". Journal of Mathematical Analysis and Applications, **499**:2 (2021) (Article 125055)


Thank you for attention!