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ON CONSTRUCTIONS OF SEMIGROUPS

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Abstract

This paper gives a brief survey of constructions of semigroups by using structures of some semigroups belonging to the class of regular semigroups, quasi-regular semigroups, and abundant semigroups. In particular, we show some basic notation and structure theorems for some semigroups, for example, Rees matrix semigroups over the 0-group G^0 and their generalizations, bands, E-ideal quasi-regular semigroups, $\mathcal{C}^*$-quasiregular semigroups, $\mathcal{L}^*$-inverse semigroups, $\mathcal{Q}^*$-inverse semigroups, and regular ortho-lc-monoids.

Key words: regular semigroups, quasi-regular semigroups, abundant semigroups, constructions.

1. Rees matrix semigroups and their generalizations

For notation and terminology not given in this paper, the reader is referred to [1–13].

A semigroup is called *completely 0-simple* if it is 0-simple and has a primitive idempotent. In 1940, Rees provided the following recipe for manufacturing completely 0-simple semigroups.

Let G be a group with identity element e, and let I, Λ be non-empty sets. Let $P = (p_{\lambda,i})$ be a $\Lambda \times I$ matrix with entries in the 0-group $G^0 (= G \cup \{0\})$, and suppose that P is regular, in the sense that no row or no column of P consists entirely of zeros. Formally,

$$\begin{aligned} &(\forall i \in I)(\exists \lambda \in \Lambda)\ \ p_{\lambda i} \neq 0, \\ &(\forall \lambda \in I)(\exists i \in I)\ \ p_{\lambda i} \neq 0. \end{aligned} \tag{1}$$

Let $S = (I \times G \times \Lambda) \cup \{0\}$, and define a multiplication on S by

$$(i, a, \lambda) \cdot (b, j, \mu) = \begin{cases} (i, a p_{\lambda j} b, \mu) & \text{if } p_{\lambda j} \neq 0, \\ 0 & \text{if } p_{\lambda j} = 0, \end{cases}$$

$$(i, a, \lambda) \cdot 0 = 0 \cdot (i, a, \lambda) = 0 \cdot 0 = 0. \tag{2}$$

The semigroup constructed with this recipe will be denoted by $\mathcal{M}^0[G; I, \Lambda; P]$ and will be called the $I \times \Lambda$ *matrix semigroup over the 0-group* G^0 *with the regular sandwich matrix* P.

Theorem 1 (The Rees Theorem, [14]). *Let G^0 be a 0-group, let I, Λ be non-empty sets, and let $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix with entries in G^0. Suppose that P is regular in the sense of* (1)*. Let $S = (I \times G \times \Lambda) \cup \{0\}$, and define a multiplication on S by* (2)*. Then S is a completely 0-simple semigroup.*

Conversely, every completely 0-simple semigroup is isomorphic to one constructed in this way.

In 1990, M.V. Lawson in [15] gave another abstract characterization of Rees matrix semigroups as follows.

Let S be a monoid with identity 1 and zero element 0, having group of units $G(S)$. Let Λ and I be non-empty sets, and let P be a $\Lambda \times I$ matrix over S with entries $p_{\lambda i}$ where $(\lambda, i) \in \Lambda \times I$. *The matrix semigroup* $M = M^0(S; I, \Lambda; P)$ is the set of triples $I \times S \times \Lambda$ with a zero $\mathbf{0}$ adjoined and where we identify all the elements of the form $(i, 0, \lambda)$ with $\mathbf{0}$, under a multiplication given by

$$(i, x, \lambda) \cdot (j, y, \mu) = \begin{cases} (i, xp_{\lambda j}y, \mu) & \text{if } p_{\lambda j} \neq 0, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Theorem 2 [15]. *Let S be a Rees semigroup with $e \in U \setminus \{0\}$. Then*
(i) S is abundant if and only if eSe is abundant;
(ii) S is regular if and only if eSe is regular;
(iii) S is inverse if and only if S is reduced, eSe is inverse and $Reg_U(S)$ is a subsemigroup (for details, see [15]).

To further generalize the Rees matrix semigroup constructed above, we have recently established in [16] the following construction of semigroups by using semigroupoids.

A *semigroupoid* is a pair (S, S^0) consisting of a set S of morphisms and a set S^0 of objects, together with the functions $\tau : S \to S^0$ and $\omega : S \to S^0$, and a function μ which is so called "multiplication" from the set $S * S = \{(x, y) \in S \times S \mid \tau(x) = \omega(y)\}$ to S; we usually write xy instead of $\mu(x, y)$, and if $(x, y) \in S * S$, then we write $\exists xy$; in addition, the following two axioms hold:

(C1) If $\exists xy$, then $\tau(xy) = \tau(y)$ and $\omega(xy) = \omega(x)$;

(C2) $x(yz) = (xy)z$ whenever the products are defined.

Let $A, B \in S^0$. Then, in this case, the set Mor $(A, B) = \{x \in S \mid \tau(x) = A$ and $\omega(x) = B\}$ is called the *Mor-set* from A to B. A semigroupoid S is said to be *strongly connected* if each Mor-set (A, B) is non-empty.

Let I and Λ be two non-empty sets and S a strongly connected semigroupoid. Define two surjective functions $F : I \to S^0$ and $G : \Lambda \to S^0$. Now,let $p : \Lambda \times I \to S$ be a function such that

$$p_{(\lambda,i)} \in \mathrm{Mor}\,(F(i), G(\lambda)).$$

We simply write $p_{(\lambda,i)} = p_{\lambda i}$ so that the entries of the $\Lambda \times I$ matrix $P = (p_{\lambda i})$ are $p_{\lambda i}$. Let $M = M(S, F, G; P)$ be the following set

$$M = \{(i, x, \lambda) \in I \times S \times \Lambda \mid x \in \mathrm{Mor}\,(G(\lambda), F(i))\},$$

equipped with the multiplication given by $(i, x, \lambda)(j, y, \mu) = (i, xp_{\lambda j}y, \mu)$.

Then it is easy to check that the set $M = \{(i, x, \lambda) \in I \times S \times \Lambda \mid x \in \mathrm{Mor}\,(G(\lambda), F(i))\}$ forms a semigroup under the above multiplication. We call M a *Rees matrix semigroup over a semigroupoid* (for details, see [16]).

2. Presentations of bands and their generalizations

We say that a semigroup S is a band if every element of S is idempotent. In 1971, Petrich in [17] gave the general structure theorem for bands.

Theorem 3 [14, Theorem 4.4.5]. *Let Y be a semilattice, and let $\{E_\alpha \mid \alpha \in Y\}$ be a family of pairwise disjoint rectangular bands indexed by Y. For each α, let $E_\alpha = I_\alpha \times \Lambda_\alpha$, and for each pair α, β of elements of Y such that $\alpha \geqslant \beta$, let $\Phi_{\alpha,\beta} : E_\alpha \to \mathcal{T}_{I_\beta}{}^* \times \mathcal{T}_{\Lambda_\beta}$ be a morphism, where*

$$a\Phi_{\alpha,\beta} = (\phi^a_\beta, \psi^a_\beta) \quad (a \in E_\alpha).$$

Suppose also that

(i) if $a=(i,\mu)\in E_\alpha$*, then* ${\phi_\alpha}^a$ *and* ${\psi_\alpha}^a$ *are constant maps, and*

$$\langle \phi_\alpha^{(i,\mu)}\rangle = i, \quad \langle \psi_\alpha^{(i,\mu)}\rangle = \mu;$$

(ii) if $a\in S_\alpha,\ b\in S_\beta$ *and* $\alpha\beta=\gamma$*, then* $\phi_\gamma^a\phi_\gamma^b$ *and* $\psi_\gamma^a\psi_\gamma^b$ *are constant maps;*

(iii) if $\langle\phi_\gamma^a\phi_\gamma^b\rangle$ *is denoted by* j *and* $\langle\psi_\gamma^a\psi_\gamma^b\rangle$ *by* ν*, then for all* $\delta\leqslant\gamma$*,*

$$\phi_\delta^{(j,\nu)} = \phi_\delta^a\phi_\delta^b, \quad \psi_\delta^{(j,\nu)} = \psi_\delta^a\psi_\delta^b.$$

Let $B=\bigcup\{E_\alpha \mid \alpha\in Y\ \}$*, and define the product of* a *in* E_α *and* $b\in E_\beta$ *by*

$$a*b = (\langle\phi_\gamma^a\phi_\gamma^b\rangle, \langle\psi_\gamma^a\psi_\gamma^b\rangle),$$

where $\gamma=\alpha\beta$*. Then* $(B,*)$ *is a band, whose* $\mathcal{J}$*-classes are the rectangular bands* E_α*.*

Conversely, every band is determined in this way by a semilattice Y*, a family of rectangular bands* $E_\alpha = I_\alpha\times\Lambda_\alpha$ *indexed by* Y*, and a family of morphisms* $\Phi_{\alpha,\beta}$: $E_\alpha\to {\mathcal{T}_{I_\beta}}^*\times\mathcal{T}_{\Lambda_\beta}$ $(\alpha,\beta\in Y,\ \alpha\geqslant\beta)$ *satisfying (i), (ii) and (iii).*

An element a of a semigroup S is called regular if there exists $x\in S$ such that $a=axa$; an element a of S is called quasi-regular if there exists a natural number n such that a^n is regular. A semigroup S is called regular (quasi-regular) if every element of S is regular (quasi-regular). It is easy to see that quasi-regular semigroups are generalizations of regular semigroups. As a generalization of bands, in 1989 Ren and Guo introduced and studied E-ideal quasi-regular semigroups.

According to [18], a semigroup S is called an *E-ideal quasi-regular semigroup* if S is quasi-regular and $E(S)$ is an ideal of S.

For E-ideal quasi-regular semigroups, Ren and Guo [18] have given the following constructions.

The set Q with a partial operation is called a *partial power breaking semigroup* if there is a partial binary operation on the set Q such that for any $p,q,r\in Q$, $(pq)r\in Q$ (well-defined) if and only if $p(qr)\in Q$; in this case, $(pq)r=p(qr)$ holds, and for every $a\in Q$, there exists $n\in N$ such that $a^n\notin Q$.

Let Y be a semilattice, and let $\{E_\alpha=I_\alpha\times\Lambda_\alpha \mid \alpha\in Y\}$ be a family of pairwise disjoint rectangular bands. Let Q be a partial power breaking semigroup together with the mapping $\varphi: Q\to \bigcup\limits_{\alpha\in Y} E_\alpha$ satisfying the following properties.

(i) For any $a,\,b\in Q$, if $\varphi(a)\in E_\alpha,\ \varphi(b)\in E_\beta$ and $\alpha\beta=\gamma$, then $ab\in Q$ implies $\varphi(ab)\in E_\gamma$. For every pair $\alpha,\,\beta\in Y$ with $\alpha\geqslant\beta$, we can construct two mappings:

$$\Psi_{\alpha,\beta}: \varphi^{-1}(E_\alpha)\longrightarrow {\mathcal{T}_{I_\beta}}^*\times\mathcal{T}_{\Lambda_\beta},$$
$$a\mapsto(\phi_\beta^a,\psi_\beta^a)$$

and

$$\Phi_{\alpha,\beta}: E_\alpha\longrightarrow {\mathcal{T}_{I_\beta}}^*\times\mathcal{T}_{\Lambda_\beta},$$
$$e\mapsto(\phi_\beta^e,\psi_\beta^e)$$

that satisfy the following properties:

(ii) If $e=(i,j)\in E_\alpha$, then $\phi_\alpha^e,\psi_\alpha^e$ are constant transformations on I_α and Λ_α, respectively, and $\langle\phi_\alpha^e\rangle=i, \langle\psi_\alpha^e\rangle=j$. Here we denote the values of the constant transformations by $\langle\phi_\alpha^e\rangle$ and $\langle\psi_\alpha^e\rangle$, respectively.

(iii) 1° If $e\in E_\alpha,\ f\in E_\beta$, and $\delta\leqslant\gamma=\alpha\beta$, then $\phi_\gamma^e\phi_\gamma^f$ and $\psi_\gamma^e\psi_\gamma^f$ are transformations on I_γ and Λ_γ, respectively. Let $\langle\phi_\gamma^e\phi_\gamma^f\rangle=i, \langle\psi_\gamma^e\psi_\gamma^f\rangle=j$, we have

$$\phi_\delta^{(i,j)}=\phi_\delta^e\phi_\delta^f, \psi_\delta^{(i,j)}=\psi_\delta^e\psi_\delta^f.$$

2° If $e \in E_\alpha$, $a \in Q$, $\varphi(a) \in E_\beta$, and $\delta \leqslant \gamma = \alpha\beta$, then $\phi^e_\gamma\phi^a_\gamma$, $\phi^a_\gamma\phi^e_\gamma$ and $\psi^e_\gamma\psi^a_\gamma$, $\psi^a_\gamma\psi^e_\gamma$ are constant transformations on I_γ and Λ_γ respectively. Let $\langle\phi^e_\gamma\phi^a_\gamma\rangle = k, \langle\psi^e_\gamma\psi^a_\gamma\rangle = l$, $\langle\phi^a_\gamma\phi^e_\gamma\rangle = k'$, and $\langle\psi^a_\gamma\psi^e_\gamma\rangle = l'$, we have

$$\phi^{(k,l)}_\delta = \phi^e_\delta\phi^a_\delta, \psi^{(k,l)}_\delta = \psi^e_\delta\psi^a_\delta,$$

$$\phi^{(k',l')}_\delta = \phi^a_\delta\phi^e_\delta, \psi^{(k',l')}_\delta = \psi^a_\delta\psi^e_\delta.$$

3° If $a, b \in Q$, $ab \notin Q$, $\varphi(a) \in E_\alpha$, $\varphi(b) \in E_\beta$, and $\delta \leqslant \gamma = \alpha\beta$, then $\phi^a_\gamma\phi^b_\gamma$, $\psi^a_\gamma\psi^b_\gamma$ are constant transformations on I_γ and Λ_γ, respectively. Let $\langle\phi^a_\gamma\phi^b_\gamma\rangle = u, \langle\psi^a_\gamma\psi^b_\gamma\rangle = v$, we have

$$\phi^{(u,v)}_\delta = \phi^a_\delta\phi^b_\delta, \psi^{(u,v)}_\delta = \psi^a_\delta\psi^b_\delta.$$

(iv) If $a, b \in Q$, $ab \in Q$, $\varphi(a) \in E_\alpha$, $\varphi(b) \in E_\beta$, and $\delta \leqslant \gamma = \alpha\beta$, then

$$\phi^{ab}_\delta = \phi^a_\delta\phi^b_\delta, \psi^{ab}_\delta = \psi^a_\delta\psi^b_\delta.$$

We now write $\sum = Q\dot{\bigcup} \dot{\bigcup}_{\alpha\in Y} E_\alpha$ and define an operation $*$ on $\sum$ as follows:

a) If $a, b \in Q$ and $ab \in Q$, then $a * b = ab$. If $a, b \in Q$, $\varphi(a) \in E_\alpha$, $\varphi(b) \in E_\beta$ and $\alpha\beta = \gamma$, but $ab \notin Q$, then

$$a * b = (\langle\phi^a_\gamma\phi^b_\gamma\rangle, \langle\psi^a_\gamma\psi^b_\gamma\rangle).$$

b) If $e \in E_\alpha$, $a \in Q$, $\varphi(a) \in E_\beta$, and $\alpha\beta = \gamma$, then

$$a * e = (\langle\phi^a_\gamma\phi^e_\gamma\rangle, \langle\psi^a_\gamma\psi^e_\gamma\rangle),$$

$$e * a = (\langle\phi^e_\gamma\phi^a_\gamma\rangle, \langle\psi^e_\gamma\psi^a_\gamma\rangle).$$

c) If $e \in E_\alpha, f \in E_\beta$, and $\alpha\beta = \gamma$, then

$$e * f = (\langle\phi^e_\gamma\phi^f_\gamma\rangle, \langle\psi^e_\gamma\psi^f_\gamma\rangle).$$

We denote the above system consisting of $\sum$ and the operation $*$ on $\sum$ by $\sum = \sum(Q, \dot{\bigcup}_{\alpha\in Y} E_\alpha, \Psi, \Phi, \varphi)$. It is easy to show that $\sum = \sum(Q, \dot{\bigcup}_{\alpha\in Y} E_\alpha, \Psi, \Phi, \varphi)$ is a semigroup, i.e. the above operation $*$ on $\sum$ is associative.

Theorem 4 [18]. *Let S be a semigroup. Then S is an E-ideal quasi-regular semigroup if and only if S is isomorphic to some semigroup of type $\sum = \sum(Q, \dot{\bigcup}_{\alpha\in Y} E_\alpha, \Psi, \Phi, \varphi)$.*

3. Δ-products and generalized Δ-products

A regular semigroup S is called a *left C-semigroup* (in short, LC-semigroup) if for any $a \in S$, $aS \subseteq Sa$. In 1991, Zhu, Guo and Shum in [19] gave the following characterizations for left C-semigroups.

Theorem 5 [19]. *Suppose that S is an orthodox semigroup with a band E of idempotents. Then the following statements on S are equivalent:*

(i) S is a left C-semigroup;

(ii) $(\forall e \in E)\ eS \subseteq Se$;

(iii) $(\forall e \in E)(\forall a \in S)\ eae = ea$;

(iv) $\mathcal{D}^S \cap (E \times E) = \mathcal{L}^E$;

(v) S is a semilattice of left groups;

(vi) $\mathcal{L} = \mathcal{J}$ is a semilattice congruence on S.

In studying of structure theory of left C-semigroups, Guo, Ren and Shum [20] introduced the concept of Δ-products of semigroups as follows.

Let Y be a semilattice. Let $T = \bigcup_{\alpha \in Y} T_\alpha$ be a semilattice of semigroups T_α and $I = \bigcup_{\alpha \in Y} I_\alpha$ be a semilattice partition of the set I on the semilattice Y. For each $\alpha \in Y$, write $S_\alpha = T_\alpha \times I_\alpha$; for any $\alpha, \beta \in Y, \alpha \geqslant \beta$, define the mapping

$$\Psi_{\alpha,\beta} : S_\alpha \longrightarrow \mathcal{T}_{I_\beta},$$

$$a \mapsto \psi^a_{\alpha,\beta},$$

satisfying the following conditions:

(P_1) If $(u,i) \in S_\alpha,\ i' \in I_\alpha$, then $\psi^{(u,i)}_{\alpha,\alpha} i' = i$.

(P_2) If $(u,i) \in S_\alpha,\ (v,j) \in S_\beta$, then

(a) $\psi^{(u,i)}_{\alpha,\alpha\beta}\psi^{(v,j)}_{\beta,\alpha\beta}$ are constant value mappings on $I_{\alpha\beta}$, denote the value by $\langle\psi^{(u,i)}_{\alpha,\alpha\beta}\psi^{(v,j)}_{\beta,\alpha\beta}\rangle$;

(b) if $\alpha\beta \geqslant \delta$, $\langle\psi^{(u,i)}_{\alpha,\alpha\beta}\psi^{(v,j)}_{\beta,\alpha\beta}\rangle = k$, we have $\psi^{(uv,k)}_{\alpha\beta,\delta} = \psi^{(u,i)}_{\alpha,\delta}\psi^{(v,j)}_{\beta,\delta}$.

Define a multiplication on the set S by

$$(u,i) * (v,j) = (uv, \langle\psi^{(u,i)}_{\alpha,\alpha\beta}\psi^{(v,j)}_{\beta,\alpha\beta}\rangle), \quad (u,i) \in S_\alpha, (v,j) \in S_\beta.$$

where uv is the product of u and v in the semigroup T.

It is easy to verify that $S = \bigcup_{\alpha \in Y} S_\alpha$ with the operation above forms a semigroup. The semigroup S constructed above is called a Δ-product of a semigroup T and a set I with respect to semilattice Y, denoted by $S = T\Delta_{Y,\Psi}I$.

Theorem 6 [21]. *Let $T = [Y; G_\alpha, \varphi_{\alpha,\beta}]$ be a strong semilattice of group G_α, and let $I = \bigcup_{\alpha \in Y} I_\alpha$ be a semilattice decomposition of a left regular band I for left zero bands I_α. Then the Δ-product $S = T\Delta_{Y,\Psi}I$ of T and I with respect to Y is a LC-semigroup.*

Conversely, every LC-semigroup S can be constructed in this way.

According to [22], a quasi-regular semigroup S is called a C^**-quasiregular semigroup* if for any $e \in E(S)$, the mapping $\psi_e : S^1 \to eS^1e$ defined by $x \mapsto exe$ is a semigroup homomorphism and $RegS$ is an ideal of S.

Some characterizations of C^*-quasiregular semigroups were given by Shum, Ren and Guo in [22].

Theorem 7 [22]. *The following statements are equivalent for a semigroup S:*

(i) S is a C^*-quasiregular semigroup;

(ii) S is a quasi-completely regular semigroup in which $RegS$ is an ideal of S and $E(S)$ is a regular band;

(iii) S is a quasi-completely regular semigroup such that $eS \cup Se \subseteq RegS$ and the mapping $\varphi_e : E(S) \to eE(S)e$ defined by $f \mapsto efe$ is a semigroup homomorphism for all $e \in E(S)$;

(iv) S is a semilattice of quasi-rectangular groups such that

$$(\forall a \in S)(\exists m \in N)\ a^m S \cup Sa^m \subseteq RegS$$

and $E(S)$ is a regular band;

(v) S is a nil-extension of a quasi-C-semigroup.

It is well-known that the structure of completely regular semigroups can be described by the translational hull of semigroups (see M. Petrich in [23]). Inspired by the work of M. Petrich, we can also construct quasi-completely regular semigroups by using translations on semigroups.

To obtain structure of C^*-quasiregular semigroups, we consider a more general construction for semigroups rather than the Δ-product structure. We call this new structure the generalized Δ-product structure.

We first cite the following concepts.

A mapping θ from a power breaking partial semigroup Q to another one is called a partial homomorphism if $(ab)\theta = a\theta b\theta$, whenever $a, b, ab \in Q$.

Write $\mathcal{T}(X)(\mathcal{T}^*(X))$ for the semigroup of all left (right) transformations on a set X. Also, we use the symbol $\langle\varphi\rangle$ to denote the value of a constant mapping φ acting on the set X.

We are now ready to state the definition of generalized Δ-product of semigroups.

(I) Let τ be a partial homomorphism from a power breaking partial semigroup Q to a semilattice Y; write $Q_\alpha = \tau^{-1}(\alpha)$, for any $\alpha \in Y$.

(II) Let $T = [Y, T_\alpha, \xi_{\alpha\beta}]$ be a strong semilattice of semigroups T_α, where $\xi_{\alpha\beta}$ is the structure homomorphism. Let $I = \bigcup_{\alpha\in Y} I_\alpha$ and $\Lambda = \bigcup_{\alpha\in Y} \Lambda_\alpha$ be a semilattice partition for the set I and for the set Λ on the semilattice Y, respectively. It is well-known that if T_α are groups, then the strong semilattice $T = [Y, T_\alpha, \xi_{\alpha\beta}]$ is a Clifford semigroup.

For every $\alpha \in Y$, form the following three sets, namely, the sets

$$S_\alpha^0 = Q_\alpha \cup T_\alpha,$$
$$S_\alpha^\ell = Q_\alpha \cup (I_\alpha \times T_\alpha),$$
$$S_\alpha^r = Q_\alpha \cup (T_\alpha \times \Lambda_\alpha).$$

(III) For any $\alpha, \beta \in Y$ with $\alpha \geq \beta$, define the mapping

$$\theta_{\alpha,\beta} : S_\alpha^0 \to T_\beta \text{ by } a \mapsto a\theta_{\alpha,\beta},$$

and we require that $\theta_{\alpha,\beta}$ satisfies the following condition.

(P1) (i) $\theta_{\alpha,\beta}|_{T_\alpha} = \xi_{\alpha,\beta}$.

(ii) if $a \in Q_\alpha$ and $\alpha \geqslant \beta \geqslant \gamma$, then $a\theta_{\alpha,\beta}\theta_{\beta\gamma} = a\theta_{\alpha,\gamma}$.

(iii) if $a \in Q_\alpha$, $b \in Q_\beta$ and $ab \in Q_{\alpha\beta}$ with $\alpha\beta \geqslant \delta$, then $(ab)\theta_{\alpha\beta,\delta} = a\theta_{\alpha,\delta} b\theta_{\beta,\delta}$.

(IV) For $\alpha, \beta \in Y$ with $\alpha \geqslant \beta$, define the following two mappings $\varphi_{\alpha,\beta}$ and $\psi_{\alpha,\beta}$:

$$\varphi_{\alpha,\beta} : S_\alpha^\ell \to \mathcal{T}(I_\beta) \quad \text{by} \quad a \mapsto \varphi_{\alpha,\beta}^a;$$
$$\psi_{\alpha,\beta} : S_\alpha^r \to \mathcal{T}^*(\Lambda_\beta) \quad \text{by} \quad a \mapsto \psi_{\alpha,\beta}^a.$$

Let $\varphi_{\alpha,\beta}$ and $\psi_{\alpha,\beta}$ satisfy the following conditions (P1), (P2), (P2)* and (P3)*, respectively.

(P2) If $(i, g) \in I_\alpha \times T_\alpha$ and $j \in I_\alpha$, then $\varphi_{\alpha,\alpha}^{(i,g)} j = i$.

(P2)* If $(g, \lambda) \in T_\alpha \times \Lambda_\alpha$ and $\mu \in \Lambda_\alpha$, then $\mu\psi_{\alpha,\alpha}^{(g,\lambda)} = \lambda$.

For the sake of convenience, we write $(i, g)\theta_{\alpha,\beta} = g\theta_{\alpha,\beta}$ and $(g, \lambda)\theta_{\alpha,\beta} = g\theta_{\alpha,\beta}$ for any $(i, g) \in I_\alpha \times T_\alpha$ and $(g, \lambda) \in T_\alpha \times \Lambda_\alpha$.

(P3) Let α, β and $\delta \in Y$ with $\alpha\beta \geqslant \delta$.

(i) If $a \in S_\alpha^\ell$, $b \in S_\beta^\ell$, and $ab \in Q_{\alpha\beta}$, then $\varphi_{\alpha\beta,\delta}^{ab} = \varphi_{\alpha,\delta}^a \varphi_{\beta,\delta}^b$.

(ii) If $a \in S_\alpha^\ell$, $b \in S_\beta^\ell$ and $ab \notin Q_{\alpha\beta}$, then $\varphi_{\alpha,\alpha\beta}^a \varphi_{\beta,\alpha\beta}^b$ is a constant mapping acting on the set $I_{\alpha\beta}$.

Let $k = \langle \varphi_{\alpha,\alpha\beta}^a \varphi_{\beta,\alpha\beta}^b \rangle$ be the constant value of $\varphi_{\alpha,\alpha\beta}^a \varphi_{\beta,\alpha\beta}^b$ and $g = a\theta_{\alpha,\alpha\beta} b\theta_{\beta,\alpha\beta}$. Then

$$\varphi_{\alpha\beta,\delta}^{(k,g)} = \varphi_{\alpha,\delta}^a \varphi_{\beta,\delta}^b.$$

(P3)* Let α, β and $\delta \in Y$ with $\alpha\beta \geqslant \delta$.

(i) If $a \in S^r_\alpha$, $b \in S^r_\beta$, and $ab \in Q_{\alpha\beta}$, then $\psi^{ab}_{\alpha\beta,\delta} = \psi^a_{\alpha,\delta}\psi^b_{\beta,\delta}$.

(ii) If $a \in S^r_\alpha$, $b \in S^r_\beta$, and $ab \notin Q_{\alpha\beta}$, then $\psi^a_{\alpha,\alpha\beta}\psi^b_{\beta,\alpha\beta}$ is a constant mapping acting on the set $\Lambda_{\alpha\beta}$.

Let $u = \langle\psi^a_{\alpha,\alpha\beta}\psi^b_{\beta,\alpha\beta}\rangle$ be the constant value of $\psi^a_{\alpha,\alpha\beta}\psi^b_{\beta,\alpha\beta}$ and $\nu = a\theta_{\alpha,\alpha\beta}b\theta_{\beta,\alpha\beta}$. Then

$$\psi^{(u,\nu)}_{\alpha\beta,\delta} = \psi^a_{\alpha,\delta}\psi^b_{\beta,\delta}.$$

(V) Now, form the set $S = \bigcup_{\alpha\in Y} S_\alpha = \bigcup_{\alpha\in Y} (Q_\alpha \cup (I_\alpha \times T_\alpha \times \Lambda_\alpha))$ and define a binary operation "$*$" on S satisfying the following conditions.

[M1] If $a \in Q_\alpha$, $b \in Q_\beta$, and $ab \in Q_{\alpha\beta}$, then $a * b = ab$.

[M2] If $a \in Q_\alpha$, $b \in Q_\beta$, and $ab \notin Q_{\alpha\beta}$, then

$$a * b = (\langle\varphi^a_{\alpha,\alpha\beta}\varphi^b_{\beta,\alpha\beta}\rangle, a\theta_{\alpha,\alpha\beta}b\theta_{\beta,\alpha\beta}, \langle\psi^a_{\alpha,\alpha\beta}\psi^b_{\beta,\alpha\beta}\rangle).$$

[M3] If $a \in Q_\alpha$, $(i,g,\lambda) \in I_\beta \times T_\beta \times \Lambda_\beta$, then

$$a * (i,g,\lambda) = (\langle\varphi^a_{\alpha,\alpha\beta}\varphi^{(i,g)}_{\beta,\alpha\beta}\rangle, a\theta_{\alpha,\alpha\beta}g\theta_{\beta,\alpha\beta}, \langle\psi^a_{\alpha,\alpha\beta}\psi^{(g,\lambda)}_{\beta,\alpha\beta}\rangle),$$

$$(i,g,\lambda) * a = (\langle\varphi^{(i,g)}_{\beta,\alpha\beta}\varphi^a_{\alpha,\alpha\beta}\rangle, g\theta_{\beta,\alpha\beta}a\theta_{\alpha,\alpha\beta}, \langle\psi^{(g,\lambda)}_{\beta,\alpha\beta}\psi^a_{\alpha,\alpha\beta}\rangle).$$

[M4] If $(i,g,\lambda) \in I_\alpha \times T_\alpha \times \Lambda_\alpha$, $(j,h,\mu) \in I_\beta \times T_\beta \times \Lambda_\beta$, then

$$(i,g,\lambda) * (j,h,\mu) = (\langle\varphi^{(i,g)}_{\alpha,\alpha\beta}\varphi^{(j,h)}_{\beta,\alpha\beta}\rangle, g\xi_{\alpha,\alpha\beta}h\xi_{\beta,\alpha\beta}, \langle\psi^{(g,\lambda)}_{\alpha,\alpha\beta}\psi^{(h,\mu)}_{\beta,\alpha\beta}\rangle).$$

It can be verified, by routine checking, that $(S,*)$ is a semigroup.

Now, we write $\Sigma = \{\varphi_{\alpha,\beta}, \psi_{\alpha,\beta}, \theta_{\alpha,\beta} \mid \alpha,\beta \in Y, \alpha \geqslant \beta\}$ and call it the structure mapping of the semigroup $S = \bigcup_{\alpha\in Y} (Q_\alpha \cup (I_\alpha \times T_\alpha \times \Lambda_\alpha))$.

Summarizing all the above steps, we give the following definition.

Definition 1 [22]. The above constructed semigroup S is called the generalized Δ-product of the power breaking partial semigroup Q, the semigroup T, the sets I and Λ with respect to the semilattice Y and the structure mapping Σ. Denote this semigroup by $S = \Delta_{Y,\Sigma}(Q,I,T,\Lambda)$.

Now, we state a construction theorem for a C^*-quasiregular semigroup.

Theorem 8 [22]. *Let Y be a semilattice, Q be a power breaking partial semigroup, $G = [Y, G_\alpha, \xi_{\alpha,\beta}]$ be a strong semilattice of groups G_α, $I = \bigcup_{\alpha\in Y} I_\alpha$, and $\Lambda = \bigcup_{\alpha\in Y} \Lambda_\alpha$ be a left regular band and a right regular band, respectively. Then the generalized Δ-product $\Delta_{Y,\Sigma}(Q,I,G,\Lambda)$ is a C^*-quasiregular semigroup.*

Conversely, every C^*-quasiregular semigroup can be constructed by a generalized Δ-product $\Delta_{Y,\Sigma}(Q,I,G,\Lambda)$.

4. Left wreath products and wreath products

In 1982, an abundant semigroup was first introduced and studied by J.B. Fountain [5]. To show the definition of an abundant semigroup, we first cite a set of relations called Green's star relations on a semigroup S:

$$\mathcal{L}^* = \{(a,b) \in S \times S \mid (\forall x,y \in S^1) ax = ay \Leftrightarrow bx = by\},$$
$$\mathcal{R}^* = \{(a,b) \in S \times S \mid (\forall x,y \in S^1) xa = ya \Leftrightarrow xa = yb\},$$

$$\mathcal{H}^* = \mathcal{L}^* \wedge \mathcal{R}^*,$$
$$\mathcal{D}^* = \mathcal{L}^* \vee \mathcal{R}^*,$$
$$\mathcal{J}^* = \{(a,b) \in S \times S \mid J^*(a) = J^*(b)\},$$

where $J^*(a)$ denote the principal *-ideal generated by the element a in S (see [5] and [24]).

Clearly, on any semigroup S we have $\mathcal{L} \subseteq \mathcal{L}^*$ and $\mathcal{R} \subseteq \mathcal{R}^*$. It is easy to see that for regular elements $a, b \in S$, $(a,b) \in \mathcal{L}^*$ if and only if $(a,b) \in \mathcal{L}$. Moreover, we can easily see that $\mathcal{L}^*$ is a right congruence and $\mathcal{R}^*$ is a left congruence on S, respectively.

An abundant semigroup S is a semigroup in which each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class contains an idempotent. It is clear that a regular semigroup is abundant. In fact, abundant semigroups can be regarded as natural generalizations of regular semigroups.

An abundant semigroup S is called an *$\mathcal{L}^*$-inverse semigroup* if S is an IC semigroup whose idempotents form a left regular band (for details, see [25]).

To obtain structure of $\mathcal{L}^*$-inverse semigroups, the concept of left wreath product of semigroups was introduced by Ren and Shum in [25].

Let Γ be a type-A semigroup with semilattice Y of idempotents. Let $B = \cup_{\alpha \in Y} B_\alpha$ be a semilattice decomposition of a left regular band B into left zero bands B_α.

Because the type-A semigroup Γ is abundant, we can always identify the element $\gamma \in \Gamma$ by its corresponding idempotent $\gamma^\dagger \in R^*_\gamma(\Gamma) \cap E$ or by $\gamma^* \in L^*_\gamma(\Gamma) \cap E$, respectively. Moreover, since the type-A semigroup Γ is also an IC abundant semigroup, there is a connecting isomorphism $\eta : \langle \omega^\dagger \rangle \to \langle \omega^* \rangle$ such that $\alpha\omega = \omega(\alpha\eta)$ for any $\alpha \in \langle \omega^\dagger \rangle$ and $\omega \in \Gamma$.

Now, we form the set $B \bowtie \Gamma = \{(e,\gamma) | \ e \in B_{\gamma^\dagger}, \gamma \in \Gamma\}$. In order to make this set $B \bowtie \Gamma$ a semigroup, we need to introduce a multiplication "$*$" defined on the set $B \bowtie \Gamma$ by the following mapping. Firstly, we define a mapping $\varphi : \Gamma \to End(B)$ by $\gamma \mapsto \sigma_\gamma$, where $\sigma_\gamma \in End(B)$, which is the endomorphism semigroup on B. This mapping satisfies the following properties.

(P1) Absorbing: for each $\gamma \in \Gamma$ and $\alpha \in Y$, we have $B_\alpha \sigma_\gamma \subseteq B_{(\gamma\alpha)^\dagger}$. In particular, if $\gamma \in Y$, then σ_γ is an inner endomorphism on B such that $e^{\sigma_\gamma} = fe$ for some $f \in B_\gamma$ and all $e \in B$.

(P2) Focusing: for α, $\beta \in \Gamma$ and $f \in B_{(\alpha\beta)^\dagger}$, we have $\sigma_\beta \sigma_\alpha \delta_f = \sigma_{\alpha\beta}\delta_f$, where δ_f is an inner endomorphism induced by f on B satisfying $h^{\delta_f} = fh$ for all $h \in B$.

(P3) Homogenizing: for $e \in B_{\omega^\dagger}$, $g \in B_{\tau^\dagger}$, and $h \in B_{\xi^\dagger}$, if $\omega\tau = \omega\xi$ and $eg^{\sigma_\omega} = eh^{\sigma_\omega}$, then $fg^{\sigma_{\omega^*}} = fh^{\sigma_{\omega^*}}$, for any $f \in B_{\omega^*}$.

(P4) Idempotent connecting: assume that for any $\omega \in \Gamma$, η is the connecting isomorphism which maps $\langle \omega^\dagger \rangle$ to $\langle \omega^* \rangle$ by $\alpha \mapsto \alpha\eta$. If $(e, \omega^\dagger)$ and $(f, \omega^*) \in B \bowtie \Gamma$, then there is a bijection $\theta : \langle e \rangle \to \langle f \rangle$ such that

(i) $e\theta = f$ and $g = e(g\theta)^{\sigma_\omega}$, for $g \in \langle e \rangle$;

(ii) for $g \in \langle e \rangle$ and $\alpha \in \langle \omega^\dagger \rangle$, $(g, \alpha) \in B \bowtie \Gamma$ if and only if $(g\theta, \alpha\eta) \in B \bowtie \Gamma$.

Equipped with the above mapping φ, we now define a multiplication "$*$" on $B \bowtie \Gamma$ by

$$(e, \omega) * (f, \tau) = (ef^{\sigma_\omega}, \omega\tau)$$

for any (e,ω), $(f,\tau) \in B \bowtie \Gamma$, where $f^{\sigma_\omega} = f\sigma_\omega$.

It can be verified that the multiplication "$*$" defined above for the set $B \bowtie \Gamma$ is associative. We call the semigroup a left wreath product of a left regular band B and a type-A semigroup Γ under a mapping φ, denoted by $B \bowtie_\varphi \Gamma$.

We are now going to establish a structure theorem for $\mathcal{L}^*$-inverse semigroups.

Theorem 9 [25, Theorem 4.1]. *A semigroup S is an $\mathcal{L}^*$-inverse semigroup if and only if S is a left wreath product of a left regular band B and a type-A semigroup Γ.*

In [26], we call an IC abundant semigroup S a $\mathcal{Q}^*$*-inverse semigroup* if the set of its idempotents E forms a regular band, i.e. E satisfies the identity $efege = efge$, for all e, f and g in E.

Suppose that S is a $\mathcal{Q}^*$-inverse semigroup whose set of idempotents E forms a regular band. Denote the $\mathcal{J}$-class containing the element $e \in E$ by $E(e)$. We first have the following result.

Theorem 10 [26, Theorem 3.2]. *If an equivalence relation δ on S is defined by $a\delta b$ if and only if $b = eaf$ and $a = gbh$ for some $e \in E(a^+)$, $f \in E(a^*)$, $g \in E(b^+)$ and $h \in E(b^*)$, then the equivalence relation δ is the smallest type-A good congruence on S.*

Let S be a $\mathcal{Q}^*$-inverse semigroup with a regular band of idempotents E. Define relations μ_l and μ_r on S as follows:

$$(a,b) \in \mu_l \Leftrightarrow (xa, xb) \in \mathcal{L}^* \quad (x \in E),$$

$$(a,b) \in \mu_r \Leftrightarrow (ax, bx) \in \mathcal{R}^* \quad (x \in E).$$

Put $\rho_1 = \delta \cap \mu_r$ and $\rho_2 = \delta \cap \mu_l$ on S (see [26]). We are now able to establish the following theorem for $\mathcal{Q}^*$-inverse semigroups.

To obtain structure theory for $\mathcal{Q}^*$-inverse semigroups, the concept of the *wreath product* of semigroups was introduced by Ren and Shum in [26] as follows.

In the *wreath product* of semigroups, we need the following ingredients:

(a) Y: a semilattice.

(b) Γ: a type-A semigroup whose set of idempotents is the semilattice Y.

(c) I: a left regular band such that $I = \bigcup_{\alpha \in Y} I_\alpha$, where I_α is a left zero band for all $\alpha \in Y$.

(d) Λ: a right regular band such that $\Lambda = \bigcup_{\alpha \in Y} \Lambda_\alpha$, where Λ_α is a right zero band for all $\alpha \in Y$.

We now form the following sets:

$$I \bowtie \Gamma = \{(e,\omega)|\ \omega \in \Gamma,\ e \in I_{\omega^+}\},$$

$$\Gamma \bowtie \Lambda = \{(\omega,i)|\ \omega \in \Gamma,\ i \in \Lambda_{\omega^*}\},$$

and

$$I \bowtie \Gamma \bowtie \Lambda = \{(e,\omega,i)|\ \omega \in \Gamma, e \in I_{\omega^+} \text{ and } i \in \Lambda_{\omega^*}\}.$$

Since $\omega \in \Gamma$ and Γ is a type-A semigroup, there are some idempotents $\omega^\dagger \in R^*_\omega(\Gamma) \cap E(\Gamma)$ and $\omega^* \in L^*_\omega(\Gamma) \cap E(\Gamma)$. Also since the set of idempotents of Γ forms a semilattice, $\omega^\dagger$ and ω^* are in Y. This illustrates that the sets $I \bowtie \Gamma$, $\Gamma \bowtie \Lambda$, and $I \bowtie \Gamma \bowtie \Lambda$ are well-defined. We only need to define an associative multiplication on the set $I \bowtie \Gamma \bowtie \Lambda$ so that the set $I \bowtie \Gamma \bowtie \Lambda$ under the multiplication turns out to be a semigroup.

Before we define a multiplication on $I \bowtie \Gamma \bowtie \Lambda$, we need to give a description for the structure mappings.

Define a mapping $\varphi : \Gamma \to End(I)$ by $\gamma \mapsto \sigma_\gamma$ for $\gamma \in \Gamma$ and $\sigma_\gamma \in End(I)$ satisfying the following conditions.

(P1) For each $\gamma \in \Gamma$ and $\alpha \in Y$, we have $I_\alpha \sigma_\gamma \subseteq I_{(\gamma\alpha)^\dagger}$. In particular, if $\gamma \in Y$, then σ_γ is an inner endomorphism on I such that there exists $g \in I_\gamma$ with $e^{\sigma_\gamma} = ge$, for all $e \in I$, where e^{σ_γ} denotes $e\sigma_\gamma$.

(P2) For α, $\beta \in \Gamma$ and $f \in I_{(\alpha\beta)^\dagger}$, we have $\sigma_\beta \sigma_\alpha \delta_f = \sigma_{\alpha\beta}\delta_f$, where δ_f is an inner endomorphism induced by f on I satisfying $h^{\delta_f} = fh = fhf$, for all $h \in I$.

(P3) For $e \in I_{\omega^\dagger}$, $g \in I_{\tau^\dagger}$ and $h \in I_{\xi^\dagger}$, if $\omega\tau = \omega\xi$ and $eg^{\sigma_\omega} = eh^{\sigma_\omega}$, then $fg^{\sigma_{\omega^*}} = fh^{\sigma_{\omega^*}}$, for all $f \in I_{\omega^*}$.

(P4) Assume that for any $\omega \in \Gamma$, η is the connecting isomorphism which maps $\langle \omega^\dagger \rangle$ to $\langle \omega^* \rangle$ by $\alpha \mapsto \alpha\eta$. If $(e, \omega^\dagger)$ and $(f, \omega^*) \in I \bowtie \Gamma$, then there is a bijection $\theta : \langle e \rangle \to \langle f \rangle$ such that

(i) $e\theta = f$ and $ge^{\sigma_\alpha} = e(g\theta)^{\sigma_\omega}$ for any $g \in \langle e \rangle$ and $\alpha \in \langle \omega^\dagger \rangle$;

(ii) for any $g \in \langle e \rangle$ and $\alpha \in \langle \omega^\dagger \rangle$, $(g, \alpha) \in I \bowtie \Gamma$ if and only if $(g\theta, \alpha\eta) \in I \bowtie \Gamma$.

Similarly, define a mapping $\psi : \Gamma \to End(\Lambda)$ by $\gamma \mapsto \rho_\gamma$ for $\gamma \in \Gamma$ and $\rho_\gamma \in End(\Lambda)$ satisfying the following conditions.

(P1)$'$ For each $\gamma \in \Gamma$ and $\alpha \in Y$, we have $\Lambda_\alpha \rho_\gamma \subseteq \Lambda_{(\alpha\gamma)^*}$. In particular, if $\gamma \in Y$, then ρ_γ is an inner endomorphism on Λ such that there exists $i \in \Lambda_\gamma$ with $j^{\rho_\gamma} = ji$ for all $j \in \Lambda$, where j^{ρ_γ} denotes $j\rho_\gamma$.

(P2)$'$ For $\alpha, \beta \in \Gamma$ and $i \in \Lambda_{(\alpha\beta)^*}$, we have $\rho_\alpha\rho_\beta\varepsilon_i = \rho_{\alpha\beta}\varepsilon_i$, where ε_i is an inner endomorphism induced by i on Λ such that $j^{\varepsilon_i} = ji = iji$ for any $j \in \Lambda$.

(P3)$'$ For $i \in \Lambda_{\omega^*}$, $j \in \Lambda_{\tau^*}$, and $k \in \Lambda_{\xi^*}$, if $\tau\omega = \xi\omega$ and $j^{\rho_\omega} i = k^{\rho_\omega} i$, then $j^{\rho_{\omega^\dagger}} m = k^{\rho_{\omega^\dagger}} m$ for all $m \in \Lambda_{\omega^\dagger}$.

(P4)$'$ Assume that for any $\omega \in \Gamma$, η is the connecting isomorphism which maps $\langle \omega^\dagger \rangle$ to $\langle \omega^* \rangle$ by $\alpha \mapsto \alpha\eta$. If $(\omega^\dagger, j)$ and $(\omega^*, i) \in \Gamma \bowtie \Lambda$, then there is a bijection $\theta' : \langle i \rangle \to \langle j \rangle$ such that the following conditions hold:

(i) $j\theta' = i$, $k^{\rho_\omega} i = i^{\rho_{\alpha\eta}}(k\theta')$, for any $k \in \langle j \rangle$ and $\alpha \in \langle \omega^\dagger \rangle$;

(ii) for any $k \in \langle i \rangle$ and $\alpha \in \langle \omega^\dagger \rangle, (\alpha, k) \in \Gamma \bowtie \Lambda$ if and only if $(\alpha\eta, k\theta') \in \Gamma \bowtie \Lambda$.

After gluing up the above components I, Γ and Λ together with the mappings φ and ψ, we now define a multiplication on the set $I \bowtie \Gamma \bowtie \Lambda$ by

$$(e, \omega, i) * (f, \tau, j) = (ef^{\sigma_\omega}, \omega\tau, i^{\rho_\tau} j), \tag{3}$$

for any $(e, \omega, i), (f, \tau, j) \in I \bowtie \Gamma \bowtie \Lambda$, where $f^{\sigma_\omega} = f\sigma_\omega$ and $i^{\rho_\tau} = i\rho_\tau$.

Using the properties $(P1)$, $(P2)$, $(P1)'$ and $(P2)'$, we can easily verify that the above multiplication "$*$" on $I \bowtie \Gamma \bowtie \Lambda$ is associative. We now call the above constructed semigroup a *wreath product* of I, Γ and Λ with respect to φ and ψ, and denote it by $Q = I \bowtie_\varphi \Gamma \bowtie_\psi \Lambda$.

Theorem 11 [26, Theorem 4.4]. *The wreath product $I \bowtie_\varphi \Gamma \bowtie_\psi \Lambda$ of a left regular band I, a type-A semigroup Γ and a right regular band Λ with respect to the mappings φ and ψ is a $\mathcal{Q}^*$-inverse semigroup.*

Conversely, every $\mathcal{Q}^*$-inverse semigroup S can be expressed by a wreath product of $I \bowtie_\varphi \Gamma \bowtie_\psi \Lambda$.

Remark 1. The class of $\mathcal{Q}^*$-inverse semigroups contains several interesting classes of semigroups as its special subclasses. We only discuss some of these special subclasses as follows.

(a) $\mathcal{L}^*$-inverse semigroups and $\mathcal{R}^*$-inverse semigroups

By Theorem 11, a $\mathcal{Q}^*$-inverse semigroup S can be expressed as a wreath product $I \bowtie_\varphi \Gamma \bowtie_\psi \Lambda$ of I, Γ and Λ with respect to the mappings φ and ψ, where Γ is a type-A semigroup, I and Λ are respectively a left regular band and a right regular band. In Theorem 11, if $\Lambda = \emptyset$, then $S = I \bowtie_\varphi \Gamma$, which is an $\mathcal{L}^*$-inverse semigroup. Similarly, if we let $I = \emptyset$, then $\Gamma \bowtie_\psi \Lambda$ becomes an $\mathcal{R}^*$-inverse semigroup. Thus, the class of $\mathcal{L}^*$-inverse semigroups and the class of $\mathcal{R}^*$-inverse semigroups are two special subclasses of the class of $\mathcal{Q}^*$-inverse semigroups. In this case, we can easily re-obtain Theorem 10 for structure of $\mathcal{L}^*$-inverse semigroups, as a corollary of Theorem 11.

(b) Quasi-inverse semigroups

We know that a quasi-inverse semigroup is a regular semigroup whose set of idempotent forms a regular band. It is clear that a quasi-inverse semigroup is a special $\mathcal{Q}^*$-inverse semigroup.

When S is a quasi-inverse semigroup, we can define a relation δ on S by $a\delta b$ if and only if $b = eaf$ for some $e \in E(aa')$ and $f \in E(a'a)$, where a' is an inverse element of a. It can be immediately seen from [25] that δ is the smallest inverse semigroup congruence on S, and so $\Gamma = S/\delta$ is the greatest inverse semigroup homomorphism image of S. Obviously, the inverse semigroup $\Gamma = S/\delta$ must be a type-A semigroup whose set of idempotents forms a semilattice. As a result, a wreath product $I \bowtie_\varphi \Gamma \bowtie_\psi \Lambda$ of S, regarded as a Q^*-inverse semigroup, can be simplified by using the so-called *half-direct product* (in short, $H.D.$-product) of a quasi-inverse semigroup given by M. Yamada in [27] as described in the following Theorem 12.

Theorem 12 [27, Theorem 6]. *Let S be a quasi-inverse semigroup whose set of idempotents forms a regular band E. Let δ be the smallest inverse congruence on S such that $\Gamma = S/\delta$ is the greatest inverse semigroup induced by δ, and let Y be the semilattice of Γ. Define the congruences η_1, η_2 on E by $e\eta_1 f$ if and only if $e\mathcal{R}f$; $e\eta_2 f$ if and only if $e\mathcal{L}f$, respectively.*

For $X \subseteq E$, write $\widetilde{X} = \{\widetilde{e} \mid e \in X\}$ and $\widehat{X} = \{\widehat{e} \mid e \in X\}$, where $\widetilde{e}$ and $\widehat{e}$ are the η_1-class and the η_2-class containing $e \in X$, respectively. Then the following statements hold:

(i) $E/\eta_1 = \widetilde{E}$ is a left regular band such that $\widetilde{E} = \bigcup_{\alpha\in Y} \widetilde{E_\alpha}$, where every $\widetilde{E}_\alpha$ is a left zero band; $E/\eta_2 = \widehat{E}$ is a right regular band such that $\widehat{E} = \bigcup_{\alpha\in Y} \widehat{E}_\alpha$, where each $\widehat{E}_\alpha$ is a right zero band, for every $\alpha \in Y$.

(ii) S is isomorphic to an H.D.-product of $\widetilde{E}, \Gamma$ and $\widehat{E}$ with respect to the mappings φ' and ψ', respectively. Conversely, any H.D.-product of a left regular band $I = \bigcup_{\alpha\in Y} \Lambda_\alpha$, an inverse semigroup Γ and a right regular band $\Lambda = \bigcup_{\alpha\in Y} \Lambda_\alpha$ with respect to the mappings φ' and ψ' is a quasi-inverse semigroup S, where Γ is the greatest inverse semigroup homomorphic image of S and Y is the semilattice of idempotents of Γ.

In 1958, Kimura first considered [7] the spined product of semigroups as follows. If S and T are two semigroups having a common homomorphic image H, and if $\phi : S \to H$ and $\psi : T \to H$ are homomorphisms onto H, then the spined product of S and T with respect to H, ϕ and ψ is defined by

$$Y = \{(s,t) \in S \times T \mid s\phi = t\psi\}.$$

For $\mathcal{Q}^*$-inverse semigroups, we have also the following another constructions.

Theorem 13 [26, Theorem 5.1]. *A semigroup S is a $\mathcal{Q}^*$-inverse semigroup if and only if S is a spined product of an $\mathcal{L}^*$-inverse semigroup $S_1 = I \bowtie_\varphi \Gamma$ and an $\mathcal{R}^*$-inverse semigroup $S_2 = \Gamma \bowtie_\psi \Lambda$ with respect to a type-A semigroup Γ.*

5. Semi-spined product of semigroups

In generalizing regular semigroups, apart from weakening the definition of regularity, one of the most suitable approach is to modify the usual Green's relations on semigroups. During the recent 40 years, a series of generalized Green's relations have been established, such as $(*)$-Green's relations, (l)-Green's relations, $(*,\sim)$-Green's relations, $(\sim,\sim)$-Green's relations and $(\sim_U)$-Green's relations (see [28]). In this section, we only introduce $(*,\sim)$-Green's relations.

According to Fountain, a semigroup S is *rpp* if all of its principal right ideals $aS^1(a \in S)$, regarded as the right S^1-systems, are projective (see [29] and [24]). It was shown in [24] that a semigroup S is *rpp* if and only if for any $a \in S$, the set

$$\mathcal{M}_a \stackrel{d}{=} \{e \in E(S) | S^1 a \subseteq Se \ \ \& \ \ Ker\, a_l \subseteq Ker\, e_l\}$$

is non-empty, where $E(S)$ is the set of all idempotents of S. An *rpp* semigroup is said to be strongly *rpp* [20] if

$$(\forall a \in S) \ \ (\exists!\ e \in \mathcal{M}_a) \ \ ea = a.$$

In [20] and [30], (l)-Green's relations have been applied to the study of *rpp* semigroups, especially strongly *rpp* semigroups. However, strongly *rpp* semigroups do not form a satisfactory generalization of completely regular semigroups in the class of *rpp* semigroups.

In order to get a satisfactory generalization of completely regular semigroups in the class of *rpp* semigroups, Guo, Shum and Gong [31] introduced the so-called $(*, \sim)$-Green's relations on a semigroup S:

$$\mathcal{L}^{*,\sim} \stackrel{d}{=} \mathcal{L}^*,$$

$$\mathcal{R}^{*,\sim} \stackrel{d}{=} \widetilde{\mathcal{R}},$$

$$\mathcal{H}^{*,\sim} \stackrel{d}{=} \mathcal{L}^{*,\sim} \wedge \mathcal{R}^{*,\sim} = \mathcal{L}^{*,\sim} \cap \mathcal{R}^{*,\sim},$$

$$\mathcal{D}^{*,\sim} \stackrel{d}{=} \mathcal{L}^{*,\sim} \vee \mathcal{R}^{*,\sim},$$

$$a\mathcal{J}^{*,\sim} b \stackrel{d}{\Longleftrightarrow} \text{“}J^{*,\sim}(a) = J^{*,\sim}(b)\text{”},$$

where, for any $a, b \in S$,

$$a\widetilde{\mathcal{R}}b \stackrel{d}{\Longleftrightarrow} (\forall e \in E(S))\text{“}ea = a \longleftrightarrow eb = b\text{” [15]},$$

and $J^{*,\sim}(a)$ is the smallest ideal containing and saturated by $\mathcal{L}^{*,\sim}$ and $\mathcal{R}^{*,\sim}$.

Let S be a semigroup and $\mathcal{E}(S)$ the lattice of all equivalences on S. For any $\sigma \in \mathcal{E}(S)$, call $A \subseteq S$ a *subset saturated by* σ if A is a union of some σ-classes of S; call S σ*-abundant* if every σ-class of S contains idempotents of S.

A semigroup S is called *r-wide* [31] if S is $\mathcal{L}^{*,\sim}$-abundant and $\mathcal{R}^{*,\sim}$-abundant. An r-wide semigroup is called a *super-r-wide semigroup* [31] if S is $*, \sim$-abundant. Call a semigroup S an *ortho-lc-monoid* if S is a super-r-wide semigroup with $E(S) \leq S$ [31]. An ortho-lc-monoid S is called a *regular ortho-lc-monoid* if $E(S)$ forms a regular band. It is clear that an ortho-lc-monoid is strongly *rpp*, and each $\mathcal{H}^{*,\sim}$-class is a left cancellative monoid (in short, lc-monoid).

For $(*, \sim)$-Green's relations, we have the following results.

Theorem 14 [31]. *On a semigroup S, we have*

(i) $\mathcal{R}^{*,\sim}$ *is usually not a left congruence even if S is an* $\mathcal{R}^{*,\sim}$*-abundant semigroup.*

(ii) In general, we have $\mathcal{L}^{*,\sim} \circ \mathcal{R}^{*,\sim} \neq \mathcal{R}^{*,\sim} \circ \mathcal{L}^{*,\sim}$.

But

(iii) If S is super-r-wide, then $\mathcal{R}^{*,\sim}$ *is a left congruence and* $\mathcal{D}^{*,\sim} = \mathcal{L}^{*,\sim} \circ \mathcal{R}^{*,\sim}$ $(= \mathcal{R}^{*,\sim} \circ \mathcal{L}^{*,\sim}$, *of course).*

Thus,

(iv) When S is super-r-wide, the corresponding Green's Lemma holds for the $(*, \sim)$*-Green's relations.*

By using $(*,\sim)$-Green's relations, we can give some characterizations of super-r-wide semigroups and ortho-lc-monoids.

Theorem 15 [31]. *Let S be an r-wide semigroup. Then S is a super-r-wide semigroup if and only if S is a strongly rpp semigroup on which $*,\sim = \mathcal{L}^{*,\sim} \circ \mathcal{R}^{*,\sim} = \mathcal{R}^{*,\sim} \circ \mathcal{L}^{*,\sim}$ holds.*

Call that a semigroup is a *rectangular lc-monoid* if it is isomorphic to the direct product of a rectangular band and a left cancellative monoid.

Theorem 16 [31]. *A semigroup S is an ortho-lc-monoid if and only if S is rpp and a semilattice of rectangular lc-monoids.*

Based on the semilattice decomposition of ortho-lc-monoids, the semi-spined product structure of regular ortho-lc-monoids was provided in [31].

Let $M = [Y; M_\alpha, \varphi_{\alpha,\beta}]$ be an lc-Clifford semigroup, i.e. a strong semilattice Y of left cancellative monoids $M_\alpha' s$, $I = \bigcup_{\alpha\in Y} I_\alpha$ a semilattice decomposition of the left regular band I into left zero bands $I_\alpha{}'s$, and $\Lambda = \bigcup_{\alpha\in Y} \Lambda_\alpha$ a semilattice decomposition of the right regular band Λ into right zero bands $\Lambda_\alpha{}'s$. We define the following mappings

$$\delta : \Lambda = \bigcup_{\alpha\in Y} I_\alpha \times M_\alpha \longrightarrow \mathcal{T}_l(I)$$

and

$$\varepsilon : \bigcup_{\alpha\in Y} (M_\alpha \times \Lambda_\alpha) \longrightarrow \mathcal{T}_r(\Lambda)$$

satisfying the following conditions:

(P1) If $(i,a) \in I_\alpha \times M_\alpha$ and $j \in I_\beta$, then $\delta(i,a)j \in I_{\alpha\beta}$;

(Q1) If $(b,\mu) \in M_\alpha \times \Lambda_\alpha$ and $\lambda \in \Lambda_\beta$, then $\lambda\varepsilon(b,\mu) \in \Lambda_{\alpha\beta}$;

(P2) If $\alpha \le \beta$ holds in (P1) for $\alpha,\ \beta \in Y$, then $\delta(i,a)j = i$;

(Q2) If $\alpha \le \beta$ holds in (Q1) for $\alpha,\ \beta \in Y$, then $\lambda\varepsilon(b,\mu) = \mu$;

(P3) If $(i,a) \in I_\alpha \times M_\alpha$ and $(j,b) \in I_\beta \times M_\beta$, then

$$\delta(i,a)\delta(j,b) = \delta(\delta(i,a)j, a\varphi_{\alpha,\alpha\beta} b\varphi_{\beta,\alpha\beta});$$

(Q3) If $(a,\lambda) \in M_\alpha \times \Lambda_\alpha$ and $(b,\mu) \in M_\beta \times \Lambda_\beta$, then

$$\varepsilon(a,\lambda)\varepsilon(b,\mu) = \varepsilon(a\varphi_{\alpha,\alpha\beta} b\varphi_{\beta,\alpha\beta}, \lambda\varepsilon(b,\mu));$$

(P4) Let $(i,a) \in I_\alpha \times M_\alpha$, $j \in I_\beta$ and $k \in I_\gamma$. If

$$\delta(i,a)j = \delta(i,a)k,$$

then

$$\delta(i,1_\alpha)j = \delta(i,1_\alpha)k.$$

Define a multiplication "$\circ$" on the set

$$S = \bigcup_{\alpha\in Y} (I_\alpha \times M_\alpha \times \Lambda_\alpha)$$

by

$$(i,a,\lambda)(j,b,\mu) = (\delta(i,a)j, a\varphi_{\alpha,\alpha\beta} b\varphi_{\beta,\alpha\beta}, \lambda\varepsilon(b,\mu)). \tag{4}$$

It is easy to prove that S forms a semigroup under the multiplication (4).

Definition 2 [31]. The semigroup S constructed above is called the semi-spined product of the lc-Clifford semigroup $M = [Y; M_\alpha, \varphi_{\alpha,\beta}]$, the left regular band $I = \bigcup\limits_{\alpha \in Y} I_\alpha$ and the right regular band $\Lambda = \bigcup\limits_{\alpha \in Y} \Lambda_\alpha$ with respect to the semilattice Y and structure mappings δ and ε.

By using the above definition of semi-spined product of semigroups, we obtain the following structure theorem for regular ortho-lc-monoids.

Theorem 17 [31]. *The semi-spined product of semigroups described in Definition 2 is a regular ortho-lc-monoid. Conversely, every regular ortho-lc-monoid can be constructed in this manner.*

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Резюме

К.П. Шум, С.М. Рен, Ч.М. Гун. О методах построения полугрупп.

В статье представлен краткий обзор методов построения полугрупп с использованием структур некоторых полугрупп, относящихся к классам регулярных, квазирегулярных и избыточных полугрупп. В частности, приведены основные обозначения и структурные теоремы для некоторых полугрупп, таких как рисовские полугруппы матричного типа над 0-группой G^0 и их обобщения, связки, E-идеальные квазирегулярные полугруппы, $\mathcal{C}^*$-квазирегулярные полугруппы, $\mathcal{L}^*$-инверсные и $\mathcal{Q}^*$-инверсные полугруппы и регулярные орто-lc-моноиды.

Ключевые слова: регулярные полугруппы, квазирегулярные полугруппы, избыточные полугруппы, конструкции.

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