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GLOBAL WARMING IN THE LIGHT OF AN ANALYTIC MODEL OF THE EARTH'S ATMOSPHERE

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Abstract

Within a simplified atmospheric model the greenhouse effect is treated by analytic methods starting from physical first principles. The influence of solar radiation, absorption cross-sections of the greenhouse molecules, and convection on the earth's temperature is shown and discussed explicitly by mathematical formulae in contrast to the climate simulations. The analytic results are applied on the present atmospheric situation.

Key words: global warming, greenhouse gases, modeling of atmospheric processes, numerical simulation.

Introduction

It may be historically interesting that the influence of atmospheric absorbing molecules on the earth's temperature has been investigated already very early in 1827 by J. Fourier [1] and in 1838 by Cl. Pouillet [2] and especially later in 1896 by S. Arrhenius [3]. However, their results could be only preliminary because of the incomplete physical basis at that time. Today the change of the climate of the earth is treated usually by numerical simulations with the aim to take into account all imaginable influences in order to get a detailed picture of the behaviour of the climate, for instance in consequence of the production of greenhouse gases. But by this procedure the survey is lost. On the other hand, this is guaranteed, if one restricts oneself to an atmospheric model considering only the most important properties, which can be solved by analytic methods. This is the idea of my contribution, so that everybody with a sufficient knowledge in physics and in higher mathematics can understand qualitatively as well as quantitatively the behaviour of the atmosphere in consequence of an enlargement of its content of greenhouse gases; for this I start from "first principles" of physics. Moreover, by such an analytic way, which corresponds to A.Z. Petrov's intention, the influence of the solar radiation, the absorption cross-sections of the greenhouse molecules and of the gravitational field of the earth on the earth's temperature can be studied and discussed explicitly.

1. The model

In view of the solar constant, my model starts from a nearly constant mean energy flux J of the solar radiation on the surface of the earth; this radiation has short wave lengths ($\lambda_{\max} \simeq 4.8 \cdot 10^{-5}$ cm) and reaches the surface more or less immediately. In consequence of the absorption of this radiation the earth's surface will be heated and radiates infrared rays with wave lengths of around $1.7 \cdot 10^{-3}$ cm into the atmosphere. The mean temperature in the atmosphere may be T, its value at the surface T_0; the mean temperature T_E of the earth's surface itself will be determined separately in Section 3.

As long as the mean free path length l of the infrared photons is small compared with the thickness of the atmosphere as a consequence of absorption and re-emission by the molecules of the greenhouse gases, we have an energy transport by radiation in the form of diffusion of the infrared photons connected with a radiation energy flux density (1. Fick's law corresponding to the 2. law of thermodynamics):

$$\vec{j} = -\lambda \, \vec{\nabla} T \quad (\lambda > 0), \tag{1}$$

where

$$\lambda \sim l = \left(\sum_Q n_Q \, \sigma_Q \right)^{-1} \tag{2}$$

is the "radiation conductivity" of the atmosphere; n_Q is the number density of the molecules of the greenhouse gases; σ_Q is their effective absorption cross-section for infrared photons and Q indicates the different greenhouse gases. The absorption cross-section σ_Q is determined in the first step by the quantum mechanical transition probabilities and is given in spectral decomposition by (sharp line of frequency $\nu_{Qnm} = |E_{Qm} - E_{Qn}|/h$)

$$\sigma_Q(\nu) = \frac{2\pi}{3} \frac{|E_{Qm} - E_{Qn}|}{\hbar^2 c} |\vec{d}_{Qmn}|^2 \delta(\nu - \nu_{Qnm}) \tag{3}$$

with the dipole matrix element $\vec{d}_{Qmn}$ of the oscillating and rotating molecules. However, in (2) the effective absorption cross-section of the greenhouse molecules is needed, i.e. (3) must be additionally multiplied with the probability $w(E_{Qm}, T)$, so that the absorbing energy eigenstate E_{Qm} of the molecule is occupied. For the case of thermodynamic equilibrium w is given by the Boltzmann distribution. Furthermore, we average in the following the spectral absorption cross-section $\sigma_Q(\nu)$ with respect to the frequency range of the infrared radiation of the earth, so that only the relevant absorption lines are taken into account. Thus we get:

$$\sigma_Q(T) = \overline{\sigma_Q(\nu)}^\nu = \sum_{m,n} \int_0^\infty w_Q(E_{Qm}, T) \sigma_Q(\nu) I(\nu, T) \, d\nu \Big/ \int_0^\infty I(\nu, T) \, d\nu, \tag{4}$$

where $I(\nu, T)$ represents the radiation spectrum of the earth for the temperature T, which we approximate in good agreement with the observation by that of the black body radiation. Simultaneously, Eq. (4) shows that by introduction of the temperature-dependent effective absorption cross-section $\sigma_Q(T)$ the absorption of the single spectral lines is replaced by the continuous absorption of the black body radiation in such a way, that the absorbed energy is exactly the same for both cases. By this procedure, the calculation is extremely simplified compared with a calculation in spectral decomposition. However, even in that simple case we cannot calculate $\sigma_Q(T)$ "ab initio" directly according to (3) and (4). Therefore we approximate $\sigma_Q(T)$ within the relevant temperature range by the tangent of the real course in the double-logarithmic representation, i.e. by the power law

$$\sigma_Q(T) = \tilde{\sigma}_Q / T^\kappa, \quad \tilde{\sigma}_Q = \text{const} \tag{5}$$

with $\kappa \leq 4$ in view of the fact that the denominator in (4) is proportional to T^4 (Stefan − Boltzmann law); such an approximation is very reliable for a large temperature range. The exact value of the exponent κ, which is only very weekly dependent on Q, will be determined later by fitting of the results to the observation. In contrast to this, the radiation conductivity in (1) can be given exactly in the case of black body radiation following the consideration, that the (by $\sum_Q n_Q \, \sigma_Q(T)$) absorbed black radiation energy

flux density $\vec{j}$ produces the black radiation pressure, or more precisely its force density; in this way one finds [4]

$$\lambda = \frac{16}{3}\,\sigma l T^3, \tag{6}$$

where $\sigma = \pi^2 k^4/60c^2\hbar^3 = 5.67\cdot10^{-8}$ W/m^{2}K^4 is the Stefan$-$Boltzmann-constant. The energy flux density (1) goes over into free radiation propagation without scattering, if the free path length l of the infrared photons is sufficiently large; then the energy flux J of the sun will be re-emitted into the universe. Yet, before the balance equation is valid (radiation energy conservation)

$$J = \oint \mathbf{j}\,d\mathbf{f} \tag{7}$$

for every closed surface around the earth.

According to this model, the temperature distribution of the atmosphere is determined by the absorption and the thermalization of the infrared radiation. The collision time of the molecules in the atmosphere is shorter than the natural life-time of the excited energy-eigenstates by several orders of magnitude. Thus we suppose there is a local thermodynamic equilibrium in the atmosphere. Heat conductivity and convection are neglected in the first step; however, heat conductivity could be taken into account very easily by an additional term in (6) but does not play any role; convection will be treated subsequently in Section 4. On the other hand, the day-night change, the summer-winter differences and the variations with respect to the geographic altitude as well as the influences of winds and oceanic streams are neglected completely. For the case that the free path length of the infrared photons is comparable with the thickness of the atmosphere, the model loses its applicability.

2. Temperature, density and pressure distributions of the atmosphere

With respect to the conservation of the radiation energy the integral (7) is valid for any closed surface around the earth, for instance, for any sphere of radius r within the atmosphere. Herewith we find using (1), (2), (5), and (6):

$$\partial T/\partial r = T' = -\frac{3I\sum\limits_{Q}\widetilde{\sigma}_Q\,n_Q}{16\sigma\,r^2\,T^{3+\kappa}}, \quad I = J/4\pi. \tag{8}$$

Additionally, there exists hydrodynamical equilibrium in the atmosphere; i.e. the static Euler-equation is valid (differential barometric equation):

$$\vec{\nabla}p + \varrho\vec{\nabla}\phi = 0, \quad \phi = -\frac{MG}{r} \quad \text{(without self-gravitation of the air)}, \tag{9}$$

where M is the mass of the earth and G is the Newtonian gravitational constant. Finally we define the concentration of the greenhouse molecules as

$$x_Q = n_Q/n_L, \tag{10}$$

i.e. as the ratio between the number density n_Q of the greenhouse molecules in question and that of the air molecules n_L (78% N$_2$, 20% O$_2$). Now

$$n = n_L + \sum_Q n_Q \tag{11}$$

represents the total number density of molecules, according to which

$$n_L = n \Big/ \Big(1 + \sum_Q x_Q \Big), \quad n_Q = n x_Q \Big/ \Big(1 + \sum_{Q'} x_{Q'} \Big) \tag{12}$$

is valid, and for density ϱ and pressure p of the atmosphere we find, under the assumption of an ideal gas for the atmosphere,

$$\varrho = n \, \frac{m_L + \sum\limits_Q x_Q \, m_Q}{1 + \sum\limits_Q x_Q}, \quad p = nkT, \tag{13}$$

where k is the Boltzmann constant, m_Q is the mass of the greenhouse molecules in question, and m_L is the mean mass of the air molecules (mean molecular weight is 28.8). The influence of the small radiation pressure is neglected. Insertion of (13) in (9) gives additionally to (8) a second differential equation

$$n'kT + nkT' + \frac{m_L + \sum\limits_Q x_Q \, m_Q}{1 + \sum\limits_Q x_Q} \, MG \, \frac{n}{r^2} = 0 \tag{14}$$

assuming spherical symmetry. Here the concentration x_Q (see (10)) is considered as a constant parameter, which is in good agreement with the observations. From (8) and (14) both variables $n(r)$ and $T(r)$ are to be determined.

Solving (8) with respect to n/r^2 and inserting into (14) results in the exact differential equation

$$n'kT + nkT' - \frac{m_L + \sum\limits_Q x_Q \, m_Q}{\sum\limits_Q x_Q \, \widetilde{\sigma}_Q} \, \frac{16\sigma MG}{3I} \, T^{3+\kappa} \, T' = 0 \tag{15}$$

with the solution

$$n = \frac{a^2}{k} \, T^{3+\kappa} - \frac{A^2}{kT}, \tag{16}$$

where A^2 is the integration constant and a^2 has the meaning

$$a^2 = \frac{m_L + \sum\limits_Q x_Q \, m_Q}{\sum\limits_Q x_Q \, \widetilde{\sigma}_Q} \, \frac{4\sigma MG}{3(1 + \kappa/4)I}. \tag{17}$$

The sign of the integration constant A^2 is chosen in such a way, that n is a monotonic function of T and that the atmosphere possesses a well defined outer border ($n = 0$, $\varrho = 0$, $p = 0$) with the border temperature

$$T_G = \left(\frac{A^2}{a^2} \right)^{1/(4+\kappa)}. \tag{18}$$

With the boundary condition $T_G = 0$ it follows that $A^2 = 0$.

Insertion of (16) in (8) or (14) results immediately in the following differential equation for $T(r)$:

$$(4 + \kappa)kT' + \frac{m_L + \sum\limits_Q x_Q \, m_Q}{1 + \sum\limits_Q x_Q} \, \frac{MG}{r^2} = 0. \tag{19}$$

By separation of the variables we find with the integration constant B the solution:

$$kT = \frac{m_L + \sum\limits_Q x_Q \, m_Q}{(4 + \kappa)(1 + \sum\limits_Q x_Q)} \frac{MG}{r} - B. \tag{20}$$

Furthermore, we have the solution (16) in the form $(A^2 = 0)$:

$$n = \frac{a^2}{k} T^{3+\kappa}, \quad T_G = 0 \tag{21}$$

and, as border of the atmosphere $(T \to T_G = 0)$

$$R = r_{\max} = b^2 / B \tag{22}$$

with the gravitational coupling strength

$$b^2 = \frac{m_L + \sum\limits_Q x_Q \, m_Q}{(4 + \kappa)(1 + \sum\limits_Q x_Q)} MG \tag{23}$$

connected with a^2 (see (17)) according to

$$a^2 = \frac{1 + \sum\limits_Q x_Q}{\sum\limits_Q x_Q \, \widetilde{\sigma}_Q} b^2 \frac{16\sigma}{3I}. \tag{24}$$

For $x_Q = 0$ we find $(1 + \kappa/4)b^2 = 4.8 \cdot 10^{-3} \mathrm{g} \cdot \mathrm{cm}^3 \cdot \mathrm{sec}^{-2}$ $(M = 6 \cdot 10^{27}$ g), which is, in view of $x_Q \ll 1$ (trace gases), a very good approximation.

The remaining integration constant B in (20) will be determined finally by the total absorption cross-section $\sum\limits_Q \int n_Q \sigma_Q \, d^3x$ of all greenhouse molecules. From (5), (12), (20), and (21) it follows:

$$\sum_Q \int n_Q \sigma_Q \, d^3x = 4\pi b^2 \frac{16\sigma}{3k^4 I} \int\limits_{R_0}^{R} \left(\frac{b^2}{r} - B\right)^3 r^2 \, dr, \tag{25}$$

where R_0 is the radius of the earth. The calculation of the integral gives, after insertion of the upper limit according to (22),

$$\int\limits_{R_0}^{R} \left(\frac{b^2}{r} - B\right)^3 r^2 \, dr = b^6 [\ln \frac{b^2}{BR_0} - \frac{11}{6}] + 3b^4 BR_0 - \frac{3}{2}b^2(BR_0)^2 + \frac{1}{3}(BR_0)^3. \tag{26}$$

Herewith Eq. (25) must be solved with respect to B or BR_0. But this is not exactly possible in view of the logarithmic term in (26). Therefore, we make the ansatz

$$R = R_0(1 + \epsilon), \quad \epsilon > 0, \tag{27}$$

according to which (see (22))

$$BR_0 = b^2 / (1 + \epsilon), \tag{28}$$

and consider $\epsilon \ll 1$. The thickness of the atmosphere is, according to the experience, small compared with the earth's radius. Herewith we obtain from (26)

$$\int_{R_0}^{R} \left(\frac{b^2}{r} - B\right)^3 r^2 \, dr = \frac{1}{4} \, b^6 \epsilon^4 + O(\epsilon^5) \tag{29}$$

and (25) results in

$$\epsilon = \frac{k}{2b^2} \left(\frac{3I \sum\limits_{Q} \int n_Q \, \sigma_Q \, d^3x}{\pi \sigma}\right)^{1/4}, \tag{30}$$

whereby also BR_0 (see (28) and (31)) is given. The condition $\epsilon \ll 1$ means an upper limit for the total absorption cross-section of the greenhouse molecules (see below).

Now we are able to determine the radial behaviour of the atmosphere. From (20) and (28) it follows, with the use of (30),

$$BR_0 = b^2 \left[1 - \frac{k}{2b^2} \left(\frac{3I \sum\limits_{Q} \int n_Q \, \sigma_Q \, d^3x}{\pi \sigma}\right)^{1/4}\right] \tag{31}$$

and[1]

$$kT = b^2 \left[\frac{1}{r} - \frac{1}{R_0} + \frac{1}{R_0} \frac{k}{2b^2} \left(\frac{3I \sum\limits_{Q} \int n_Q \, \sigma_Q \, d^3x}{\pi \sigma}\right)^{1/4}\right] \tag{32}$$

with the temperature of the atmosphere at the earth's surface $T_0 = T(r = R_0)$:

$$T_0 = \frac{1}{2R_0} \left(\frac{3I \sum\limits_{Q} \int n_Q \, \sigma_Q \, d^3x}{\pi \sigma}\right)^{1/4}. \tag{33}$$

Inversely (33) reads in view of (30) and (23):

$$\epsilon = \frac{kT_0 R_0}{b^2} = \frac{(4 + \kappa)\left(1 + \sum\limits_{Q} x_Q\right) k T_0 R_0}{\left(m_L + \sum\limits_{Q} x_Q \, m_Q\right) M G}. \tag{34}$$

Because of $x_Q \ll 1$ it is possible to estimate the value of ϵ by (34); one finds, with $T_0 = 300$ K ($R_0 = 6370$ km),

$$\epsilon = 5.5 \cdot 10^{-3}(1 + \kappa/4) \quad (x_Q = 0), \tag{35}$$

so that the assumption $\epsilon \ll 1$ is justified. With (32) and (21), also the particle number density (density) and pressure are given according to (5) as

$$n = \frac{1 + \sum\limits_{Q} x_Q}{\sum\limits_{Q} x_Q \, \widetilde{\sigma}_Q} \, b^2 \, \frac{16\sigma}{3kI} \, T^{3+\kappa}, \quad p = \frac{1 + \sum\limits_{Q} x_Q}{\sum\limits_{Q} x_Q \, \widetilde{\sigma}_Q} \, b^2 \, \frac{16\sigma}{3I} \, T^{4+\kappa} \tag{36}$$

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[1]By the substitution $r = R_0 + h$ with $h \ll R_0$, it follows that T decreases linearly in first approximation with increasing h; one finds $dT/dh = -b^2/(kR_0^2) = -0.9/(1 + \kappa/4)\,^\circ\mathrm{C}/100$ m.

as well as the border of the atmosphere, in view of (27) and (30), as

$$R = R_0 \left[1 + \frac{k}{2b^2} \left(\frac{3I \sum\limits_Q \int n_Q \sigma_Q \, d^3x}{\pi\sigma} \right)^{1/4} \right] \tag{37}$$

and the thickness of the atmosphere as

$$H = R - R_0 = R_0 \epsilon = \frac{kR_0}{2b^2} \left(\frac{3I \sum\limits_Q \int n_Q \, \sigma_Q \, d^3x}{\pi\sigma} \right)^{1/4}. \tag{38}$$

With increasing values of n_Q (or x_Q) the atmosphere expands. Together with (35) we can estimate the thickness of the atmosphere as

$$H \simeq 35(1 + \kappa/4) \text{ km}. \tag{39}$$

Accordingly, the atmosphere would reach only the mesosphere because of $\kappa \leq 4$. The fact that the atmosphere is actually higher and that the temperature increases again in the upper regions in contrast to (32) depends on the additional heating of the upper atmosphere in consequence of the solar ultraviolet absorption by O_3, which is neglected in my model[2].

Evidently, the features of the atmosphere are determined by the solar radiation I and the total absorption cross-section $\sum\limits_Q \int n_Q \, \sigma_Q \, d^3x$ of the greenhouse molecules as well as by the gravitational force of the earth ($b^2 \sim MG$). However, in T_0 (see (33)) b^2 drops out (!), so that the atmospheric temperature at the surface is determined only by the product $I \sum\limits_Q \int n_Q \, \sigma_Q \, d^3x$. The results (33) and (38) reflect very well the influence of the greenhouse molecules. When $n_Q \to 0$ the temperature T_0 and the thickness H go to zero. Simultaneously, one finds by logarithmic differentiation of (33) immediately the enlargement ΔT_0 of the atmospheric temperature at the surface as a result of an increasing $\Delta \int n_Q \, \sigma_Q \, d^3x$ of the total absorption cross-section of the greenhouse molecules:

$$\frac{\Delta T_0}{T_0} = \frac{1}{4} \frac{\sum\limits_Q \Delta \int n_Q \, \sigma_Q \, d^3x}{\sum\limits_Q \int n_Q \, \sigma_Q \, d^3x} \tag{40}$$

as well as, in consequence of an increasing ΔI of the radiation power I of the sun,

$$\frac{\Delta T_0}{T_0} = \frac{1}{4} \frac{\Delta I}{I}. \tag{41}$$

3. The temperature of the surface of the earth

The temperature T_E of the earth's surface is determined by the fact that in the stationary case the surface must re-emit the infalling radiation power. This consists,

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[2]This is supported by the fact that according to the experience the lower region of the atmosphere up to approximately 30 km contains already 99% of the air.

first, of the radiation flux J of the sun and, second, of the infrared photons backscattered by the greenhouse molecules in the lower region of the atmosphere with the thickness of a mean free path length of the infrared photons. Also the warm atmosphere radiates.

For calculation of the backscattered infrared radiation we have to determine, at first, the thickness $R_c - R_0$ of the radiating region by the integral

$$\int\limits_{R_0}^{R_c} l^{-1}\, dr = 1, \tag{42}$$

where l^{-1} is given according to (2), (5), (12), (21), and (32) by

$$l^{-1} = b^8\, \frac{16\sigma}{3k^4 I} \left(\frac{1}{r} - \frac{\beta}{R_0}\right)^3, \quad \beta = (1+\epsilon)^{-1}. \tag{43}$$

Only solutions with $R_c < R$ are useful; if they do not exist, the whole model is not applicable (cf. (48)). Performing the integral we get:

$$b^8\, \frac{16\sigma}{3k^4 I} \left[\frac{1}{2}\left(\frac{1}{R_0^2} - \frac{1}{R_c^2}\right) + 3\frac{\beta}{R_0}\left(\frac{1}{R_c} - \frac{1}{R_0}\right) + 3\frac{\beta^2}{R_0^2}\ln\frac{R_c}{R_0} - \frac{\beta^3}{R_0^3}(R_c - R_0)\right] = 1. \tag{44}$$

This equation must be solved with respect to R_c, which is, however, impossible to be done exactly because of the logarithmic term. Therefore, we make analogously to (27) the ansatz

$$R_c = R_0(1 + \delta), \quad \delta \ll 1 \quad (\delta > 0) \tag{45}$$

and expand Eq. (44) with respect to δ and ϵ. In this way we find:

$$\delta\epsilon^3 - \frac{3}{2}\,\delta^2\epsilon^2 + \delta^3\epsilon - \frac{1}{4}\,\delta^4 = \frac{3k^4 I R_0^2}{16\sigma b^8}. \tag{46}$$

This equation of order 4 in δ can be solved easily owing to the binomial series on the left-hand side; the 4 roots are:

$$\begin{aligned}
\delta_{1,2} &= \epsilon \pm \left(\epsilon^4 - \frac{3k^4 I R_0^2}{4\sigma b^8}\right)^{1/4}, \\
\delta_{3,4} &= \epsilon \pm i\left(\epsilon^4 - \frac{3k^4 I R_0^2}{4\sigma b^8}\right)^{1/4},
\end{aligned} \tag{47}$$

from which, however, in view of $0 \leqslant \delta \leqslant \epsilon$, only

$$\delta = \epsilon - \left(\epsilon^4 - \frac{3k^4 I R_0^2}{4\sigma b^8}\right)^{1/4} = \epsilon\left[1 - \left(1 - \frac{4\pi R_0^2}{\sum\limits_Q \int n_Q\,\sigma_Q\,d^3x}\right)^{1/4}\right] \tag{48}$$

is useful. The atmosphere must be higher than the free path length of the infrared photons given by

$$R_c - R_0 = R_0\delta = \frac{kR_0}{2b^2}\left(\frac{3I\sum\limits_Q \int n_Q\,\sigma_Q\,d^3x}{\pi\sigma}\right)^{1/4}\left[1 - \left(1 - \frac{4\pi R_0^2}{\sum\limits_Q \int n_Q\,\sigma_Q\,d^3x}\right)^{1/4}\right]. \tag{49}$$

Accordingly, $\sum\limits_Q \int n_Q\,\sigma_Q\,d^3x \geq 4\pi R_0^2$ must be fulfilled for δ real valued and $\delta \leqslant \epsilon$.

On the other hand from $\epsilon \ll 1$ it follows that $\sum\limits_Q \int n_Q\,\sigma_Q\,d^3x \ll 16\pi\sigma b^8/(3k^4 I)$, which is, however, realized very well. The temperature of the atmosphere at $r = R_c$ reads

$$T_c = \frac{1}{2R_0}\left[\frac{3I\sum\limits_Q \int n_Q\,\sigma_Q\,d^3x}{\pi\sigma}\left(1-\frac{4\pi R_0^2}{\sum\limits_Q \int n_Q\,\sigma_Q\,d^3x}\right)\right]^{1/4} =$$

$$= T_0\left(1-\frac{4\pi R_0^2}{\sum\limits_Q \int n_Q\,\sigma_Q\,d^3x}\right)^{1/4}. \quad (50)$$

Now the backscattered radiation flux J_R will be calculated in such a way that every greenhouse molecule in the lower region of the atmosphere (see (49)) radiates with its mean absorption cross-section $\sigma_Q(T)$ (see (4)) as a black body with the atmospheric temperature $T(r)$ in the direction of the earth's surface (Kirchhoff's law). This gives with respect to (2):

$$J_R = 4\pi \int\limits_{R_0}^{R_c} \sigma\,T^4\,r^2\,\frac{dr}{l}. \quad (51)$$

The right hand side of (51) can be read also in such a way that the spherical layer of thickness l within the region $R_0 \leq r \leq R_c$ radiates as a black body in the direction of the earth. Insertion of $T(r)$ and l according to (20), (23), (28), and (43) yields:

$$J_R = 4\pi b^{16}\frac{16\sigma^2}{3k^8 I}\int\limits_{R_0}^{R_c} r^2\left[\frac{1}{r}-\frac{\beta}{R_0}\right]^7 dr = 4\pi b^{16}\frac{16\sigma^2}{3k^8 I}\left[\frac{1}{4}\left(\frac{1}{R_0^4}-\frac{1}{R_c^4}\right) + \right.$$

$$+\frac{7}{3}\frac{\beta}{R_0}\left(\frac{1}{R_c^3}-\frac{1}{R_0^3}\right) - \frac{21}{2}\frac{\beta^2}{R_0^2}\left(\frac{1}{R_c^2}-\frac{1}{R_0^2}\right) + 35\frac{\beta^3}{R_0^3}\left(\frac{1}{R_c}-\frac{1}{R_0}\right) +$$

$$\left. +35\frac{\beta^4}{R_0^4}\ln\frac{R_c}{R_0} - 21\frac{\beta^5}{R_0^5}(R_c-R_0) + \frac{7}{2}\frac{\beta^6}{R_0^6}(R_c^2-R_0^2) - \frac{1}{3}\frac{\beta^7}{R_0^7}(R_c^3-R_0^3)\right]. \quad (52)$$

Since R_c is known only approximately (see (45) and (48)), it is necessary to expand also the right hand side of (52) with respect to δ and ϵ (see (43) and (45)). Considering only the leading terms we obtain:

$$J_R = 4\pi b^{16}\frac{2\sigma^2}{3k^8 I R_0^4}\,\epsilon^8\left[1-\left(1-\frac{\delta}{\epsilon}\right)^8\right]. \quad (53)$$

After insertion of ϵ and δ according to (30) and (48) we find the simple result:

$$I_R = \frac{J_R}{4\pi} = \frac{3}{4}I\left[\frac{\sum\limits_Q \int n_Q\,\sigma_Q\,d^3x}{4\pi R_0^2} - \frac{1}{2}\right]. \quad (54)$$

The energy balance for determination of the surface temperature T_E of the earth reads now, under the assumption of black body radiation of the earth's surface,

$$\sigma R_0^2 T_E^4 = I + I_R \tag{55}$$

and results, after insertion of (54), in:

$$T_E = \left\{ \frac{I}{\sigma R_0^2} \left[1 + \frac{3}{4} \left(\frac{\sum\limits_Q \int n_Q \, \sigma_Q \, d^3 x}{4\pi R_0^2} - \frac{1}{2} \right) \right] \right\}^{1/4} . \tag{56}$$

Obviously, the second term within the brackets represents the greenhouse effect. Of course, the limiting case $n_Q \to 0$ is not allowed because of $\delta \le \epsilon$; however, we see from (55) that without backscattered infrared photons the surface temperature of the earth would be

$$T_E(0) = T_E(n_Q = 0) = \left(\frac{I}{\sigma R_0^2} \right)^{1/4} . \tag{57}$$

Surprisingly, the temperatures T_0, T_c and T_E are independent from the value of the exponent κ of the power law (5) and independent from the gravitational force ($b^2 \sim MG$), but only determined by the energy flux I of the sun and by the influence of the greenhouse molecules $\int n_Q \, \sigma_Q \, d^3 x$, and increase with increasing n_Q similar to the thickness of the atmosphere (ϵ); however, the free path length (δ) decreases. Obviously, the presupposition of the model that the thickness of the atmosphere must be larger than the free path length of the infrared photons will be fulfilled better and better with increasing number of greenhouse molecules. On the other hand, the thickness of the atmosphere and the free path length of the infrared photons depend also on the exponent κ and increase with increasing values of κ by the factor $1 + \kappa/4$.

For the determination of T_0, T_c and T_E according to (33), (50) and (56) the knowledge of the value $\sum\limits_Q \int n_Q \, \sigma_Q \, d^3 x$ is necessary. Since this value is unknown, we estimate it by the present temperature data. Without greenhouse molecules the mean temperature of the earth's surface would be, according to (57), $T_E(0) = -18\,°\mathrm{C}$ by the use of the solar constant. The primary solar constant amounts to $1.368\,\mathrm{KW/m^2}$; subtraction of the albedo yields $957.6\,\mathrm{W/m^2}$ at the earth's surface. However, the mean surface temperature of the earth amounts approximately to $T_E = +18\,°\mathrm{C}$. Herewith we find from (56) and (57)

$$\frac{\sum\limits_Q \int n_Q \, \sigma_Q \, d^3 x}{4\pi R_0^2} = 1.43 \tag{58}$$

and from (33) und (50) it follows:

$$T_0 = T_E(0) \left(\frac{3}{4} \frac{\sum\limits_Q \int n_Q \, \sigma_Q \, d^3 x}{4\pi R_0^2} \right)^{1/4} = -13.6\,°\mathrm{C}; \quad T_c = -81\,°\mathrm{C}. \tag{59}$$

The magnitude of T_c is in good agreement with the temperature at the tropopause. Now the exact value of ϵ, and herewith of the thickness H of the atmosphere, can be determined from (34) and (38); one thus finds:

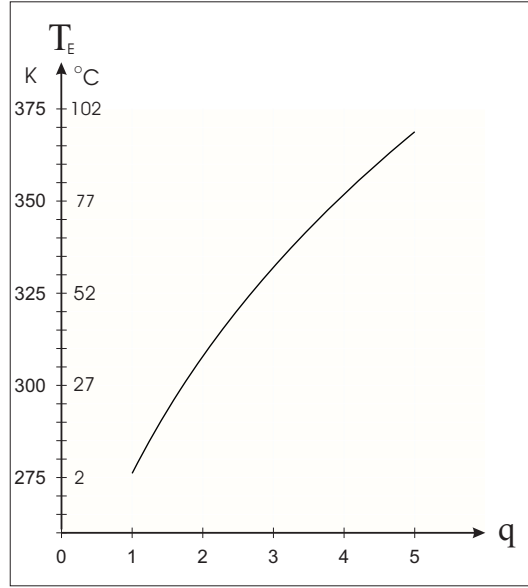$$\epsilon = 4.8 \cdot 10^{-3} (1 + \kappa/4), \quad H = 30.4(1 + \kappa/4)\,\mathrm{km} \tag{60}$$

Fig. 1. Dependence of the temperature T_E of the earth's surface on the parameter $q =$ $= \sum_Q \int n_Q \, \sigma_Q \, d^3x / 4\pi R_0^2$ according to (56)

instead of the rough estimations (35) and (39), and from (48) and (49) we get:

$$\delta = 1.25 \cdot 10^{-3} \, (1 + \kappa/4), \quad R_c - R_0 = R_0 \delta = 7.9(1 + \kappa/4) \text{ km}, \tag{61}$$

where the last value is again in accordance with the height of the tropopause. However, the altitude H (see (60)) is too small in comparison with the observation, if we do not take into account the κ-correction. Assuming a mean altitude of the atmosphere of 55 km, which corresponds to the stratopause, we obtain from (60)

$$\kappa = 3.2. \tag{62}$$

The height of the tropopause amounts then to 14.2 km, which is in good agreement with the observations. Because the free path length of the infrared photons reaches a height of approximately 14 km in relation to a height of the total atmosphere of 55 km, the model lies at the limit of its validity. The temperature T_E of the earth's surface in dependence of $\sum_Q \int n_Q \sigma_Q \, d^3x / 4\pi R_0^2$ is shown in Fig. 1.

For the relative change of the earth's surface temperature T_E in consequence of a small change of the number of the greenhouse molecules or of a small change of the solar radiation we find:

$$\frac{\Delta T_E}{T_E} = \frac{3}{2} \left[5 + 6 \frac{\sum_Q \int n_Q \, \sigma_Q \, d^3x}{4\pi R_0^2} \right]^{-1} \frac{\sum_Q \Delta \int n_Q \, \sigma_Q \, d^3x}{4\pi R_0^2}, \tag{63}$$

$$\frac{\Delta T_E}{T_E} = \frac{1}{4} \frac{\Delta I}{I}. \tag{64}$$

The solar radiation fluctuates at the earth's surface in the range of 0.3 W/m^2 during approximately 10 years in consequence of the activity of the sunspots; this gives

according to (64) a temperature change of $2.3 \cdot 10^{-2}\,°\mathrm{C}$. On the other hand, it follows from (63) together with (58) for the present situation:

$$\frac{\Delta T_E}{T_E} = 0.16 \, \frac{\sum\limits_Q \Delta \int n_Q \, \sigma_Q \, d^3x}{\sum_Q \int n_Q \, \sigma_Q \, d^3x} = 2.18 \cdot 10^{-20} \sum_Q \Delta \int n_Q \, \sigma_Q \, d^3x \qquad (65)$$

(σ_Q in cm^2). As a result we state that changes of the intensity of the solar radiation give rise to temperature changes at the earth's surface by a factor of 0.25 and changes of the absorption of the greenhouse molecules by a factor of 0.16. In the latter case, the knowledge of the absorption cross-sections is very essential. In order to calculate quantitatively the increase of the surface temperature of the earth due to an increase of the concentration of the greenhouse gases, e.g. that of CO_2, the exact knowledge of the proper absorption cross-section $\sigma_Q(T)$ is necessary according to (4).

A very rough estimation of $\sigma_Q(T_0)$ is possible by the air pressure $p_0 = p(T_0)$ at the earth's surface. From (36) it follows immediately that

$$\sum_Q x_Q \, \sigma_Q(T_0) = \frac{1 + \sum\limits_Q x_Q}{p_0} \, b^2 \, \frac{16\sigma}{3I} \, T_0^4.\qquad (66)$$

Insertion of the known values of p_0, T_0, I and b^2, results for the case when $\kappa = 3$ in ($x_Q \ll 1$):

$$\sum_Q x_Q \, \sigma_Q(T_0) = 3.78 \cdot 10^{-26} \ \mathrm{cm}^2.\qquad (67)$$

With $x_Q \simeq 3 \cdot 10^{-4} \,\hat{=}\, 0.03\,\%$ we find:

$$\sum_Q \sigma_Q(T_0) \simeq 1.26 \cdot 10^{-22} \ \mathrm{cm}^2.\qquad (68)$$

If one distributes this total absorption cross-section in very rough approximation by equal parts on the 4 main greenhouse gases (H_2O, CO_2 CH_4, N_2O), the present enlargement of $20 \cdot 10^9$ t of CO_2 per year corresponding to $\Delta N_{CO_2} = 2.7 \cdot 10^{38}$ leads according to (65) to an increase of the surface temperature

$$\frac{\Delta T_E}{T_E} = 2.78 \cdot 10^{-4} \Rightarrow \Delta T_E = 8.09 \cdot 10^{-2}\,°\mathrm{C}\qquad (69)$$

per year.

For calculating this, it can be shown for $\kappa = 3$ that $\Delta \int n_{CO_2} \, \sigma_{CO_2} \, d^3x \simeq$
$\simeq 1.5 \Delta N_{CO_2} \sigma_{CO_2}(T_0)$ and furthermore, that $N_L = \frac{4}{7} \sum\limits_Q \int n_Q \, \sigma_Q \, d^3x \Big/ \sum\limits_Q x_Q \, \sigma_Q(T_0)$
and $p_0 = (m_L + \sum\limits_Q x_Q \, m_Q) \, \frac{MG}{4\pi R_0^4} N_L$. As a result we find that $N_L = 1.1 \cdot 10^{44} \,\hat{=}\, M_L =$
$= 5.2 \cdot 10^{21}$ g, which gives very good agreement with the observation. The total number of the CO_2-molecules (0.038%) amounts to $4.2 \cdot 10^{40}$.

A more precise determination of ΔT_E in terms of an increasing CO_2-concentration per year is possible by a half-empirical calculation of $\sigma_{CO_2}(T_0)$ by the integral (4). Taking into account the radiation temperature of the earth only one absorption line of CO_2 is important, namely at the wave length of $1.4 \cdot 10^{-3}$ cm ($\hat{=}\, \nu = 2 \cdot 10^{13}$ Hz).
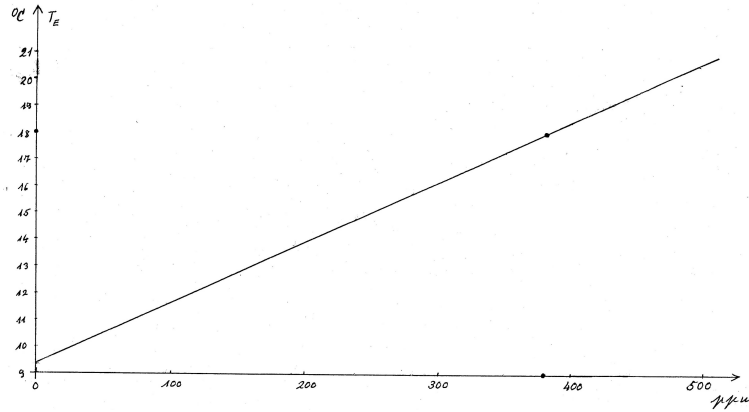
Fig. 2. Surface temperature T_E of the earth as function of the CO_2-concentration

According to the HITRAN database [5] the corresponding spectral absorption cross-section amounts to $\sigma_{CO_2}(\nu) = 5 \cdot 10^{-18}$ cm^2. Assuming not a sharp line as in (3) but a Doppler- and impact-broadened line with a line width of $\Delta\nu \simeq 10^8$ Hz we get from (4):

$$\sigma_{CO_2}(T_0) = 1.8 \cdot 10^{-23} \text{ cm}^2, \tag{70}$$

which is not extremely different from the rough estimation (68). Correspondingly, the temperature rise per year amounts now to:

$$\Delta T_E = 4.6 \cdot 10^{-2} \,^\circ\text{C}, \tag{71}$$

taking into account the mentioned CO_2 production rate. However, the half of this rate is absorbed today by the oceans, so that the actual temperature rise lies at $\Delta T_E = 2.3 \cdot 10^{-2} \,^\circ\text{C}$, corresponding to an increase of 1.22 ppm per year, which is in good agreement with numerical simulations, for example with the IPCC-reports [6]. The increase of the temperature of the lower atmosphere, cf. (40) and (65), is always larger; it is $\Delta T_0 = 6.5 \cdot 10^{-2} \,^\circ\text{C}$ and $\Delta T_0 = 3.2 \cdot 10^{-2} \,^\circ\text{C}$ per year respectively. The weakness of any prediction of a temperature rise depending on the production of greenhouse gases is based on the fact that the absorption cross-section $\sigma_Q(T)$ cannot be determined very exactly by the present observational data.

With the knowledge of the present number of the atmospheric CO_2-molecules, according to their present concentration (380 ppm), we can also calculate the mean surface temperature T_E of the earth as a function of a variable CO_2-concentration. The result is shown in Fig. 2. If we eliminated all CO_2-molecules out of the atmosphere, the surface temperature of the earth would decrease down to 9.3 °C. Its increase as a result of the CO_2-production, especially since the industrial revolution (from 290 ppm till 380 ppm), is in agreement with the observations (see Fig. 3). However, by comparison of the theoretical and the empirical temperature curves, one gets the impression that the global warming is retarded (especially between 1945 and 1975), perhaps because of formation of clouds or solar influences [7].

4. The convection

The fact that $T_E > T_0$ (cf. (33) and (56)) implies a convection in the lowest region of the earth's atmosphere, by which also a continuous temperature transition between the earth's surface and the atmosphere is established. Bubbles of atmospheric gas will
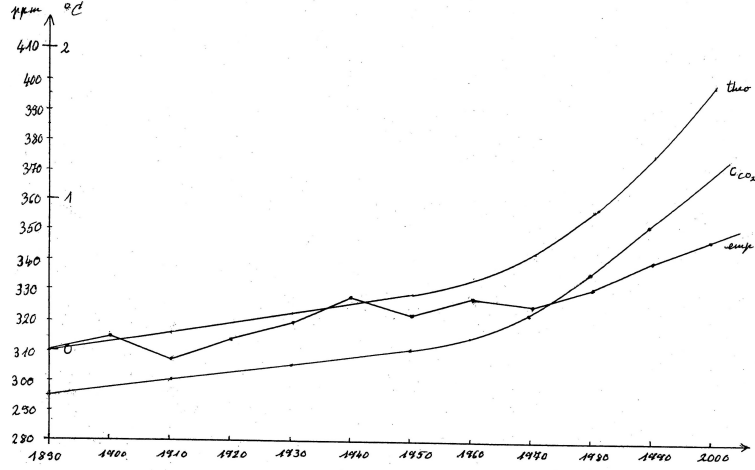
Fig. 3. Comparison of the theoretical and empirical temperature evolution with the increasing CO_2-concentration

be heated at the earth's surface to the temperature T_E and ascend within the cooler atmosphere under nearly adiabatic cooling until the surrounding atmospheric temperature T is reached. For the adiabatic cooling of the gas bubbles with the volume V, the following is valid (T_g is temperature of the gas bubble):

$$T_g V^{2/3} = \text{const} \tag{72}$$

According to the ideal gas equation, the following is valid for the changing volume:

$$V = NkT_g/p, \tag{73}$$

where p is the pressure in the atmosphere equalling the pressure in the gas bubble. According to (36), we find:

$$p = \text{const}\, T^{4+\kappa}. \tag{74}$$

Insertion of (73) and (74) into (72) yields

$$T_g^{5/3} = \text{const}\, T^{(8+2\kappa)/3}. \tag{75}$$

The constant in (75) will be determined at the earth's surface, where $T_g = T_E$ and $T = T_0$ is valid; thus it follows:

$$T_g^{5/3} = T_E^{5/3} \left(\frac{T}{T_0}\right)^{(8+2\kappa)/3}. \tag{76}$$

The ascent of the gas bubbles, i.e. the convection is stopped, when $T_g = T = T_K$ is reached, which means:

$$T_K^{3+2\kappa} = T_0^{8+2\kappa}/T_E^5. \tag{77}$$

With the values of T_0 and T_E (see (59)) and with $\kappa = 3$, we obtain

$$T_K = 243.3 \text{ K} \,\hat{=}\, -29.7\,^\circ\text{C}. \tag{78}$$

This temperature corresponds according to the temperature behaviour (32) to the height ($r(T_K) = R_K$):

$$R_K - R_0 = 3.3 \text{ km}. \tag{79}$$

Up to this mean altitude, which is much lower than the tropopause, convection is active, which is in good agreement with the observation (weather). At the same time, however, this result shows that convection is not important for the energy transport into the higher atmosphere. Therefore, the neglection of the convective energy transport within the model may be justified retrospectively.

Резюме

Х. Денен. Глобальное потепление в свете аналитической модели атмосферы Земли.

В статье рассматривается парниковый эффект в рамках упрощенной модели атмосферы с использованием аналитических методов и с опорой на основные физические принципы. При помощи математических формул (в отличие от климатического моделирования) подробно показано влияние солнечного излучения, поглощения молекул парниковых газов, а также конвекции на температуру Земли. Результаты анализа применены к современнму состоянию атмосферы.

Ключевые слова: глобальное потепление, парниковые газы, моделирование атмосферных процессов, численное моделирование.

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