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A CAUCHY INTEGRAL METHOD TO SOLVE THE 2D DIRICHLET AND NEUMANN PROBLEMS FOR IRREGULAR SIMPLY-CONNECTED DOMAINS

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Abstract

A method for construction of solutions to the continuous approximate 2D Dirichlet and Neumann problems in the arbitrary simply-connected domain with a smooth boundary has been discussed. The numerical finite difference method for solving the Dirichlet problem for an irregular domain meets the difficulties connected with construction of an adequate difference scheme for this domain and its discretization. We have reduced the solving of the Dirichlet problem to the solving of a linear integral equation. Unlike in the case of the Fredholm's solution to the problem, we have applied the properties of Cauchy integral boundary values rather than the logarithmic potential of a double layer. We have searched the solution to the integral equation in the form of a Fourier polynomial with the coefficients being the solution of a linear equation system. The continuous solution to the Dirichlet problem has the form of the Cauchy integral real part. The values near the boundary of the domain have been obtained with the help of analytic continuation of the Cauchy integral over an inner curve. Comparison of the Dirichlet problem exact solution and the continuous approximate solution has shown an error less than 10^{-5}. The Neumann problem solution has been reduced to the Dirichlet problem solution for the conjugate harmonic function. Comparison of the Neumann problem exact solution and the continuous approximate solution has shown an error less than 10^{-4}.

Keywords: Cauchy integral, Fourier polynomial, Dirichlet problem, Neumann problem, Fredholm integral equation, simply-connected domain

Introduction

The Laplace equation arises in different areas, such as electrostatics (where it describes the electrostatic potential in a charge-free region), gravitation (to indicate the gravitational potential in free space), steady-state flow of inviscid fluids, and steady-state heat conduction. Many authors (for example, [1–4]) introduced integral equation methods for solving the two-dimensional Laplace equation in order to calculate the potential field.

There exist three main groups of methods for solving the Dirichlet problem: the Fourier method and Green function method for domains of special types [5]; the method of double layer logarithmic potential with the density which is the solution of the Fredholm integral equation [3, 4, 6]; numerical methods (finite difference method, finite element method) [7]. The numerical finite difference method for solving the Dirichlet problem for an irregular domain meets the difficulties connected with construction of an adequate difference scheme for this domain. The numerical finite element method for solving the Dirichlet problem for an irregular domain meets the difficulties connected with adequate discretization for this domain.

The 2D Dirichlet problem is reduced to a linear integral equation. We apply the properties of Cauchy integral boundary values rather than a double layer logarithmic potential. We search the solution of the integral equation in the form of a Fourier polynomial with the coefficients being the solution of a linear equation system. The paper is organized as follows. In section 2, we describe the 2D problem of Laplace equation with Dirichlet and Neumann boundary conditions in the simply connected domain. In section 3, the algorithm for solving the problem is introduced. In section 4, we formulate our technique based on analytic continuation of Taylor series to improve the Cauchy integral solution at points near the boundaries. In section 5, numerical experiments are tested to demonstrate the viability of the proposed method.

1. Formulation of the problem

1.1. 2D Dirichlet problem for the Laplace equation. Let Ω be a simply connected domain, $\partial\Omega = \{(x(t), y(t)), t \in [0, 2\pi]\}$ be the smooth boundary of Ω. Then, the corresponding Dirichlet problem for the Laplace equation is to find the doubly differentiable in Ω function $u(x, y)$, which is continuous in $\Omega \cup \partial\Omega$ and satisfies the Laplace equation

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0, \quad (x,y) \in \Omega, \tag{1}$$

and the Dirichlet boundary conditions defined in the parametric form:

$$u(x(t), y(t)) = f_0(t), \quad t \in [0, 2\pi]. \tag{2}$$

1.2. 2D Neumann problem for the Laplace equation. Let Ω be a simply connected domain, $\partial\Omega = \{(x(t), y(t)), t \in [0, 2\pi]\}$ be the smooth boundary of Ω. Then, the corresponding Neumann problem for the Laplace equation is to find the doubly differentiable in Ω function $u(x, y)$, which is continuous in $\Omega \cup \partial\Omega$ and satisfies the Laplace equation

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0, \quad (x,y) \in \Omega, \tag{3}$$

and the Neumann boundary conditions defined in the parametric form:

$$\frac{\partial u}{\partial n} = f(t), \quad \text{on} \quad \partial\Omega, \quad t \in [0, 2\pi], \tag{4}$$

where n is a derivative in the external direction normal to the boundary of Ω.

2. Cauchy integral method solution for the 2D Laplace problem in simply connected domain

2.1. Solution of the 2D Laplace problem with Dirichlet boundary conditions for the simply connected domain. We denote the function to be found as $u(x, y) = \operatorname{Re}(B(x + iy))$, where $B(z)$ is an analytic function in the given simply connected domain Ω. So, the problem is as follows: given the function $f_0(t) = \operatorname{Re}(B(z(t)))\,|_{z(t)=(x(t)+iy(t))}$, $t \in [0, 2\pi]$, it is necessary to find the function $u(x, y) = \operatorname{Re}(B(x + iy))$, $(x, y) \in \Omega$.

By denoting $g_0(t) = \operatorname{Im}(B(z))\,|_{z(t)=x(t)+iy(t)\in\partial\Omega}$ the boundary value of analytic in Ω function $B(z)$, $z = x + iy$, can be presented as follows:

$$B(z(t)) = f_0(t) + ig_0(t), \quad z(t) = x(t) + iy(t), \quad t \in [0, 2\pi].$$

If the functions $f_0(t)$ and $g_0(t)$ are from Hölder class, then, according to [8, 9], the boundary value of the analytic function $B(x,y)$ in Ω meets the relation

$$f_0(t) + ig_0(t) = \frac{1}{i\pi} \int_0^{2\pi} \frac{f_0(\tau) + ig_0(\tau)}{z(\tau) - z(t)} z'(\tau)\, d\tau, \tag{5}$$

where the singular integral on the right-hand side is the principal value integral and $z(t) = x(t) + iy(t)$ represents the complex form of the simply-connected domain Ω boundary smooth curve $\partial\Omega$. Let us assume that the given boundary curve can be represented in the form of or approximated by the finite Fourier polynomial as follows:

$$z(t) = \sum_{k=-m}^{m} c_k e^{ikt}. \tag{6}$$

After separating the imaginary part of equation (5) we obtain the following Fredholm integral equation:

$$g_0(t) = -\frac{1}{\pi} \int_0^{2\pi} f_0(\tau) \left(\log\left[z(\tau) - z(t)\right]\right)'_\tau d\tau + \frac{1}{\pi} \int_0^{2\pi} g_0(\tau) \left(\arg\left[z(\tau) - z(t)\right]\right)'_\tau d\tau. \tag{7}$$

We consider the factor $(e^{i\tau} - e^{it})$ in the expression $(z(\tau) - z(t))$ in order to separate the improper VP integral in the Fredholm equation of the second kind as follows [10]:

$$\begin{aligned} \log\left[z(\tau) - z(t)\right] = \log(2i) + \log\left(\sin\left(\frac{\tau - t}{2}\right)\right) + i\frac{\tau + t}{2} \\ + \log\left(\sum_{k=1}^{m} c_k e^{ikt} \sum_{l=0}^{k-1} e^{il(\tau - t)} - \sum_{k=1}^{m} c_{-k} e^{-ik\tau} \sum_{l=0}^{k-1} e^{il(\tau - t)}\right), \end{aligned} \tag{8}$$

so that equation (7) takes the form:

$$\begin{aligned} g_0(t) = \frac{1}{2\pi} \int_0^{2\pi} g_0(\tau) d\tau + \frac{1}{\pi} \int_0^{2\pi} g_0(\tau) K(\tau, t)\, d\tau \\ - \frac{1}{2\pi} \int_0^{2\pi} f_0(\tau) \cot\frac{\tau - t}{2} d\tau - \frac{1}{\pi} \int_0^{2\pi} f_0(\tau) L(\tau, t)\, d\tau. \end{aligned} \tag{9}$$

Here,

$$K(\tau, t) = \operatorname{Im}\left[\log\left(\sum_{k=1}^{m} c_k e^{ikt} \sum_{l=0}^{k-1} e^{il(\tau - t)} - \sum_{k=1}^{m} c_{-k} e^{-ik\tau} \sum_{l=0}^{k-1} e^{il(\tau - t)}\right)\right]'_\tau,$$

$$L(\tau, t) = \operatorname{Re}\left[\log\left(\sum_{k=1}^{m} c_k e^{ikt} \sum_{l=0}^{k-1} e^{il(\tau - t)} - \sum_{k=1}^{m} c_{-k} e^{-ik\tau} \sum_{l=0}^{k-1} e^{il(\tau - t)}\right)\right]'_\tau.$$

Let us search the solution of Fredholm integral equation (9) in the form of Fourier series as follows:

$$g_0(t) = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos(nt) + \beta_n \sin(nt) \tag{10}$$

According to [10, 11], equation (9) has a unique solution if we set the summand

$$\alpha_0 = \frac{1}{2\pi}\int\limits_0^{2\pi} g_0(\tau)\,d\tau = 0.$$

The solvability of equation (9) is proved in [10], where the Fourier series solution form of the function $g_0(t)$ leads us to an infinite linear system of equations which can be reduced to a finite one according to the following lemma:

Lemma 1 [10]. *Let the numbers $j,p > 1$ and a constant $U > 0$ exist, so that $|\partial^{j+p}G(\tau,t)/\partial t^j\partial\tau^p| \le U$ and the function $Y(t)$ possesses the bounded second derivative: $|Y''(t)| < T$. Then, the approximate solution of the uniquely resolvable Fredholm integral equation of the second kind*

$$X(t) = \int\limits_0^{2\pi} G(\tau,t)X(\tau)\,d\tau + Y(t),$$

where $Y(t)$ is 2π periodic and $G(\tau,t)$ is 2π periodic with respect to both variables, can be reduced to the solving of a finite linear system of equations with the error estimated by $\mathcal{O}\left(1/N^2\right)$ where N is the finite linear system rank.

According to Lemma 1, the approximate solution (10) can be a finite sum with $\alpha_0 = 0$ as follows:

$$g_0(t) = \sum_{n=1}^{N} \alpha_n\cos(nt) + \beta_n\sin(nt) \tag{11}$$

As in [10], we substitute (11) into the equation (9) and apply the following Hilbert formula:

$$\frac{1}{2\pi}\int\limits_0^{2\pi} f_0(\tau)\cot\frac{\tau - t}{2}\,d\tau = \sum_{k=1}^{\infty} -\gamma_k\sin kt + \delta_k\cos kt, \tag{12}$$

if the coefficients γ_k, δ_k are the coefficients of the function $f_0(t)$ of Fourier series decomposition:

$$f_0(t) = \sum_{k=1}^{\infty}\gamma_k\cos kt + \delta_k\sin kt.$$

Then, the truncated linear equation system can be written in the following matrix form:

$$\begin{pmatrix} AA & AB \\ BA & BB \end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \begin{pmatrix} p \\ q\end{pmatrix}, \tag{13}$$

where

$$\alpha = \begin{pmatrix}\alpha_1 \\ \vdots \\ \alpha_N\end{pmatrix}, \quad \beta = \begin{pmatrix}\beta_1 \\ \vdots \\ \beta_N\end{pmatrix}, \quad p = \begin{pmatrix}p_1 \\ \vdots \\ p_N\end{pmatrix}, \quad q = \begin{pmatrix}q_1 \\ \vdots \\ q_N\end{pmatrix},$$

$$\{p_l\}_{l=1}^{N} = -\delta_l - \frac{1}{\pi^2}\int\limits_0^{2\pi} f_0(\tau)\,d\tau\int\limits_0^{2\pi} L(\tau,t)\cos(lt)\,dt,$$

$$\{q_l\}_{l=1}^{N} = \gamma_l - \frac{1}{\pi^2}\int\limits_0^{2\pi} f_0(\tau)\,d\tau\int\limits_0^{2\pi} L(\tau,t)\sin(lt)\,dt.$$

Here,

$$\begin{pmatrix} AA & AB \\ BA & BB \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} aa_{11} & aa_{12} & \cdots & aa_{1N} \\ aa_{21} & aa_{22} & \cdots & aa_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ aa_{N1} & aa_{N2} & \cdots & aa_{NN} \end{pmatrix} & \begin{pmatrix} ab_{11} & ab_{12} & \cdots & ab_{1N} \\ ab_{21} & ab_{22} & \cdots & ab_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ ab_{N1} & ab_{N2} & \cdots & ab_{NN} \end{pmatrix} \\ \begin{pmatrix} ba_{11} & ba_{12} & \cdots & ba_{1N} \\ ba_{21} & ba_{22} & \cdots & ba_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ ba_{N1} & ba_{N2} & \cdots & ba_{NN} \end{pmatrix} & \begin{pmatrix} bb_{11} & bb_{12} & \cdots & bb_{1N} \\ bb_{21} & bb_{22} & \cdots & bb_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ bb_{N1} & bb_{N2} & \cdots & bb_{NN} \end{pmatrix} \end{pmatrix}.$$

The elements of the coefficient matrix are defined as follows:

$$aa_{ln} = \delta_{ln} - \frac{1}{\pi^2} \int_0^{2\pi} \cos(n\tau)\, d\tau \int_0^{2\pi} K(\tau, t) \cos(lt)\, dt,$$

$$ab_{ln} = -\frac{1}{\pi^2} \int_0^{2\pi} \sin(n\tau)\, d\tau \int_0^{2\pi} K(\tau, t) \cos(lt)\, dt,$$

$$ba_{ln} = -\frac{1}{\pi^2} \int_0^{2\pi} \cos(n\tau)\, d\tau \int_0^{2\pi} K(\tau, t) \sin(lt)\, dt,$$

$$bb_{ln} = \delta_{ln} - \frac{1}{\pi^2} \int_0^{2\pi} \sin(n\tau)\, d\tau \int_0^{2\pi} K(\tau, t) \sin(lt)\, dt,$$

and δ_{ln} is the Kronecker delta symbol. The linear system is resolvable and we can easily find the coefficients $\{\alpha_n, \beta_n\}_{n=1}^{N}$ of equation (11).

Finally, the approximate solution to the Laplace equation (1-2) is the Cauchy integral real part as a function of $z = x + iy$:

$$u(x, y) = \mathrm{Re} \left(\frac{1}{2\pi i} \int_0^{2\pi} \frac{f_0(t) + i g_0(t)}{z(t) - (x + iy)} z'(t)\, dt \right). \tag{14}$$

2.2. Solution of the 2D Laplace problem with Neumann boundary conditions for the simply connected domain. Given the function

$$f(t) = \left. \frac{\partial u}{\partial n} \right|_{(x(t), y(t)) \in \partial\Omega},$$

we need to find the harmonic function $u(x, y)$ in Ω. In order to apply our method to the Neumann problem, let $v(x, y)$ be the conjugate harmonic to $u(x, y)$ in Ω. By using the Cauchy–Riemann conditions we can deduce the following relation:

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial s},$$

where s is the direction tangent to the boundary $\partial\Omega$.

Now, the Dirichlet problem is to be solved: let us find the harmonic in Ω function $v(x,y)$ with the corresponding boundary condition

$$v(x(t),y(t)) = \int_0^t f(t)|z'(t)|\,dt, \quad \text{on} \;\; \partial\Omega, \quad t\in[0,2\pi]. \tag{15}$$

Evidently, $v\left(x(0),y(0)\right) = v\left(x(2\pi),y(2\pi)\right)$, so we can write the necessary condition of solvability of this problem as follows:

$$\int_0^{2\pi} f(t)|z'(t)|\,dt = 0,$$

and this agrees with the necessary condition of solvability of the Neumann problem. After applying the Cauchy integral method and obtaining the approximate harmonic function $v(x,y)$, the final solution of equation (3)–(4) will be easily obtained with the help of the Cauchy–Riemann conditions of the analytic function. We obtain the solution to the Neumann problem within an arbitrary summand.

3. Approximation of the Dirichlet problem solution at the points near the boundaries

Let $z_0 = (x_0 + iy_0)$ be the point in Ω near the boundary curve $\partial\Omega$. The harmonic function solution at this point is approximated by the real part of the Cauchy integral formula

$$u(x_0,y_0) = \mathrm{Re}\left(\frac{1}{2\pi i}\int_0^{2\pi}\frac{B(z(t))z'(t)}{z(t)-(x_0+iy_0)}\,dt\right), \tag{16}$$

where $B(z(t))$ is the boundary values of the analytic in Ω function. Evidently, we have singularity at the points near the boundaries. By taking arbitrary disk with radius R and center σ in Ω we can write the analytic function at any point z inside this disk as follows:

$$B(z) = \frac{1}{2\pi i}\int_0^{2\pi}\frac{B(R\,e^{it}+\sigma)iR\,e^{it}}{(R\,e^{it}+\sigma)-z}\,dt, \tag{17}$$

where $B(R\,e^{it}+\sigma)$ is the boundary values of the analytic function in this disk. After applying the Taylor expansion we get

$$B(z) = \sum_{k=0}^{\infty}\frac{1}{2\pi R^k}\left[\int_0^{2\pi} B(R\,e^{it}+\sigma)e^{-ikt}\,dt\right](z-\sigma)^k. \tag{18}$$

The approximate harmonic function at points near the boundaries of $\partial\Omega$ can be found using analytic continuation [12] of the Taylor expansion (18) of $B(z)$ to the disk of a radius more that R.

4. Numerical examples

In this section, we will present four of our numerical results of the Cauchy integral method. The first two examples solve the Dirichlet problem for simply connected domains with irregular boundaries. The second two examples solve the Neumann problem

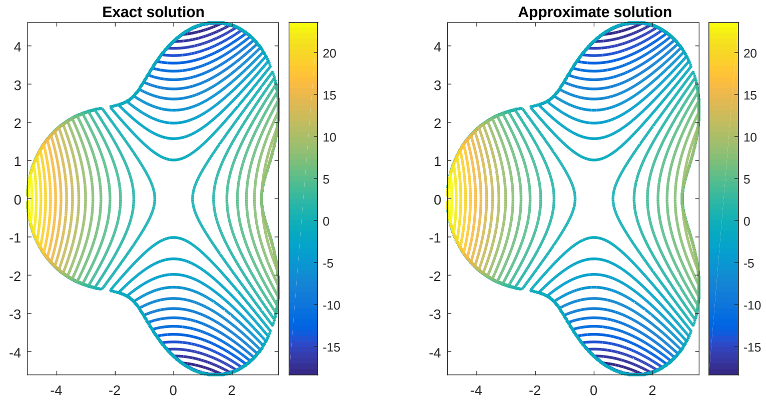

Fig. 1. The contour plots of the exact and approximate solutions of the 2D Dirichlet problem of Example 1

for the same domains. The performance of the Cauchy integral method is measured by the maximum absolute error

$$\varepsilon_k = |u_{\text{exact}} - u_{\text{approx}}|.$$

Example 1. Let us introduce the simply-connected domain with the boundary defined in the following parametric form:

$$z(t) = r(t)\left(\cos(t) + i\sin(t)\right),$$

where

$$r(t) = \sqrt{(a+b)^2 + 1 - 2(a+b)\cos\frac{at}{b}}, \quad t \in [0, 2\pi].$$

The method was applied to the Dirichlet problem with $a = 3,\ b = 1$. The exact solution to the proposed problem is $u(x, y) = x^2 - y^2$. The full domain with the contour plots of the exact and approximate solutions is shown in Fig. 1. The figure shows that the results are very close and the maximum absolute error is $= 9.350 \cdot 10^{-7}$.

Example 2. Let us consider, as in [11], the simply-connected domain with a non-starlike boundary, which is defined in the following parametric form:

$$z(t) = -0.5 + e^{it} + 0.5e^{2it} + 0.2ie^{-2it}, \quad t \in [0, 2\pi].$$

The exact solution to the proposed problem is $u(x, y) = e^x \cos y$. The full domain with the contour plots of the exact and approximate solutions is shown in Fig. 2. The figure shows that the results are very close and the maximum absolute error is $= 2.668 \cdot 10^{-6}$.

Example 3. Let us apply the method for solving the Neumann problem to the simply connected domain proposed in example (1) with $a = 3,\ b = 1$. The exact solution to the proposed problem is $u(x, y) = x^3 - 3xy^2$. The full domain with the contour plots of the exact and approximate solutions is shown in Fig. 3. The figure shows that the results are very close and the maximum absolute error is $= 4.969 \cdot 10^{-5}$.

Example 4. For the simply connected domain with non-star like boundaries proposed in example (2). The Cauchy integral method was applied to find the approximate solution of the Neumann problem. The exact solution to this problem is $u(x, y) = x^2 - y^2$. The full domain with the contour plots of the exact and approximate solutions are shown in Fig. 4. The figure shows that the results are very close and the maximum absolute error is $= 1.222 \cdot 10^{-12}$.

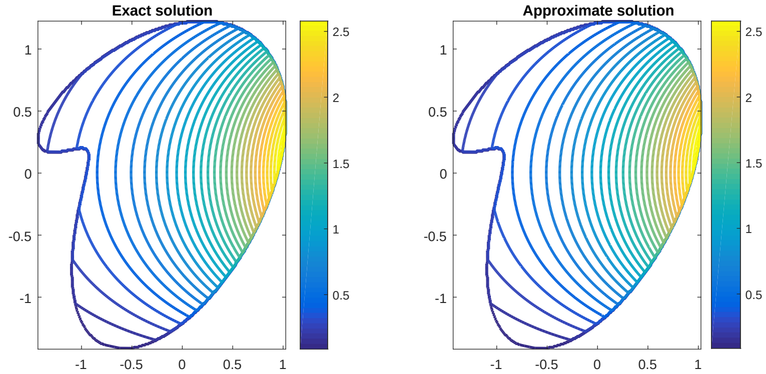

Fig. 2. The contour plots of the exact and approximate solutions of the 2D Dirichlet problem of Example 2

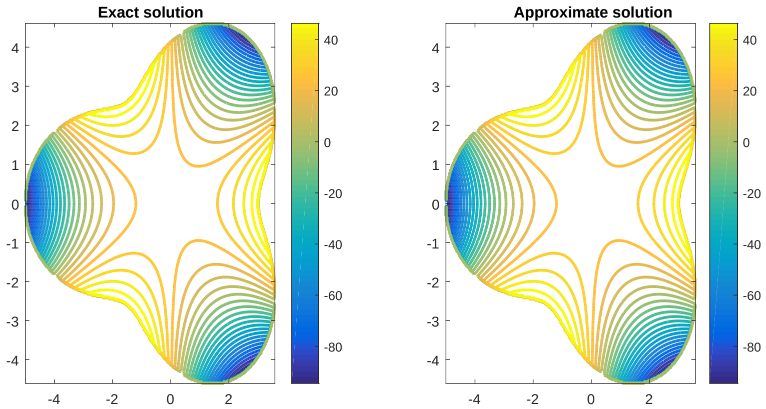

Fig. 3. The contour plots of the exact and approximate solutions of the 2D Neumann problem of Example 3

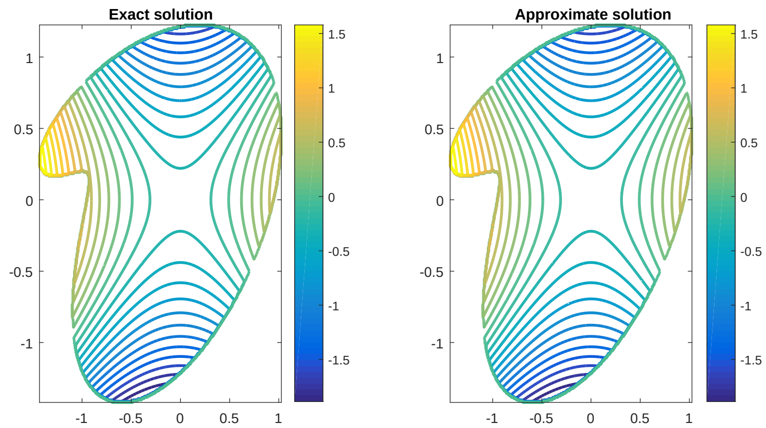

Fig. 4. The contour plots of the exact and approximate solutions of the 2D Neumann problem of Example 4

Conclusions

The Cauchy integral method provides highly accurate results for the solution of the 2D Dirichlet and Neumann problem for the irregular simply connected domains.

The method is applicable for domains bounded by any smooth curve approximated by the Fourier polynomial. The method was programmed using MATLAB R2012a and several examples are given to verify the efficiency of the method.

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Решение двумерных задач Дирихле и Неймана для односвязных областей сложной формы с помощью интеграла Коши

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Аннотация

Предлагается способ построения непрерывного приближенного решения двумерных задач Дирихле и Неймана в произвольной односвязной области с гладкой границей. Построение численных решений задачи Дирихле в произвольной области связано либо с трудностями построения адекватных разностных схем (метод конечных разностей), либо с трудностями дискретизации такой области (метод конечных элементов).

В работе решение задачи Дирихле сводится к решению линейного интегрального уравнения. При этом, в отличие от решения Фредгольма, используется не логарифмический потенциал двойного слоя, а свойства граничных значений интеграла Коши. Приближенное решение интегрального уравнения ищется в виде полинома Фурье, для нахождения коэффициентов которого решается система линейных уравнений.

Предлагаемое непрерывное решение задачи Дирихле представляет собой реальную часть интеграла Коши, взятого по границе области. Значения решения вблизи границы области получаются с помощью аналитического продолжения значений интеграла Коши, представленного в виде ряда Тейлора для внутренних точек. Сравнение полученных приближенных решений задачи Дирихле с точными решениями в примерах демонстрируют ошибку, меньшую, чем 10^{-5}. Решение задачи Неймана сводится к решению задачи Дирихле для сопряженной гармонической функции. Сравнение полученных приближенных решений задачи Неймана с точными решениями в примерах демонстрируют ошибку, меньшую, чем 10^{-4}.

Ключевые слова: интеграл Коши, многочлен Фурье, задача Дирихле, задача Неймана, интегральное уравнение Фредгольма, односвязанная область

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