# NEW TRENDS ON FRACTIONAL INTEGRAL AND DIFFERENTIAL EQUATIONS 

A.A. Kilbas


#### Abstract

One- and multi-dimensional integral equations and ordinary and partial differential equations with fractional integrals and derivatives by Riemann-Liouville, Liouville, Caputo, Hadamard and Riesz are considered. The method based on the reduction of the Cauchytype and Cauchy problems for the one-dimensional nonlinear fractional differential equations to Volterra integral equations is discussed. A unified approach is presented to solve in close form of some classes of one- and multi-dimensional linear integral equations and linear ordinary and partial differential equations of fractional order. This approach is based on compositional relations, operational calculus and integral transforms by Laplace, Fourier and Mellin. Problems and new trends of research are discussed.


## 1. Introduction

Integral and differential equations of fractional order, in which an unknown function is contained under the operations of integrals and derivatives of fractional order, have been of great interest recently. It coursed both by intensive development of the theory of fractional calculus itself and by the applications of such constructions in various sciences. In this connection we note the books [1-9], the papers [10-12] and Proceedings of the first Workshop on Fractional Differentiation and its Applications, July 19-21, Boreaux, France, Bordeaux Univ., Bordeaux, 2004.

In the above monographs and papers one may find various applications of fractional integral and differential equations in physics, mechanics, chemistry, engineering and other disciplines together with bibliography in these fields.

The fractional integral and differential equations have the following general forms

$$
\begin{equation*}
F\left[x, y(x), I^{\alpha_{1}} y(x), I^{\alpha_{2}} y(x), \ldots, I^{\alpha_{m}} y(x)\right]=f(x) \tag{1.1}
\end{equation*}
$$

and

$$
\begin{equation*}
F\left[x, y(x), D^{\alpha_{1}} y(x), D^{\alpha_{2}} y(x), \ldots, D^{\alpha_{m}} y(x)\right]=f(x) . \tag{1.2}
\end{equation*}
$$

Here $x$ is a point in $m$-dimensional Euclidean space $\mathbb{R}^{n}(n \in \mathbb{N}=\{1,2, \ldots\})$, $F\left[x, y, y_{1}, \ldots, y_{m}\right]$ and $f(x)$ are given functions, and $I^{\alpha_{k}}$ and $D^{\alpha_{k}}$ are the operators of fractional integration and differentiation with real $\alpha_{k}>0$ or complex $\alpha_{k}$, $\operatorname{Re} \alpha_{k}>0(k=1,2, \ldots, m)$. The corresponding linear equations with given functions $c_{k}(x) \quad(k=0,1, \ldots, m)$ and $f(x)$ are represented as

$$
\begin{align*}
& c_{0}(x) y(x)+\sum_{k=1}^{m} c_{k}(x)\left(I^{\alpha_{k}} y\right)(x)=f(x)  \tag{1.3}\\
& c_{0}(x) y(x)+\sum_{k=1}^{m} c_{k}(x)\left(D^{\alpha_{k}} y\right)(x)=f(x) \tag{1.4}
\end{align*}
$$

The fractional integration and differentiation operators in (1.1)-(1.4) can have different forms. A survey of methods and results on fractional integral equations was given in the books by Gorenflo, Vessela [2] and Samko, Kilbas, Marichev [4], while on fractional differential equations in two survey papers by the author and Trujillo [13, 14].

Among these equations the one-dimensional linear fractional integral and differential equations (1.3), (1.4) and the "model" nonlinear linear differential equation of the form

$$
\begin{equation*}
D^{\alpha} y(x)=f[x, y(x)] \tag{1.5}
\end{equation*}
$$

with real $\alpha>0$ or complex $\alpha(\operatorname{Re} \alpha)>0$, containing the Riemann-Liouville fractional integrals and derivatives $I^{\alpha} y=I_{a+}^{\alpha} y$ and $D^{\alpha} y=D_{a+}^{\alpha} y, a \in \mathbb{R}$, were studied more. For complex $\alpha \in \mathbb{C}, \operatorname{Re} \alpha>0$, such fractional integrals and derivatives of order $\alpha$ are defined by

$$
\begin{equation*}
\left(I_{a+}^{\alpha} y\right)(x)=\frac{1}{\Gamma(\alpha)} \int_{a}^{x} \frac{y(t) d t}{(x-t)^{1-\alpha}} \quad(x>a ; \alpha \in \mathbb{C}, \operatorname{Re} \alpha)>0 \tag{1.6}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(D_{a+}^{\alpha} y\right)(x)=\left(\frac{d}{d x}\right)^{n}\left(I_{a+}^{n-\alpha} y\right)(x) \quad(x>a ; n=[\operatorname{Re} \alpha]+1) \tag{1.7}
\end{equation*}
$$

respectively, $\Gamma(\alpha)$ being the Euler Gamma-function. It should be noted that the Riemann - Liouville approach (1.6) to the definition of fractional integration is a generalization of the integration $\int_{a+}^{x}$ applied $n$ times:

$$
\begin{equation*}
\int_{a}^{x} d t \int_{a}^{t} d t_{1} \ldots \int_{a}^{t_{n-2}} y\left(t_{n-1}\right) d t_{n-1}=\frac{1}{(n-1)!} \int_{a}^{x}(x-t)^{n-1} y(t) d t \tag{1.8}
\end{equation*}
$$

if we use the formula $(n-1)!=\Gamma(n)$ and replaced $n \in \mathbb{N}$ by $\alpha \in \mathbb{C}(\operatorname{Re} \alpha>0)$, then (1.8) yields (1.6). The fractional differentiation operator $D_{a+}^{\alpha}$ is inverse to the fractional integration one from the left:

$$
\begin{equation*}
\left(D_{a+}^{\alpha} I_{a+}^{\alpha} y\right)(x)=y(x) \quad(\alpha \in \mathbb{C}, \quad \operatorname{Re} \alpha>0) \tag{1.9}
\end{equation*}
$$

for suitable function $y(x)$. In particular if $0<\operatorname{Re} \alpha<1$,

$$
\begin{equation*}
\left(D_{a+}^{\alpha} y\right)(x)=\frac{d}{d x} \frac{1}{\Gamma(1-\alpha)} \int_{a}^{x} \frac{y(t) d t}{(x-t)^{\alpha}} \tag{1.10}
\end{equation*}
$$

and if $\alpha=n \in \mathbb{N}$, then $\left(D_{a+}^{n} y\right)(x) \equiv\left(D^{n} y\right)(x)(D=d / d x)$ is the usual derivative of order $n$.

Integral equations (1.3) with Riemann - Liouville fractional integrals (1.6) are the Volterra integral equations with power singularities, generalizing the classical Abel equation, and therefore these equations are called Abel - Volterra integral equations. As in the theory of ordinary differential equations, the methods to the investigation of differential equations of fractional order (1.4) and (1.5), with the Riemann - Liouville fractional derivative (1.7), are mainly based on the reduction of these equations to Volterra integral equations of the second kind. This approach was used by many authors to investigate the uniqueness and the existence of the solution of the Cauchy-type problem for the nonlinear equation (1.5) on a finite interval of the real line and to obtain the explicit
solution of such a problem for the linear equation (1.4). A survey of results in this field was presented in the above paper by Kilbas and Trujillo [13, Sections 4 and 5].

Here we discuss some results in this connection and show that such a method can be also applied to investigate the Cauchy-type and Cauchy problems to the one-dimensional equations (1.5) and (1.4) with the so-called Hadamard and Caputo fractional derivatives $D^{\alpha} y={ }^{H} D_{a+}^{\alpha} y$ and $D^{\alpha} y={ }^{C} D_{a+}^{\alpha} y$ of order $\alpha \in \mathbb{C}, \operatorname{Re} \alpha>0$. The Hadamard fractional derivative $\left({ }^{H} D_{a+}^{\alpha} y\right)(x)$ is defined by

$$
\begin{equation*}
\left({ }^{H} D_{a+}^{\alpha} y\right)(x)=\delta^{n}\left(\mathcal{J}_{a+}^{n-\alpha} y\right)(x) \quad(x>a ; n=[\operatorname{Re} \alpha]+1), \tag{1.11}
\end{equation*}
$$

where $\delta=x D, D=d / d x$, is the so-called $\delta$-derivative, and $\left(\mathcal{J}_{a+}^{\alpha} y\right)(x)$ is the Hadamard fractional integral of order $\alpha$ :

$$
\begin{equation*}
\left(\mathcal{J}_{a+}^{\alpha} y\right)(x)=\frac{1}{\Gamma(\alpha)} \int_{0}^{x}\left(\log \frac{x}{t}\right)^{\alpha-1} \frac{y(t) d t}{t} \quad(x>a ; \alpha \in \mathbb{C}, \operatorname{Re} \alpha>0) \tag{1.12}
\end{equation*}
$$

Such an integral is a generalization of the integration $\int_{a+}^{x} \frac{1}{x}$ applied $n$ times:

$$
\begin{equation*}
\int_{a}^{x} \frac{d t}{t} \int_{a}^{t} \frac{d t_{1}}{t_{1}} \ldots \int_{a}^{t_{n-2}} y\left(t_{n-1}\right) \frac{d t_{n-1}}{t_{n-1}}=\frac{1}{(n-1)!} \int_{a}^{x}\left(\log \frac{x}{t}\right)^{n-1} \frac{y(t) d t}{t} \tag{1.13}
\end{equation*}
$$

compare with (1.8). When $\alpha=n \in \mathbb{N}$, then $\left({ }^{H} D_{a+}^{n} y\right)(x) \equiv\left(\delta^{n} y\right)(x)$ is $\delta$ derivative of order $n$.

The Caputo derivative $\left({ }^{C} D_{a+}^{\alpha} y\right)(x)$ is defined via the Riemann-Liouville derivative (1.7) by

$$
\begin{equation*}
\left({ }^{C} D_{a+}^{\alpha} y\right)(x)=\left(D_{a+}^{\alpha}\left[y(t)-\sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!}(t-a)^{k}\right]\right)(x) \quad(n=[\operatorname{Re} \alpha]+1) \tag{1.14}
\end{equation*}
$$

where $n=[\operatorname{Re} \alpha]+1$ for $\alpha \notin \mathbb{N}$ while $n=\alpha$ for $\alpha \in \mathbb{N}$. When $\alpha \notin \mathbb{N}$, there holds the relation

$$
\begin{equation*}
\left({ }^{C} D_{a+}^{\alpha} y\right)(x)=\frac{1}{\Gamma(\alpha)} \int_{a}^{x} \frac{y^{(n)}(t) d t}{(x-t)^{1-\alpha}} \quad n=[\operatorname{Re}(\alpha)]+1 \tag{1.15}
\end{equation*}
$$

for suitable functions $y$. For $n-1<\alpha<n$ the derivative ${ }^{C} D_{a+}^{\alpha} y$ in the form (1.15) was defined by Caputo in [15] and presented in his book [16]. Therefore the constructions (1.14) and (1.15) are called Caputo derivatives.

For one-dimensional linear differential equations of fractional order, as in the case of linear ordinary differential equations, the same methods can be applied to study different aspects of these equations. In particular, methods based on operational calculus, compositional relations and Laplace transform can be used to find their explicit solutions. Here we discuss some results in this connection and show that the Fourier and Mellin transforms can be also used to deduce explicit solutions of linear fractional integral and differential equations of the form (1.3) and (1.4) with constant coefficients $c_{k} \in \mathbb{R}$ and with the Liouville and Hadamard fractional integrals $I^{\alpha} y=I_{-\infty,+}^{\alpha} y \equiv I_{+}^{\alpha} y$ and $I^{\alpha} y=\mathcal{J}_{0+}^{\alpha} y$, and fractional derivatives $D^{\alpha} y=D_{-\infty,+}^{\alpha} y \equiv D_{+}^{\alpha} y$ and $D^{\alpha} y={ }^{H} D_{0+}^{\alpha} y$, defined on the real line $\mathbb{R}$ and on the half axis $\mathbb{R}_{+}=(0, \infty)$, respectively.

The multi-dimensional Fourier transform can be also applied to solve in closed form of the linear integral and differential equations (1.3) and (1.4) with constant coefficients
$c_{k} \in \mathbb{R}$ with the Riesz fractional integral $I^{\alpha} y=\mathbf{I}^{\alpha} y$ and derivative $D^{\alpha} y=\mathbf{D}^{\alpha} y$ of complex order $\alpha \in \mathbb{C}(\operatorname{Re} \alpha>0)$. Such integrals and derivatives are defined as negative and positive powers $(-\Delta)^{-\alpha / 2}$ and $(-\Delta)^{\alpha / 2}$ of the Laplace operator

$$
\begin{equation*}
\Delta=\frac{\partial^{2}}{\partial x_{1}^{2}}+\ldots+\frac{\partial^{2}}{\partial x_{n}^{2}}, \tag{1.16}
\end{equation*}
$$

and it can be represented in terms of the direct $\mathcal{F}$ and the inverse $\mathcal{F}^{-1}$ Fourier transforms by

$$
\begin{gather*}
\left.\left(\mathbf{I}^{\alpha} y\right)(x) \equiv(-\Delta)^{-\alpha / 2} y\right)(x)=\left(\mathcal{F}^{-1}|x|^{-\alpha}(\mathcal{F} y) x\right)  \tag{1.17}\\
\left.\left(\mathbf{D}^{\alpha} y\right)(x) \equiv(-\Delta)^{\alpha / 2} y\right)(x)=\left(\mathcal{F}^{-1}|x|^{\alpha}(\mathcal{F} y) x\right), \quad x=\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{R}^{n} \tag{1.18}
\end{gather*}
$$

It should be noted that for $0<\alpha<n$, the Riesz fractional integration $I^{\alpha}$ can be realized for suitable functions $f$ as the Riesz potential, given (for $x \in \mathbb{R}^{n}$ ) by

$$
\begin{equation*}
\left(\mathbf{I}^{\alpha} f\right)(x)=\gamma(n, \alpha) \int_{\mathbb{R}^{n}} \frac{f(y)}{|x-y|^{n-\alpha}} d y, \quad\left(\gamma(n, \alpha)=\frac{\Gamma[(n-\alpha) / 2]}{2^{\alpha} \pi^{n / 2} \Gamma(\alpha / 2]}\right) \tag{1.19}
\end{equation*}
$$

The method based on the Laplace and Fourier transforms, can be also applied to deduce explicit solutions of partial differential equations of fractional order. Here we discuss some results in this connection and show that such methods can be also applied to investigate the Cauchy-type and Cauchy problems for partial fractional differential equations with the Riemann-Liouville partial fractional derivative with respect to $t$ of order $\alpha>0$ defined by [3, Section 24.2]

$$
\begin{equation*}
\left({ }^{R L} \mathcal{D}_{0+, t}^{\alpha} u\right)(x, t)=\left(\frac{\partial}{\partial t}\right)^{[\alpha]+1} \frac{1}{\Gamma(1-\{\alpha\})} \int_{0}^{t} \frac{u(x, y) d t}{(y-t)^{\{\alpha\}}}(x>0, t>0 ; \alpha>0) \tag{1.20}
\end{equation*}
$$

$[\alpha]$ and $\{\alpha\}$ being the integral and fractional parts of $\alpha$, and with the Caputo partial fractional derivative with respect to $t$ of order $0<\alpha<1$ :

$$
\begin{equation*}
\left({ }^{c} D_{t}^{\alpha} u\right)(t, x)=\frac{1}{\Gamma(1-\alpha)} \int_{0}^{t} \frac{\partial u(\tau, x)}{\partial \tau} \frac{\partial \tau}{(t-\tau)^{\alpha}} \quad(x \in \mathbb{R}, t>0 ; 0<\alpha<1) \tag{1.21}
\end{equation*}
$$

Section 2 deals with the Cauchy-type problem for nonlinear fractional differential equations with the Riemann-Liouville and Hadamard fractional derivatives (1.7) and (1.11). Section 3 is devoted to the Cauchy problem for the nonlinear equations with the so-called sequential fractional derivatives and with the Caputo derivative (1.14). Operational and compositional methods to solution of one-dimensional fractional integral and differential equations are discussed in Sections 4 and 5, respectively. The method to solve such equations based on the Laplace transform is discussed in Section 6, while on the Fourier and Mellin transforms in Section 7. Such an integral transforms approach to solution of partial differential equations is presented in Section 8. Some problems and new trends of research are discussed in Section 9.

We also mention that many authors have applied methods of fractional integrodifferentiation to constructing solutions of ordinary and partial differential equations, to investigating integro-differential equations and to obtaining a unified theory of special functions. We do not discuss such problems here. Anyone may become acquainted with methods and results in these fields in the books [3, Chapter 8] and [17].

## 2. Cauchy-type problems for ordinary differential equations of fractional order: method of reduction to Volterra integral equations

In the beginning we indicate three first papers devoted to reduction of fractional differential equations with the Riemann-Liouville fractional derivative $D_{a+}^{\alpha} y$, given by (1.7), to the Voltera integral equations. Pitcher and Sewell [18] first considered the nonlinear fractional differential equation

$$
\begin{equation*}
\left(D_{a+}^{\alpha} y\right)(x)=f[x, y(x)] \quad(0<\alpha<1, a \in \mathbb{R}) \tag{2.1}
\end{equation*}
$$

provided that $f(x, y)$ is bounded and Lipschitzian with respect to $y$ in a special region $G \subset \mathbb{R} \times \mathbb{R}$. They tried to prove the uniqueness of a continuous solution $y(x)$ of such an equation on the basis of the corresponding result for the nonlinear integral equation

$$
\begin{equation*}
y(x)-\frac{1}{\Gamma(\alpha)} \int_{a}^{x} \frac{f[t, y(t)] d t}{(x-t)^{1-\alpha}}=0 \quad(x>a ; 0<\alpha<1) \tag{2.2}
\end{equation*}
$$

But the result of Pitcher and Sewell given in [18, Theorem 4.2] is not correct because they have used the relation $I_{a+}^{\alpha} D_{a+}^{\alpha} y=y$ instead of the correct one:

$$
\begin{equation*}
\left(I_{a+}^{\alpha} D_{a+}^{\alpha} y\right)(x)=y(x)-\frac{B}{\Gamma(\alpha)}(x-a)^{\alpha-1}, \quad B=\left(I_{a+}^{1-\alpha} y\right)(a+) \tag{2.3}
\end{equation*}
$$

However, the paper of Pitcher and Sewell contained the idea of the reduction of the fractional differential equation (2.1) to the Volterra integral equation (2.2).

Barrett [19] first considered the Cauchy-type problem for the linear differential equation

$$
\begin{equation*}
\left(D_{a+}^{\alpha} y\right)(x)-\lambda y(x)=f(x) \quad(n-1 \leq \operatorname{Re} \alpha<n ; \lambda \in \mathbb{C}) \tag{2.4}
\end{equation*}
$$

with the initial conditions

$$
\begin{equation*}
\left(D_{a+}^{\alpha-k} y\right)(a+)=b_{k} \in \mathbb{C} \quad(k=1,2, \ldots, n) \tag{2.5}
\end{equation*}
$$

on a finite interval $(a, b)$ of the real axis $\mathbb{R}$. Here $\left(D_{a+}^{\alpha-k} y\right)(a+)$ means the limit in the right neighborhood $(a, a+\epsilon)(\epsilon>0)$ of the point $a$ :

$$
\begin{gather*}
\left(D_{a+}^{\alpha-k} y\right)(a+)=\lim _{x \rightarrow a+}\left(D_{a+}^{\alpha-k} y\right)(x) \quad(1 \leq k \leq n-1)  \tag{2.6}\\
\left(D_{a+}^{\alpha-n} y\right)(a+)=\lim _{x \rightarrow a+}\left(I_{a+}^{n-\alpha} y\right)(x)(\alpha \neq n), \quad\left(D_{a+}^{\alpha-n} y\right)(a+)=y(a) \quad(\alpha=n) \tag{2.7}
\end{gather*}
$$

Barrett proved that if $f(x)$ belongs to $L(a, b)$ or $L(a, b) \bigcap C(a, b]$, then the problem (2.4)-(2.5) has a unique solution $y(x)$ in a subspace of $L(a, b)$ and this solution is given by

$$
\begin{align*}
y(x) & =\sum_{j=1}^{n} b_{j}(x-a)^{\alpha-j} E_{\alpha, \alpha-j+1}\left(\lambda(x-a)^{\alpha}\right)+ \\
& +\int_{a}^{x}(x-t)^{\alpha-1} E_{\alpha, \alpha}\left(\lambda(x-t)^{\alpha}\right) f(t) d t . \tag{2.8}
\end{align*}
$$

Here $E_{\alpha, \beta}(z)$ is an entire function, called the Mittag-Leffler function, defined by

$$
\begin{equation*}
E_{\alpha, \beta}(z)=\sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(\alpha k+\beta)} \quad(\alpha>0, \beta>0) ; \tag{2.9}
\end{equation*}
$$

see [20, Section 18.1]. Arguments of Barrett were based on the formula for the product $I_{a+}^{\alpha} D_{a+}^{\alpha} f$ generalizing (2.3):

$$
\begin{gather*}
\left(I_{a+}^{\alpha} D_{a+}^{\alpha} y\right)(x)=y(x)-\sum_{k=1}^{n} B_{k} \frac{(x-a)^{\alpha-k}}{\Gamma(\alpha-k+1)},  \tag{2.10}\\
B_{k}=y_{n-\alpha}^{(n-k)}(a), y_{n-\alpha}(x)=\left(I_{a+}^{n-\alpha} y\right)(x) \quad(\alpha \in \mathbb{C}, n=[\operatorname{Re} \alpha]+1) \tag{2.11}
\end{gather*}
$$

Barrett [19] has used implicitly the method of reduction of the Cauchy-type problem (2.4)-(2.5) to the Volterra integral equation of the second kind

$$
\begin{equation*}
y(x)=\sum_{j=1}^{n} \frac{b_{j}}{\Gamma(\alpha-j+1)}(x-a)^{\alpha-j}+\frac{\lambda}{\Gamma(\alpha)} \int_{a}^{x} \frac{y(t) d t}{(x-t)^{1-\alpha}}+\frac{1}{\Gamma(\alpha)} \int_{a}^{x} \frac{f(t) d t}{(x-t)^{1-\alpha}} \tag{2.12}
\end{equation*}
$$

and the method of successive approximations. According to this method, we set

$$
\begin{gather*}
y_{0}(x)=\sum_{j=1}^{n} \frac{b_{j}}{\Gamma(\alpha-j+1)}(x-a)^{\alpha-j}, \\
y_{m}(x)=y_{0}(x)+\frac{\lambda}{\Gamma(\alpha)} \int_{a}^{x} \frac{y_{m-1}(t) d t}{(x-t)^{1-\alpha}}+\frac{1}{\Gamma(\alpha)} \int_{a}^{x} \frac{f(t) d t}{(x-t)^{1-\alpha}} \quad(m=1,2, \ldots), \tag{2.13}
\end{gather*}
$$

and have

$$
\begin{equation*}
y_{m}(x)=\sum_{j=1}^{n} b_{j} \sum_{i=1}^{m+1} \frac{\lambda^{i-1}(x-a)^{\alpha i-j}}{\Gamma(\alpha i-j+1)}+\sum_{i=1}^{m} \frac{\lambda^{i-1}}{\Gamma(\alpha i)} \int_{a}^{x}(x-t)^{\alpha i-1} f(t) d t \tag{2.14}
\end{equation*}
$$

for $m=1,2, \ldots$ Passing to a limit, as $m \rightarrow \infty$, and taking into account (2.9) we obtain the solution (2.8) of the Cauchy-type problem (2.4)-(2.5).

Al-Bassam [21] first considered the Cauchy-type problem

$$
\begin{align*}
\left(D_{a+}^{\alpha} y\right)(x) & =f[x, y(x)] \quad(0<\alpha \leq 1)  \tag{2.15}\\
\left(D_{a+}^{\alpha-1} y\right)(a+) & \equiv\left(I_{a+}^{1-\alpha} y\right)(a+)=b_{1}, \quad b_{1} \in \mathbb{R} \tag{2.16}
\end{align*}
$$

in the space of continuous functions $C[a, b]$ provided that $f(x, y)$ is a real-valued, continuous and Lipschitzian function in a domain $G \subset \mathbb{R} \times \mathbb{R}$ such that $\sup _{(x, y) \in G}|f(x, y)|=$ $b_{0}<\infty$. Applying the operator $I_{a+}^{\alpha}$ to both sides of (2.15), using the relation (2.3) and the initial conditions (2.16), he reduced the above problem to the Volterra nonlinear integral equation

$$
\begin{equation*}
y(x)=\frac{b_{1}}{\Gamma(\alpha)}(x-a)^{\alpha-1}+\frac{1}{\Gamma(\alpha)} \int_{a}^{x} \frac{f[t, y(t)] d t}{(x-t)^{1-\alpha}} \quad(x>a ; 0<\alpha \leq 1) \tag{2.17}
\end{equation*}
$$

Using the method of successive approximations, Al-Bassam established the existence of the continuous solution $y(x)$ of the equation (2.17). Besides, he probably first indicated that the method of contracting mapping can be applied to prove the uniqueness of this solution $y(x)$ of (2.17), and gave such a formal proof. Al-Bassam also indicated - but did not prove - the equivalence of the Cauchy-type problem (2.15)-(2.16) and the integral equation (2.17), and therefore his results on the existence and uniqueness of the continuous solution $y(x)$ could be true only for the integral equation (2.17).

We also note that the conditions suggested by Al-Bassam are not suitable to solve the Cauchy-type problem (2.15)-(2.16) in the simplest linear case when $f[x, y(x)]=y(x)$.

The same remarks apply to his existence and uniqueness results formulated without proof to more general than (2.15)-(2.16) Cauchy-type problem with real $\alpha>0$ :

$$
\begin{gather*}
\left(D_{a+}^{\alpha} y\right)(x)=f[x, y(x)] \quad(n-1<\alpha \leq n, n=-[-\alpha]),  \tag{2.18}\\
\left(D_{a+}^{\alpha-k} y\right)(a+)=b_{k}, \quad b_{k} \in \mathbb{R}(k=1,2, \ldots, n), \tag{2.19}
\end{gather*}
$$

where the corresponding Volterra equation has the form (2.17):

$$
\begin{gather*}
y(x)=\sum_{j=1}^{n} \frac{b_{j}}{\Gamma(\alpha-j+1)}(x-a)^{\alpha-j}+\frac{1}{\Gamma(\alpha)} \int_{a}^{x} \frac{f[t, y(t)] d t}{(x-t)^{1-\alpha}}  \tag{2.20}\\
(x>a ; n-1<\alpha \leq n),
\end{gather*}
$$

to the system of problems (2.18)-(2.19) and to more general than (2.18) differential equations.

The approach suggested by Al-Bassam was used by many authors. However, they have not completed their investigations. Most of the researchers obtained some results not for the initial value problems, but for the corresponding Volterra integral equations. Some authors considered only particular cases. Moreover, some of the results obtained contained mistakes in the proof of the equivalence of initial value problems and the Volterra integral equations and in the proof of the uniqueness theorem. In this connection see Kilbas and Trujilo [13, Sections 4 and 5].

Kilbas, Bonilla and Trujillo [22, 23] have studied the Cauchy-type problem (2.18)(2.19) with complex $\alpha \in \mathbb{C}(\operatorname{Re}(\alpha)>0)$ on a finite interval $[a, b]$ of the real axis $\mathbb{R}$ in the space of absolutely integrable functions $L(a, b)$. The equivalence of this problem and the nonlinear Volterra integral equation (2.20) was established. The existence and uniqueness of the solution $y(x)$ of such a problem was proved by using the method of successive expansions. The results obtained were extended to the system of problems (2.18)-(2.19) in [24].

Similar results to the Cauchy-type problem (2.18)-(2.19) in the weighted space of continuous functions $C_{n-\alpha}[a, b]$ :

$$
\begin{equation*}
C_{n-\alpha}[a, b](x)=\left\{y(x):(x-a)^{n-\alpha} y(x) \in C[a, b] ; \quad \alpha \in \mathbb{C}, \operatorname{Re} \alpha>0\right\} \tag{2.21}
\end{equation*}
$$

with $n=[\operatorname{Re}(\alpha)]+1$ for $\alpha \notin \mathbb{N}$ and $n=\alpha$ for $\alpha \in \mathbb{N}$, were established in Kilbas, Bonilla and Trujillo [25] and Kilbas, Rivero and Trujillo [26]. In particular, the corresponding results were deduced to the Cauchy-type problem (2.15)-(2.16), with real $0<\alpha \leq 1$ being replaced by complex $\alpha, 0<\operatorname{Re}(\alpha) \leq 1$, and similar assertions were established to the weighted Cauchy problem

$$
\begin{equation*}
\left(D_{a+}^{\alpha} y\right)(x)=f[x, y(x)], \quad \lim _{x \rightarrow a+}\left[(x-a)^{n-\alpha} y(x)\right]=b_{1}, \quad b_{1} \in \mathbb{R} \tag{2.22}
\end{equation*}
$$

equivalent to the problem (2.15)-(2.16), in the space $C_{1-\alpha}[a, b]$.
Kilbas and Marzan [27] extended the above results to the Cauchy-type problem for more general than (2.18) nonlinear differential equation of complex order $\alpha \in \mathbb{C}$ $\left(0<\operatorname{Re} \alpha_{1}<\ldots<\operatorname{Re} \alpha_{n-1}<\operatorname{Re} \alpha\right):$

$$
\begin{equation*}
\left(D_{a+}^{\alpha} y\right)(x)=f\left[x, y(x),\left(D_{a+}^{\alpha_{1}} y\right)(x), \ldots,\left(D_{a+}^{\alpha_{n-1}} y\right)(x)\right] \tag{2.23}
\end{equation*}
$$

with the initial conditions (2.19).
3. Cauchy problems for ordinary differential equations of fractional order: method of reduction to Volterra integral equations. Continuation
Dzhrbashyan and Nersesyan [28] first studied the linear differential equation of the form

$$
\begin{equation*}
\left(D^{\sigma} y\right)(x) \equiv\left(D^{\sigma_{n}} y\right)(x)+\sum_{k=0}^{n-1} a_{k}(x)\left(D^{\sigma_{n-k-1}} y\right)(x)+a_{n}(x) y(x)=f(x) \tag{3.1}
\end{equation*}
$$

with the modified fractional derivatives $\left(D^{\sigma_{n}} y\right)(x)$ and $\left(D^{\sigma_{n-k-1}} y\right)(x)(k=0,1$, $\ldots, n-1$ ) defined in terms of the Riemann-Liouville fractional derivatives (1.7) by

$$
\begin{align*}
& D^{\sigma_{k}}=D_{0+}^{\alpha_{k}-1} D_{0+}^{\alpha_{k-1}} \ldots D_{0+}^{\alpha_{0}} \quad(k=1,2, \ldots n), \quad D^{\sigma_{0}}=D_{0+}^{\alpha_{0}-1}  \tag{3.2}\\
& \sigma_{k}=\sum_{j=0}^{k} \alpha_{j}-1(k=0,1, \ldots, n) ; \quad 0<\alpha_{j} \leq 1 \quad(j=0,1, \ldots, n)  \tag{3.3}\\
& \quad\left(\alpha_{k}=\sigma_{k}-\sigma_{k-1} \quad(k=1,2, \ldots, n), \quad \alpha_{0}=\sigma_{0}+1\right)
\end{align*}
$$

Constructions of the form (3.2) are known as sequential fractional derivatives. Special cases of such modified fractional derivatives in the form $\left(D_{0+}^{\alpha}\right)^{k}(k \in \mathbb{N})$ together with the corresponding linear fractional differential equations were investigated by Miller and Ross [4].

Dzhrbashyan and Nersesyan [28] proved that for $\alpha_{0}>1-\alpha_{n}$ the Cauchy problem

$$
\begin{equation*}
\left(D^{\sigma_{n}} y\right)(x)=f(x), \quad\left(D^{\sigma_{k}} y\right)(0+)=b_{k} \in \mathbb{C} \quad(k=0,1, \ldots, n-1) \tag{3.4}
\end{equation*}
$$

has a unique continuous solution $y(x) \in C[0, d]$ on an interval $[0, d]$ provided that the functions $a_{k}(x) \quad(0 \leq k \leq n-1)$ and $f(x)$ satisfy some additional conditions. In particular, when $a_{k}(x)=0(k=0,1, \ldots, n)$, they obtained the explicit solution

$$
\begin{equation*}
y(x)=\sum_{k=0}^{n-1} \frac{b_{k} x^{\sigma_{k}}}{\Gamma\left(1+\sigma_{k}\right)}+\frac{1}{\Gamma\left(\sigma_{n}\right)} \int_{a}^{x}(x-t)^{\sigma_{n}-1} f(t) d t \tag{3.5}
\end{equation*}
$$

of the Cauchy problem

$$
\begin{equation*}
\left(D^{\sigma_{n}} y\right)(x)=f(x), \quad\left(D^{\sigma_{k}} y\right)(0+)=b_{k} \quad(k=0,1, \ldots, n-1) \tag{3.6}
\end{equation*}
$$

Bonilla, Kilbas and Trujillo [29] constructed the theory of special classes of linear fractional differential equations with sequential fractional derivatives and with constant coefficients.

Delbosco and Rodino [30] considered the Cauchy problem for the nonlinear differential equation

$$
\begin{equation*}
\left(D_{0+}^{\alpha} y\right)(x)=f[x, y(x)](0 \leq x \leq T), \quad y^{(k)}(0)=y_{k}(0) \quad(k=0,1,2, \ldots,[\alpha]) \tag{3.7}
\end{equation*}
$$

with continuous function $f(x, y)$ on $[0,1] \times \mathbb{R}$. Using Schauder's fixed point theorem, they gave conditions for the existence of at least one and of a one continuous solution $y(x)$ on $[0, \delta]$ for the corresponding Volterra integral equation. Delbosco and Rodino showed that if additionally $f[x, y(x)]$ is weighted Lipschitzian:

$$
\begin{equation*}
|f[x, y(x)]-f[x, Y(x)]| \leq \frac{M}{x^{\sigma}}|y(x)-Y(x)| \tag{3.8}
\end{equation*}
$$

and $f[x, y(x)]=f[y(x)]$ and $f(0)=0$, then the Cauchy problem

$$
\begin{equation*}
\left(D_{0+}^{\alpha} y\right)(x)=f[y(x)], \quad y(a)=b \in \mathbb{R} \quad(0<\alpha<1, a>0) \tag{3.9}
\end{equation*}
$$

and the weighted Cauchy problem

$$
\begin{equation*}
\left(D_{0+}^{\alpha} y\right)(x)=f[y(x)], \quad \lim _{x \rightarrow 0} x^{1-\alpha} y(x)=c \in \mathbb{R} \quad(0<\alpha<1) \tag{3.10}
\end{equation*}
$$

have a unique solution $y(x)$ such that $x^{1-\alpha} y(x) \in C[0, h]$ for any $h>0$.
Hayek, Trujillo, Rivero, Bonilla and Moreno [31] investigated the Cauchy problem for a system of linear differential equations

$$
\begin{equation*}
\left(D_{0+}^{\alpha} y\right)(x)=f[x, y(x)], \quad y(a)=b \quad\left(0<\alpha \leq 1, a>0, b \in \mathbb{R}^{n}\right) \tag{3.11}
\end{equation*}
$$

with a real valued vector function $y(x)$ provided that $f(x, y)$ is continuous and Lipschitzian with respect to $y$. Applying the method of contractive mapping defined on a complete metric space, they proved the existence and uniqueness of a continuous solution $y(x)$ of this problem. In particular, they obtained such a result to the system of linear differential equations

$$
\begin{equation*}
\left(D_{0+}^{\alpha} y\right)(x)=A(x) y(x)+B(x), \quad y(a)=b \quad\left(0<\alpha \leq 1, a>0, b \in \mathbb{R}^{n}\right) \tag{3.12}
\end{equation*}
$$

with continuous matrices $A(x)$ and $B(x)$.
Kilbas, Marzan and Titioura [32] considered the Cauchy-type problem for the nonlinear differential equation of the form (1.5) with the Hadamard fractional derivative (1.11) on a finite interval $(a, b) \quad(0<a<b<\infty)$ :

$$
\begin{equation*}
\left.\left({ }^{H} D_{a+}^{\alpha} y\right)(x)=f[x, y(x)] \quad(\alpha \in \mathbb{C}, \operatorname{Re} \alpha>0) ; a>0\right), \tag{3.13}
\end{equation*}
$$

where $n=[\operatorname{Re} \alpha+1]$ for $\alpha \notin \mathbb{N}$ and $n=\alpha$ for $\alpha \in \mathbb{N}$, with the initial conditions

$$
\begin{equation*}
\left({ }^{H} D_{a+}^{\alpha-k} y\right)(a+)=b_{k}, \quad b_{k} \in \mathbb{C} \quad(k=1,2, \ldots, n) . \tag{3.14}
\end{equation*}
$$

It was proved the equivalence of this problem and the Volterra integral equation

$$
\begin{equation*}
y(x)=\sum_{j=1}^{n} \frac{b_{j}}{\Gamma(\alpha-j+1)}\left(\log \frac{x}{a}\right)^{\alpha-1}+\frac{1}{\Gamma(\alpha)} \int_{a}^{x}\left(\log \frac{x}{t}\right)^{\alpha-1} f[t, y(t)] \frac{d t}{t} \quad(x>a) \tag{3.15}
\end{equation*}
$$

in the space $X_{0}^{1}(a, b)$ of Lebesgue measurable functions $y(x)$ on $[a, b]$ such that $[y(x) / x] \in L(a, b)$. Using this fact the conditions were given for a unique solution of the problem (3.13)-(3.14) in $X_{0}^{1}(a, b)$. The method of successive approximation can be also applied to establish a unique solution of the corresponding Cauchy-type problem to the linear fractional differential equation

$$
\begin{equation*}
\left.\left({ }^{H} D_{a+}^{\alpha} y\right)(x)=\lambda y(x)+h(x) \quad(\lambda \in \mathbb{C}, \operatorname{Re} \alpha>0)\right) \tag{3.16}
\end{equation*}
$$

with the initial conditions (3.14) in the form

$$
\begin{align*}
y(x) & =\sum_{j=1}^{n} b_{j}\left(\log \frac{x}{a}\right)^{\alpha-j} E_{\alpha, \alpha-j+1}\left(\lambda\left(\log \frac{x}{t}\right)^{\alpha}\right)+ \\
& +\int_{a}^{x}\left(\log \frac{x}{t}\right)^{\alpha-1} E_{\alpha, \alpha}\left(\lambda\left(\log \frac{x}{t}\right)^{\alpha}\right) f(t) \frac{d t}{t} \tag{3.17}
\end{align*}
$$

Kilbas and Marzan [33, 34] investigated the differential equation of the form (1.5)

$$
\begin{equation*}
\left({ }^{C} D_{a+}^{\alpha} y\right)(x)=f[x, y(x)] \quad(a \leq x \leq b ; \alpha \in \mathbb{C}, \operatorname{Re} \alpha>0) \tag{3.18}
\end{equation*}
$$

with the Caputo fractional derivative (1.14) and with the initial conditions

$$
\begin{equation*}
y^{(k)}(a+)=b_{k}, \quad b_{k} \in \mathbb{C} \quad(k=0,1,2, \ldots, n-1), \tag{3.19}
\end{equation*}
$$

where $n=[\operatorname{Re} \alpha]+1$ for $\alpha \notin \mathbb{N}$ while $n=\alpha$ for $\alpha \in \mathbb{N}$. The equivalence of the Cauchy problem (3.18)-(3.19) and the corresponding Volterra equation

$$
\begin{equation*}
y(x)=\sum_{j=1}^{n-1} \frac{b_{j}}{j!}(x-a)^{\alpha-j}+\frac{1}{\Gamma(\alpha)} \int_{a}^{x} \frac{f[t, y(t)] d t}{(x-t)^{1-\alpha}} \quad(x>a ; n-1<\alpha \leq n) \tag{3.20}
\end{equation*}
$$

in the space $C^{n-1}[a, b]$ of functions $y(x)$ continuously differentiable up to $n-1$ was proved. On the basis of this fact and Schauder's fixed point theorem, the conditions for the existence of a unique solution $y(x) \in C^{n-1}[a, b]$ of the problem (3.18)-(3.19) and conditions, when this problem has at least one such a solution, were given. Using the method of successive expansions, the unique solution of the Cauchy problem for the corresponding to (3.18) linear equation

$$
\begin{equation*}
\left({ }^{C} D_{a+}^{\alpha} y\right)(x)-\lambda y(x)=f(x), \quad y^{(k)}(a+)=b_{k} \in \mathbb{C} \quad(k=0,1,2, \ldots, n-1) \tag{3.21}
\end{equation*}
$$

was deduced in the form

$$
\begin{equation*}
y(x)=\sum_{j=1}^{n-1} b_{j}(x-a)^{j} E_{\alpha, j+1}\left(\lambda(x-a)^{\alpha}\right)+\int_{a}^{x}(x-t)^{\alpha-1} E_{\alpha, \alpha}\left(\lambda(x-t)^{\alpha}\right) f(t) d t \tag{3.22}
\end{equation*}
$$

## 4. Ordinary differential equations of fractional order: operational calculus method

The usefulness of operational calculus to solve ordinary differential equations is well known [35]. The basis of such an operational calculus for the operators of differentiation was developed by Mikusinski [36]. It is based on the interpretation of the Laplace convolution

$$
\begin{equation*}
(f * g)(x)=\int_{0}^{x} f(x-t) g(t) d t \tag{4.1}
\end{equation*}
$$

as a multiplication of elements $f$ and $g$ in the ring of functions continuous on the half-axis $\mathbb{R}_{+}$. Mikusinski applied his operational calculus to solve ordinary differential equations with constant coefficients.

Mikusinski's scheme was developed by Ditkin [37], Ditkin and Prudnikov [38], Meller [39] and Rodriguez [40] to construct the operational calculus for a Bessel-type differential operators with nonconstant coefficients. Dimovski [41] considered the transform approach to the development of operational calculus. Rodriguez, Trujillo and Rivero [42] were probably the first who applied operational calculus for a Kratzel transform to solve a Bessel-type differential equation of fractional order.

A series of papers were devoted to develop the operational calculus to fractional calculus operators with applications to solution of differential equations of fractional order. Luchko and Srivastava [43] have constructed the operational calculus to the Riemann - Liouville fractional derivative $D_{0+}^{\alpha} y$ given in (1.7), in a special space $\mathcal{C}_{-1}$ of
functions $y(x)$ such as $x^{-p}\left(D_{0+}^{\alpha}\right)^{k} y(x) \in C[0, \infty)(k=1,2, \ldots, m)$ for some $p>-1$. They proved that the operation $*_{\lambda}$ defined for $\lambda \geq 1$ by

$$
\begin{equation*}
\left(f *_{\lambda} g\right)(x)=\left(I_{0+}^{\lambda-1} f * g\right)(x)=\int_{0}^{x}\left(I_{0+}^{\lambda-1} f\right)(x-t) g(t) d t \quad(\lambda \geq 1) \tag{4.2}
\end{equation*}
$$

$I_{0+}^{\lambda-1}$ being the operation of Riemann - Liouville fractional integration (1.6), is the convolution (without divisors) of the linear operator $I_{0+}^{\alpha}$ for $\alpha>0$ in the space $\mathcal{C}_{-1}$, and that the Riemann-Liouville operator $I_{0+}^{\alpha}$ has the convolution representation

$$
\begin{equation*}
\left(I_{0+}^{\alpha} f\right)(x)=\left(h *_{\lambda} f\right)(x) \quad\left(1 \leq \lambda<\alpha+1, h(x):=\frac{x^{\alpha-\lambda}}{\Gamma(\alpha-\lambda-1)}\right) \tag{4.3}
\end{equation*}
$$

They showed that the space $\mathcal{C}_{-1}$ with the operations $*_{\lambda}$ and + , having the property of distributivity

$$
\begin{equation*}
\left(f *_{\lambda}(g+h)\right)(x)=\left(f *_{\lambda} g\right)(x)+\left(f *_{\lambda} h\right)(x) \quad\left(f, g, h \in \mathcal{C}_{-1}\right), \tag{4.4}
\end{equation*}
$$

becomes a commutative ring without divisors of zero, and therefore, following Mikusinski [36], $\mathcal{C}_{-1}$ can be extended to the quotient field $\mathcal{M}$. Luchko and Srivastava indicated that the elements of the field $\mathcal{M}$ can be considered as convolution quotients $f / g$ with the operations

$$
\begin{equation*}
\frac{f}{g}+\frac{f_{1}}{g_{1}}=\frac{\left(f *_{\lambda} g_{1}\right)+\left(g *_{\lambda} f_{1}\right)}{\left(g *_{\lambda} g_{1}\right)}, \quad\left(\frac{f}{g}\right)\left(\frac{f_{1}}{g_{1}}\right)=\frac{\left(f *_{\lambda} f_{1}\right)}{\left(g *_{\lambda} g_{1}\right)}, \tag{4.5}
\end{equation*}
$$

which means that the ring $\mathcal{C}_{1}$ can be embedded in the field $\mathcal{M}$ by the map

$$
\begin{equation*}
f(x) \rightarrow \frac{\left(h *_{\lambda} f\right)(x)}{h(x)} \tag{4.6}
\end{equation*}
$$

with $h(x)$ in (4.3). On the basis of these facts they defined the algebraic inverse of the operator $I_{0+}^{\alpha}$ as an element $S$ of the field $\mathcal{M}$ which is reciprocal to the element $h(x)$ in the field $\mathcal{M}$ :

$$
\begin{equation*}
S=\frac{I}{h} \equiv \frac{h}{\left(h *_{\lambda} h\right)} \equiv \frac{h}{h^{2}}, \tag{4.7}
\end{equation*}
$$

where $I=h / h$ denotes the identity element of the field $\mathcal{M}$ with respect to the operation of multiplication.

Introducing the space

$$
\begin{equation*}
\left.\Omega_{\alpha}^{m}\left(\mathcal{C}_{-1}\right)=\left\{f(x) \in \mathcal{C}_{-1}:\left(D_{0+}^{m \alpha}\right)^{k} f\right)(x) \in \mathcal{C}_{-1}(k=1,2, \ldots, m)\right\} \tag{4.8}
\end{equation*}
$$

with $m \in \mathbb{N}$ and $\alpha>0$, Luchko and Srivastava proved the relation for $f(x) \in \Omega_{\alpha}^{m}\left(\mathcal{C}_{-1}\right)$ in the field $\mathcal{M}$

$$
\begin{equation*}
\left(\left(D_{0+}^{\alpha}\right)^{m} f\right)(x)=S^{m} f-\sum_{k=0}^{m-1} S^{m-k} F\left(D_{0+}^{\alpha}\right)^{k} f \tag{4.9}
\end{equation*}
$$

where the operator $F=E-I_{0+}^{\alpha} D_{0+}^{\alpha}$ is given by

$$
\begin{equation*}
(F f)(x):=\left(\left(E-I_{0+}^{m \alpha} D_{0+}^{\alpha}\right) f\right)(x)=\sum_{k=0}^{-[-\alpha]} \frac{x^{\alpha-k}}{\Gamma(\alpha-k+1)} \lim _{x \rightarrow 0}\left(D_{0+}^{\alpha-k} f\right)(x) \tag{4.10}
\end{equation*}
$$

and $E$ is the identity operator. This result means that the Riemann - Liouville fractional differentiation operator $D_{0+}^{\alpha}$ is reduced to the operator of multiplication in the field $\mathcal{M}$.

Such an operational calculus was applied by Luchko and Srivastava [43] to solve the following Cauchy-type problem:

$$
\begin{equation*}
\left(P_{m}\left(D_{0+}^{\alpha}\right) y\right)(x)=f(x), \quad P_{m}(z)=\sum_{k=1}^{m} c_{k} z^{k} \tag{4.11}
\end{equation*}
$$

with any $\alpha>0$ and $f(x) \in \mathcal{C}_{-1}$ in the space $\Omega_{\alpha}^{m}\left(\mathcal{C}_{-1}\right)$ with the initial conditions

$$
\begin{equation*}
\lim _{x \rightarrow 0}\left(D_{0+}^{\alpha-k}\left(D_{0+}^{\alpha}\right)^{i} y\right)(x)=b_{i k} \quad(i=0,1, \ldots m-1 ; k=1, \ldots, \eta) \tag{4.12}
\end{equation*}
$$

They reduced this problem to the algebraic equation in the field $\mathcal{M}$

$$
\begin{equation*}
P_{m}(S)=f+\sum_{i=0}^{m-1} P_{i}(S) \gamma_{i}, \quad P_{i}(S)=\sum_{j=1}^{m-i} c_{i+j} S^{j} \quad(i=0,1, \ldots, m-1) \tag{4.13}
\end{equation*}
$$

found its unique solution

$$
\begin{equation*}
y=\frac{I}{P_{m}(S)}+\sum_{i=0}^{m-1} \frac{P_{i}(S)}{P_{m}(S)} \gamma_{i} \tag{4.14}
\end{equation*}
$$

and gave the explicit solution of the Cauchy-type problem (4.11)-(4.12) in terms of the special function

$$
\begin{equation*}
E_{\alpha, \beta}^{\varrho}(z)=\sum_{k=0}^{\infty} \frac{(\varrho)_{k} z^{k}}{k!\Gamma(\alpha k+\beta)}, \tag{4.15}
\end{equation*}
$$

where $(\varrho)_{k}$ is the Pochhammer symbol defined by

$$
\begin{equation*}
(\varrho)_{k}=1,(\varrho)_{k}=\varrho(\varrho+1) \ldots(\varrho+k-1)(k=1,2, \ldots) . \tag{4.16}
\end{equation*}
$$

(4.15) is a generalization of the Mittag - Leffler function (2.9), deducing from (4.15) for $\varrho=1$.

Luchko and Yakubovich [44], Al-Bassam and Luchko ]45] and Hadid and Luchko [46] have used such a method to solve the Cauchy-type problems for fractional differential equations with constant coefficients involving the so-called Erdelyi-Kober-type fractional derivatives - see [3, Section 18.1]. The explicit solutions in these cases are expressed via the function of Mittag - Leffler type

$$
\begin{equation*}
E_{\varrho}\left((\alpha, \beta)_{n} ; z\right)=\sum_{k=0}^{\infty} \frac{(\varrho)_{k} z^{k}}{k!\prod_{i=1}^{n} \Gamma\left(\alpha_{i} k+\beta_{i}\right)}, \tag{4.17}
\end{equation*}
$$

more general then (4.15).
Luchko and Gorenflo [47] have used the operational method to prove that the Cauchy problem for the fractional differential equations with the Caputo fractional derivative (1.15)

$$
\begin{gather*}
\left({ }^{C} D^{\alpha} y\right)(x)-\lambda y(x)=f(x)  \tag{4.18}\\
y^{(k)}(0)=b_{k} \quad(k=0,1, \ldots n-1 ; n-1<\alpha \leq n) \tag{4.19}
\end{gather*}
$$

has the unique solution

$$
\begin{equation*}
y(x)=\sum_{k=0}^{n-1} b_{k} x^{k} E_{\alpha, k+1}\left(\lambda x^{\alpha}\right)+\int_{0}^{x} t^{\alpha-1} E_{\alpha, \alpha}\left(\lambda t^{\alpha}\right) f(x-t) d t \tag{4.20}
\end{equation*}
$$

in a special space of functions. They also investigated the Cauchy problem for more general fractional differential equation

$$
\begin{equation*}
\left(D_{*}^{\alpha} y\right)(x)-\sum_{k=1}^{m} c_{k}\left(D_{*}^{\alpha_{k}} y\right)(x)=f(x) \quad\left(\alpha>\alpha_{1}>\ldots>\alpha_{m} \geq 0\right) \tag{4.21}
\end{equation*}
$$

with the initial conditions (4.19) and constructed its explicit solution via multivariate Mittag-Leffler function.

The above and other results were discussed in a survey paper by Luchko [48]. We also note that Elizarraraz and Verde-Star [49] obtained the explicit general solution of the equation (4.11) and the explicit solution of the Cauchy-type problem (4.11)-(4.12) by using linear algebra construction and classical methods of operational calculus. Their approach was based on introducing divided differences of fractional order, coinciding with the Riemann-Liouville fractional differential operators in a certain space of functions, and generalized exponential polynomials, which are connected with functions of Mittag - Leffler type.

## 5. Integral and differential equations of fractional order: compositional method

The idea of the compositional method is based on the known formula for the Riemann - Liouville fractional integral (1.6) and derivative (1.7):

$$
\begin{align*}
& \left(I_{a+}^{\alpha}(t-a)^{\beta-1}\right)(x)=\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a)^{\beta+\alpha-1} \quad(\operatorname{Re} \beta>\operatorname{Re} \alpha)>0  \tag{5.1}\\
& \left(D_{a+}^{\alpha}(t-a)^{\beta-1}\right)(x)=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(x-a)^{\beta-\alpha-1} \quad(\operatorname{Re} \beta>\operatorname{Re} \alpha>0) \tag{5.2}
\end{align*}
$$

According to (1.6) and (1.7), (5.1) and (5.2) mean that the composition of the Riemann Liouville fractional integral $I_{a+}^{\alpha}$ derivative $D_{a+}^{\alpha}$ with the power function $(x-a)^{\beta-1}$ leads to the same function apart to a certain function factor. It means that

$$
\begin{equation*}
y(x)=(x-a)^{\beta-1} \tag{5.3}
\end{equation*}
$$

is a solution of the homogeneous integral equation

$$
\begin{equation*}
y(x)=\frac{\Gamma(\alpha+\beta)(x-a)^{\alpha}}{\Gamma(\alpha) \Gamma(\beta)} \int_{a}^{x} \frac{y(t) d t}{(x-t)^{1-\alpha}} \tag{5.4}
\end{equation*}
$$

and of the fractional differential equation

$$
\begin{equation*}
\left(D_{a+}^{\alpha} y\right)(x)=\frac{\Gamma(\beta)(x-a)^{-\alpha}}{\Gamma(\beta-\alpha)} y(x) \quad(\operatorname{Re} \beta>\operatorname{Re} \alpha>0) \tag{5.5}
\end{equation*}
$$

These arguments lead us to the conjecture that compositions of fractional integrals and derivatives with elementary functions can give exact solutions of integral and differential equations of fractional order. Moreover, from here we deduce another assumption about the possibility of such results for compositions of fractional calculus operators with special functions. It allows us to find the explicit solutions of new classes of differential equations of fractional order. The compositional method based on relations between the Riemann - Liouville and Liouville fractional differentiation operators with functions of Mittag - Leffler type was developed by the author together with Saigo, and
with functions of Bessel type - together with Bonilla, Rivero, Rodriguez and Trujillo. Here we characterize some of the results.

Kilbas and Saigo [50-53] and Saigo and Kilbas [54, 55] have investigated compositions of the Riemann - Liouville fractional integral (1.6) and derivative (1.7) with a special entire function of the form

$$
\begin{equation*}
E_{\alpha, m, l}(z)=\sum_{k=0}^{\infty} c_{k} z^{k} \tag{5.6}
\end{equation*}
$$

with

$$
\begin{equation*}
c_{0}=1, c_{k}=\prod_{i=0}^{k-1} \frac{\Gamma[\alpha(i m+l)+1]}{\Gamma[\alpha(i m+l+1)+1]} \quad(k=1,2, \ldots) \tag{5.7}
\end{equation*}
$$

Such a function, defined for $\alpha>0, m>0$ and $l \in \mathbb{R}$ such that $\alpha(m+l) \neq-1,-2, \ldots$ $(i+0,1,2, \ldots)$, was introduced by Kilbas and Saigo in [56] while studying asymptotic properties of solutions of linear integral equations of Abel-Volterra type. When $m=1$, this function coincides with the Mittag - Leffer function $E_{\alpha, \alpha l+1}(z)$ in (2.9) apart to the constant factor $\Gamma(\alpha l+1)$ :

$$
\begin{equation*}
E_{\alpha, 1, l}(z)=\Gamma(\alpha l+1) E_{\alpha, \alpha l+1}(z) \tag{5.8}
\end{equation*}
$$

Kilbas and Saigo proved the relations

$$
\begin{equation*}
\left(I_{0+}^{\alpha}\left[t^{\alpha l} E_{\alpha, m, l}\left(a t^{\alpha m}\right)\right]\right)(x)=\frac{1}{a} x^{\alpha(l-m+1)}\left[E_{\alpha, m, l}\left(a x^{\alpha m}\right)-1\right], \tag{5.9}
\end{equation*}
$$

provided that $\alpha>0, m>0$ and $l>-1 / l$, and

$$
\begin{gather*}
\left(D_{0+}^{\alpha}\left[t^{\alpha(l-m+1)} E_{\alpha, m, l}\left(a t^{\alpha m}\right)\right]\right)(x)= \\
=\frac{\Gamma[\alpha(l-m+1)+1]}{\Gamma[\alpha(l-m)+1]} x^{\alpha(l-m)}+a x^{\alpha l} E_{\alpha, m, l}\left(a x^{\alpha m}\right), \tag{5.10}
\end{gather*}
$$

provided that $l>m-1-1 / \alpha$; in particular

$$
\begin{equation*}
\left(D_{0+}^{\alpha}\left[t^{\alpha(l-m+1)} E_{\alpha, m, l}\left(a t^{\alpha m}\right)\right]\right)(x)=a x^{\alpha l} E_{\alpha, m, l}\left(a x^{\alpha m}\right) \tag{5.11}
\end{equation*}
$$

when $\alpha(l-m)=-j$ for some $j=1,2, \ldots,-[-\alpha]$.
On the basis of (5.9) they proved that the Abel-Volterra integral equation

$$
\begin{equation*}
y(x)=\frac{a x^{\beta}}{\Gamma(\alpha)} \int_{0}^{x} \frac{y(t) g t}{(x-t)^{1-\alpha}}=\sum_{k=0}^{n} f_{k} x^{\mu_{k}} \quad(0<x<d \leq \infty) \tag{5.12}
\end{equation*}
$$

with $\alpha>0, \beta>-\alpha$ and $\mu_{k}>-1, f_{k} \in \mathbb{R}(k=0,1, \ldots, n)$ has the solution

$$
\begin{equation*}
y(x)=\sum_{k=0}^{n} f_{k} x^{\mu_{k}} E_{\alpha, 1+\beta / \alpha, \mu_{k} / \alpha}\left(a x^{\alpha+\beta}\right) . \tag{5.13}
\end{equation*}
$$

Using (5.11), Kilbas and Saigo showed that the homogeneous fractional differential equation

$$
\begin{equation*}
\left(D_{0+}^{\alpha} y\right)(x)=a x^{\beta} y(x) \quad(0<x<d<\infty ; \alpha>0, a \neq 0, \beta>-\alpha) \tag{5.14}
\end{equation*}
$$

with $\alpha \neq 1,2, \ldots$ has linearly independent $[\alpha]+1$ solutions

$$
\begin{equation*}
y_{j}(x)=x^{\alpha-j} E_{\alpha, 1+\beta / \alpha, 1+(\beta-j) / \alpha}\left(a x^{\alpha+\beta}\right) \quad(j=1,2, \ldots,[\alpha]+1) \tag{5.15}
\end{equation*}
$$

for $\beta>\{\alpha\} \equiv \alpha-[\alpha]$. They also obtained the solution of the corresponding Cauchy-type problem for (5.14) with the initial conditions

$$
\begin{equation*}
\left.\left(D_{0+}^{\alpha-k} y\right)(x)\right|_{x=0}=b_{k} \quad(k=1,2, \ldots,[\alpha]+1) \tag{5.16}
\end{equation*}
$$

in the form

$$
\begin{equation*}
y(x)=\sum_{k=0}^{[\alpha]+1} b_{k} x^{\alpha-j} E_{\alpha, 1+\beta / \alpha, 1+(\beta-j) / \alpha}\left(a x^{\alpha+\beta}\right) . \tag{5.17}
\end{equation*}
$$

The result in (5.10) was applied to obtain the particular solution $y_{0}(x)$ of the nonhomogeneous fractional differential equation with a quasi-polynomial free term

$$
\begin{equation*}
\left(D_{0+}^{\alpha} y\right)(x)=a x^{\beta} y(x)+\sum_{i=0}^{n} f_{i} x^{\mu_{i}} \quad(0<x<d<\infty ; \alpha>0, a \neq 0, \beta \in \mathbb{R}) \tag{5.18}
\end{equation*}
$$

where $\mu_{i}, f_{i} \in \mathbb{R}(i=0,1, \ldots, n)$, and the solution of the corresponding Cauchy-type problem for the equation (5.18) with the initial conditions (5.16). In particular, explicit solutions of fractional differential equations of order $1 / 2$, arising in the theory of voltammetry at expanding electrodes [1, equation (8.6.1)] and in the theory of polarography were constructed.

Kilbas and Saigo also considered the connection of the generalized Mittag-Leffler function (5.6) with the right-sided Liouville fractional integrals $I_{-}^{\alpha} y$ and derivative $D_{-}^{\alpha} y$ of order $\alpha \in \mathbb{C}(\operatorname{Re} \alpha \geq 0)$ defined for $x \in \mathbb{R}$ by

$$
\begin{align*}
& \left(I_{-}^{\alpha} y\right)(x)=\frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} \frac{y(t) d t}{(t-x)^{1-\alpha}} \quad(\alpha \in \mathbb{C}, \operatorname{Re} \alpha>0)  \tag{5.19}\\
& \left(D_{-}^{\alpha} y\right)(x)=\left(-\frac{d}{d x}\right)^{n}\left(I_{-}^{n-\alpha} y\right)(x), \quad n=[\operatorname{Re} \alpha]+1 \tag{5.20}
\end{align*}
$$

For such fractional integrals and derivative they proved a relations similar to (5.9) and (5.10) and applied them to obtain the particular solutions of the non-homogeneous integral equation of the form (5.12), with the integral from 0 to $x$ being replaced by the integral from $x$ to $\infty$, and of the differential equation of the form (5.18), with $D_{0+}^{\alpha}$ being replaced by $D_{-}^{\alpha}$, in terms of the generalized Mittag - Leffler type functions (5.6).

Saigo and Kilbas [57] applied such an approach, based on compositions of usual derivatives with the generalized Mittag - Leffler type function (5.6), to solve in closed form of new classes of ordinary differential equations and corresponding Cauchy-type problems.

Kilbas and Saigo [58] studied the solvability of the nonlinear equation

$$
\begin{equation*}
\left(D_{0+}^{\alpha} y\right)(x)=a x^{\beta} y^{m}(x)+b x^{\gamma} \quad(0<x<d<\infty ; \alpha>0, m \in \mathbb{R}, m \neq 1) \tag{5.21}
\end{equation*}
$$

with real $a(a \neq 0), b, \mu$ and $\nu$. They proved that if $(\alpha+\beta) /(1-m)>\alpha-1$, then the corresponding homogeneous nonlinear equation $(b=0)$ has at least one solution given by

$$
\begin{equation*}
y(x)=\left[\frac{\Gamma(\mu+1)}{a \Gamma(\mu-\alpha+1)}\right]^{1 /(1-m)} x^{\mu}, \quad \mu=\frac{\alpha+\beta}{1-m} \tag{5.22}
\end{equation*}
$$

while the non-homogeneous nonlinear equation (5.21) with $\gamma=(\beta+m \alpha) /(1-m)$ has the solution

$$
\begin{equation*}
y(x)=\lambda x^{\mu}, \quad \mu=\frac{\alpha+\beta}{1-m} \tag{5.23}
\end{equation*}
$$

provided that the transcendental equation

$$
\begin{equation*}
\Gamma\left(\frac{\alpha+\beta}{1-m}+1-\alpha\right)\left[a \xi^{m}+b\right]-\Gamma\left(\frac{\alpha+\beta}{1-m}+1\right) \xi=0 \tag{5.24}
\end{equation*}
$$

is solvable and $\xi=\lambda$ is its solution. The problem of the uniqueness of the solutions (5.22) and (5.23) was also discussed in [58]. The solvability of the nonlinear equation (5.21) depends on the solvability of the transcendental equation (5.24). Positive solutions of such a transcendental equation were investigated in [59].

Kilbas, Bonilla, Rodriguez, Trujillo and Rivero [60] and Bonilla, Kilbas, Rivero, Rodriguez L. and Trujillo [61] have studied compositions of the left- and right-sided Liouville fractional integrals derivatives (5.19) and (5.20) with the special functions

$$
\begin{equation*}
Z_{\rho}^{\nu}(z)=\int_{1}^{\infty} t^{\nu-1} \exp \left(-t^{\rho}-\frac{z}{t}\right) d t \quad(z \in \mathbb{C}, \operatorname{Re} z>0 ; \rho>0, \nu \in \mathbb{C}) \tag{5.25}
\end{equation*}
$$

and

$$
\begin{gather*}
\lambda_{\gamma, \sigma}^{(\beta)}(z)=\frac{\beta}{\Gamma(\gamma+1-1 / \beta)} \int_{1}^{\infty}\left(t^{\beta}-1\right)^{\gamma-1 / \beta} t^{\sigma} e^{-z t} d t  \tag{5.26}\\
\left(z \in \mathbb{C}, \operatorname{Re} z>0 ; \beta>0 ; \gamma \in \mathbb{C}, \operatorname{Re} \gamma>\frac{1}{\beta}-1 ; \sigma \in \mathbb{R}\right)
\end{gather*}
$$

These functions are analytic with respect to $z$, and are invariant relative to the Liouville fractional integrals (5.19) and derivatives (5.20) except for a transformation of the indices:

$$
\begin{align*}
\left(I_{-}^{\alpha} Z_{\rho}^{\nu}\right)(x) & =Z_{\rho}^{\nu+\alpha}(x), \tag{5.27}
\end{align*} \quad\left(I_{-}^{\alpha} \lambda_{\gamma, \sigma}^{(\beta)}\right)(x)=\lambda_{\gamma, \sigma-\alpha}^{(\beta)}(x), ~\left(D_{-}^{\alpha} Z_{\rho}^{\nu}\right)(x)=Z_{\rho}^{\nu-\alpha}(x), \quad\left(D_{-, \sigma}^{\alpha} \lambda_{\gamma, \sigma}^{(\beta)}\right)(x)=\lambda_{\gamma, \sigma+\alpha}^{(\beta)}(x) .
$$

These relations were applied in [60] to obtain explicit solutions of differential equations of fractional order

$$
\begin{gather*}
x D_{-}^{\rho+1} y+(\nu-\rho) x D_{-}^{\rho} y-\rho y=0  \tag{5.29}\\
x^{2} D_{-}^{2 \rho+2} y+(2 \nu-3 \rho-1) x D_{-}^{2 \rho+1} y+(\nu-\rho)(\nu-2 \rho) D_{-}^{2 \rho} y+\rho^{2} y=0 \tag{5.30}
\end{gather*}
$$

in terms of the function (5.25), and in [61] to obtain the explicit solution of the integral equation of the third kind

$$
\begin{equation*}
x y(x)=\int_{x}^{\infty}\left[\gamma \beta+\sigma+\frac{(t-x)^{\beta-1}}{\Gamma(\beta)}+(\beta-\sigma-1) \frac{(t-x)^{\beta}}{\Gamma(\beta+1)}\right] y(t) d t=0 \quad(x>0) \tag{5.31}
\end{equation*}
$$

and of differential equations of fractional order

$$
\begin{gather*}
x\left[\left(D_{-}^{\alpha+\beta+1} y\right)(x)-\left(D_{-}^{\alpha+1} y\right)(x)\right]-(\gamma \beta+\sigma+\alpha)\left(D_{-}^{\alpha+\beta} y\right)(x)+ \\
+(\sigma+\alpha+1-\beta)\left(D_{-}^{\alpha} y\right)(x)=0 \quad(x>0) \tag{5.32}
\end{gather*}
$$

in terms of the function (5.26).

## 6. Fractional integral and differential equations: <br> Laplace transform method

Here we discuss the method based on the Laplace integral transform to deduce explicit solutions of linear integral and differential equations of fractional order. First we present a scheme to solve fractional integral and differential equation of the forms (1.3) and (1.4) by using the direct and inverse Laplace transforms $\mathcal{L}$ and $\mathcal{L}^{-1}$ :

$$
\begin{gather*}
(\mathcal{L} \varphi)(p)=\int_{0}^{\infty} \varphi(t) e^{-p t} d t  \tag{6.1}\\
\left(\mathcal{L}^{-1} g\right)(x)=\frac{1}{2 \pi i} \int_{\gamma-\infty}^{\gamma+\infty} e^{p x} g(p) d p \quad(\gamma=\operatorname{Re} p>\sigma, \sigma \in \mathbb{R}) . \tag{6.2}
\end{gather*}
$$

One may find the theory of Laplace transform in the books by Ditkin and Prudnikov [35], Titchmarsh [62] and Sneddon [63].

The transforms $\mathcal{L}$ and $\mathcal{L}^{-1}$ are inverse to each other for suitable functions $\varphi, g$ :

$$
\begin{equation*}
\mathcal{L}^{-1} \mathcal{L} \varphi=\varphi, \quad \mathcal{L} \mathcal{L}^{-1} g=g \tag{6.3}
\end{equation*}
$$

We consider the equations (1.3) and (1.4) with the Riemann-Liouville integrals $I^{\alpha_{k}}=I_{0+}^{\alpha_{k}}$ and derivatives $D^{\alpha_{k}}=D_{0+}^{\alpha_{k}}$ and constant coefficients $c_{k} \in \mathbb{C}$ :

$$
\begin{align*}
& c_{0} y(x)+\sum_{k=1}^{m} c_{k}\left(I_{0+}^{\alpha_{k}} y\right)(x)=f(x) \quad(x>0)  \tag{6.4}\\
& c_{0} y(x)+\sum_{k=1}^{m} c_{k}\left(D_{0+}^{\alpha_{k}} y\right)(x)=f(x) \quad(x>0) \tag{6.5}
\end{align*}
$$

It is known that for suitable functions $y(x)$ the Laplace transforms of $I_{0+}^{\alpha} y$ and $D_{0+}^{\alpha} y$ are given by

$$
\begin{equation*}
\left(\mathcal{L} I_{0+}^{\alpha} y\right)(p)=p^{-\alpha}(\mathcal{L} y)(p), \quad\left(\mathcal{L} D_{0+}^{\alpha} y\right)(p)=p^{\alpha}(\mathcal{L} y)(p) \tag{6.6}
\end{equation*}
$$

Applying the Laplace transform (6.1) to both sides of (6.4) and (6.5) and taking (6.6) into account we have respective formulas

$$
\begin{align*}
& {\left[c_{0}+\sum_{k=1}^{m} c_{k} p^{-\alpha_{k}}\right](\mathcal{L} y)(p)=(\mathcal{L} f)(p),}  \tag{6.7}\\
& {\left[c_{0}+\sum_{k=1}^{m} c_{k} p^{\alpha_{k}}\right](\mathcal{L} y)(p)=(\mathcal{L} f)(p) .} \tag{6.8}
\end{align*}
$$

Using the inverse Laplace transform (6.2) we obtain particular solutions of the equations (6.4) and (6.5) in respective forms

$$
\begin{align*}
& y(x)=\left(\mathcal{L}^{-1}\left[\frac{(\mathcal{L} f)(p)}{c_{0}+\sum_{k=1}^{m} c_{k} p^{-\alpha_{k}}}\right]\right)(x),  \tag{6.9}\\
& y(x)=\left(\mathcal{L}^{-1}\left[\frac{(\mathcal{L} f)(p)}{c_{0}+\sum_{k=1}^{m} c_{k} p^{\alpha_{k}}}\right]\right)(x) \tag{6.10}
\end{align*}
$$

We note that Laplace transform method was first applied by Hille and Tamarkin [64] (1930) to solve the Abel type integral equation of second kind

$$
\begin{equation*}
\varphi(x)-\frac{\lambda}{\Gamma(\alpha)} \int_{0}^{x} \frac{\varphi(t) d t}{(x-t)^{1-\alpha}}=f(x) \quad(x>0) \tag{6.11}
\end{equation*}
$$

in terms of the Mittag-Leffler function (2.9) with $\beta=1$;

$$
\begin{equation*}
\varphi(x)=\frac{d}{d x} \int_{0}^{x} E_{\alpha}\left[\lambda(x-t)^{\alpha}\right] f(t) d t, \quad E_{\alpha}(z)=E_{\alpha, 1}(z) \tag{6.12}
\end{equation*}
$$

One may find results and bibliography of papers devoted to solution in closed form of certain types of fractional integral equation (6.4) in the books by Gorenflo and Vessela [2] and by Samko, Kilbas and Marichev [3].

Maravall [65] probably first suggested a formal approach based on the Laplace transform to obtain the explicit solution of a particular case of the fractional differential equation (6.5), but his paper published in Spanish, was practically unknown.

The Laplace transform was used by many authors to obtain the explicit solutions of special cases of the differential equation (6.5): see Kilbas and Trujillo [13].

Miller and Ross [4] applied the Laplace transform method to solve the Cauchy problem for the particular case of (6.5) with derivative $\alpha_{k}=k \alpha$ and $1 / \alpha=q=1,2, \ldots$ :

$$
\begin{align*}
& \sum_{k=1}^{m} c_{k}\left(D_{0+}^{k \alpha} y\right)(x)+c_{0} y(x)=f(x)  \tag{6.13}\\
& y(0)=y^{\prime}(0)=\ldots y^{(m-1)}(0)=0 \tag{6.14}
\end{align*}
$$

Miller and Ross introduced a fractional analogue of the Green function $G_{\alpha}(x)$ defined via the inverse Laplace transform (6.2)

$$
\begin{equation*}
G_{\alpha}(x)=\left(\mathcal{L}^{-1}\left[\frac{1}{P\left(t^{\alpha}\right)}\right]\right)(x), \quad P(x)=c_{0}+\sum_{k=1}^{m} c_{k} x^{k} \tag{6.15}
\end{equation*}
$$

and proved that the unique solution $y(x)$ of (6.13)-(6.14) has the form of the Laplace convolution of $G_{\alpha}(x)$ and $f(x)$ :

$$
\begin{equation*}
y(x)=\int_{0}^{x} G_{\alpha}(x-t) f(t) d t \tag{6.16}
\end{equation*}
$$

These investigations were developed by Podlubny [5, Chapter 5]) who defined such a fractional analogue of Green function $G_{\alpha}(x)$ to more general equation (3.1), showed that the solution (3.5) of the Cauchy problem (3.6) can be expressed in terms of $G_{\alpha}(x)$ :

$$
\begin{equation*}
y(x)=\sum_{k=0}^{n-1} b_{k} y_{k}(x)+\int_{0}^{x} G_{\alpha}(x-t) f(t) d t, \quad y_{k}(x)=\left(D_{0+}^{\alpha_{n}} D_{0+}^{\alpha_{n-1}} \ldots D_{0+}^{\alpha_{k}} G_{\alpha}\right)(x) \tag{6.17}
\end{equation*}
$$

In particular, he found the explicit formula of $G_{\alpha}(x)$ for the equation (6.5) as a multiple series containing the Mittag - Leffler functions (2.9).

Examples of linear fractional differential equations of the form (6.5) and (6.13), solved by using Laplace transform method and analogies of Green function, were given by Miller and Ross [4, Chapters V and VI] and Podlubny [5, Sections 4.1.1 and 4.2.1].

Gorenflo and Mainardi [66] applied the Laplace transform to solve the fractional differential equation

$$
\begin{equation*}
\left({ }^{C} D_{o+}^{\alpha} y\right)(x)=-\rho^{\alpha} y(x)+f(x) \quad(\alpha>0 ; \quad \rho>0, x>0) \tag{6.18}
\end{equation*}
$$

with the Caputo derivative (1.15) and discussed the key role of the Mittag-Leffler function (2.9) in the cases $1<\alpha<2$ and $2<\alpha<3$.

Using the formula for the Laplace convolution

$$
\begin{equation*}
\left(\mathcal{L}\left(\int_{0}^{x} k(x-t) f(t) d t\right)\right)(p)=(\mathcal{L} k)(p)(\mathcal{L} f)(p) \tag{6.19}
\end{equation*}
$$

by analogy with (6.15) we introduce the Laplace fractional analogue of the Green function

$$
\begin{equation*}
G_{\alpha_{1}, \ldots, \alpha_{m}}(x)=\left(\mathcal{L}^{-1}\left[\frac{1}{P_{\alpha}(x)}\right]\right)(x), \quad P_{\alpha}(x)=c_{0}+\sum_{k=1}^{m} c_{k} x^{\alpha_{k}} . \tag{6.20}
\end{equation*}
$$

Then the solutions (6.9) and (6.10) of the integral and differential equations (6.4) and (6.5) have respective forms of the Laplace convolutions of $G_{\alpha_{1}, \ldots, \alpha_{m}}(1 / x)$ and $f(x)$ :

$$
\begin{equation*}
y(x)=\int_{0}^{x} G_{\alpha_{1}, \ldots, \alpha_{m}}\left(\frac{1}{x-t}\right) f(t) d t \tag{6.21}
\end{equation*}
$$

and of $G_{\alpha_{1}, \ldots, \alpha_{m}}(x)$ and $f(x)$ :

$$
\begin{equation*}
y(x)=\int_{0}^{x} G_{\alpha_{1}, \ldots, \alpha_{m}}(x-t) f(t) d t . \tag{6.22}
\end{equation*}
$$

## 7. Fractional calculus equations: Fourier and Mellin transforms method

Similarly to Section 6, we present a scheme to solve fractional integral and differential equations of the forms (1.3) and (1.4) by using direct and inverse Fourier transforms $\mathcal{F}$ and $\mathcal{F}^{-1}$ :

$$
\begin{gather*}
(\mathcal{F} \varphi)(x)=\int_{-\infty}^{\infty} e^{i x t} \varphi(t) d t \quad(x \in \mathbb{R})  \tag{7.1}\\
\left(\mathcal{F}^{-1} g\right)(x)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} e^{-i x t} g(t) d t \quad(x \in \mathbb{R}), \tag{7.2}
\end{gather*}
$$

and direct and inverse Mellin transforms $\mathcal{M}$ and $\mathcal{M}^{-1}$ :

$$
\begin{gather*}
(\mathcal{M} \varphi)(s)=\int_{0}^{\infty} t^{s-1} \varphi(t) d t \quad(s \in \mathbb{C})  \tag{7.3}\\
\left(\mathcal{M}^{-1} g\right)(x)=\frac{1}{2 \pi i} \int_{\gamma-\infty}^{\gamma+\infty} x^{-s} g(s) d s \quad(\gamma=\operatorname{Re} p) . \tag{7.4}
\end{gather*}
$$

One may find the theory of Fourier and Mellin transforms in the books by Titchmarsh [62], Sneddon [63] and Ditkin and Prudnikov [35]. In particular, they are inverse to each other for suitable functions $\varphi, g$ :

$$
\begin{array}{cl}
\mathcal{F}^{-1} \mathcal{F} \varphi=\varphi, & \mathcal{F} \mathcal{F}^{-1} g=g \\
\mathcal{M}^{-1} \mathcal{M} \varphi=\varphi, & \mathcal{M} \mathcal{M}^{-1} g=g \tag{7.6}
\end{array}
$$

We consider equations of the form (1.3) and (1.4) with constant coefficients $c_{k} \in \mathbb{R}$

$$
\begin{align*}
& c_{0} y(x)+\sum_{k=1}^{m} c_{k}\left(I_{+}^{\alpha_{k}} y\right)(x)=f(x) \quad\left(x>0 ; c_{k} \in \mathbb{C}, k=0,1, \ldots, m\right)  \tag{7.7}\\
& c_{0} y(x)+\sum_{k=1}^{m} c_{k}\left(D_{+}^{\alpha_{k}} y\right)(x)=f(x) \quad\left(x>0 ; c_{k} \in \mathbb{C}, k=0,1, \ldots, m\right) \tag{7.8}
\end{align*}
$$

involving the so-called left-sided Liouville fractional integrals and derivatives $I_{+}^{\alpha_{k}} y$ and $D_{+}^{\alpha} y$ of order $\alpha \in \mathbb{C}(\operatorname{Re} \alpha \geq 0)$ defined for $x \in \mathbb{R}$ by

$$
\begin{gather*}
\left(I_{+}^{\alpha} y\right)(x)=\frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} \frac{y(t) d t}{(x-t)^{1-\alpha}} \quad(\alpha \in \mathbb{C}, \operatorname{Re} \alpha>0)  \tag{7.9}\\
\left(D_{-}^{\alpha} y\right)(x)=\left(\frac{d}{d x}\right)^{n}\left(I_{+}^{n-\alpha} y\right)(x), \quad n=[\operatorname{Re} \alpha]+1 \tag{7.10}
\end{gather*}
$$

It is known [3, Section 7.1] that for suitable functions $y(x)$ the Fourier transforms of $I_{+}^{\alpha} y$ and $D_{+}^{\alpha} y$ are given by

$$
\begin{equation*}
\left(\mathcal{F} I_{+}^{\alpha} y\right)(x)=(-i x)^{-\alpha}(\mathcal{F} y)(x), \quad\left(\mathcal{F} D_{+}^{\alpha} y\right)(x)=(-i x)^{\alpha}(\mathcal{F} y)(x) \tag{7.11}
\end{equation*}
$$

Applying the Fourier transform (7.1) to both sides of (7.9) and (7.10), taking (7.11) into account and using the inverse Fourier transform (7.2), we obtain solutions of the equations (7.7) and (7.8) in respective forms

$$
\begin{align*}
y(x) & =\left(\mathcal{F}^{-1}\left[\frac{(\mathcal{F} f)(t)}{c_{0}+\sum_{k=1}^{m} c_{k}(-i t)^{-\alpha_{k}}}\right]\right)(x)  \tag{7.12}\\
y(x) & =\left(\mathcal{F}^{-1}\left[\frac{(\mathcal{F} f)(t)}{c_{0}+\sum_{k=1}^{m} c_{k}(-i t)^{\alpha_{k}}}\right]\right)(x) \tag{7.13}
\end{align*}
$$

Using the Fourier convolution formula

$$
\begin{equation*}
\left(\mathcal{F}\left(\int_{-\infty}^{\infty} k(x-t) f(t) d t\right)\right)(x)=(\mathcal{F} k)(x)(\mathcal{F} f)(x) \tag{7.14}
\end{equation*}
$$

by analogy with (6.20) we can introduce the Fourier fractional analogue of the Green function

$$
\begin{equation*}
G_{\alpha_{1}, \ldots, \alpha_{m}}^{1}(x)=\left(\mathcal{F}^{-1}\left[\frac{1}{P_{\alpha}^{1}(t)}\right]\right)(x), \quad P_{\alpha}^{1}(x)=c_{0}+\sum_{k=1}^{m} c_{k}(-i x)^{\alpha_{k}} \tag{7.15}
\end{equation*}
$$

and rewrite the solutions (7.12) and (7.13) in respective forms of the Fourier convolution of $G_{\alpha_{1}, \ldots, \alpha_{m}}(1 / x)$ and $f(x)$ :

$$
\begin{equation*}
y(x)=\int_{-\infty}^{\infty} G_{\alpha_{1}, \ldots, \alpha_{m}}^{1}\left(\frac{1}{x-t}\right) f(t) d t . \tag{7.16}
\end{equation*}
$$

and of $G_{\alpha_{1}, \ldots, \alpha_{m}}(x)$ and $f(x)$ :

$$
\begin{equation*}
y(x)=\int_{-\infty}^{\infty} G_{\alpha_{1}, \ldots, \alpha_{m}}^{1}(x-t) f(t) d t \tag{7.17}
\end{equation*}
$$

Similarly it is proved that the equations (1.3) and (1.4) with constant coefficients $c_{k} \in \mathbb{R}$

$$
\begin{align*}
& c_{0} y(x)+\sum_{k=1}^{m} c_{k}\left(I_{-}^{\alpha_{k}} y\right)(x)=f(x) \quad\left(x>0 ; c_{k} \in \mathbb{C}, k=0,1, \ldots, m\right)  \tag{7.18}\\
& c_{0} y(x)+\sum_{k=1}^{m} c_{k}\left(D_{-}^{\alpha_{k}} y\right)(x)=f(x) \quad\left(x>0 ; c_{k} \in \mathbb{C}, k=0,1, \ldots, m\right) \tag{7.19}
\end{align*}
$$

with the right-sided Liouville fractional integrals $I_{-}^{\alpha_{k}}$ and derivatives $D_{-}^{\alpha_{k}}$, defined in (5.19) and (5.20), has solutions of the forms (7.16) and (7.17):

$$
\begin{align*}
& y(x)=\int_{-\infty}^{\infty} G_{\alpha_{1}, \ldots, \alpha_{m}}^{2}(x-t) f(t) d t  \tag{7.20}\\
& y(x)=\int_{-\infty}^{\infty} G_{\alpha_{1}, \ldots, \alpha_{m}}^{2}(x-t) f(t) d t \tag{7.21}
\end{align*}
$$

where

$$
\begin{equation*}
G_{\alpha_{1}, \ldots, \alpha_{m}}^{2}(x)=\left(\mathcal{F}^{-1}\left[\frac{1}{P_{\alpha}^{2}(t)}\right]\right)(x), \quad P_{\alpha}^{2}(x)=c_{0}+\sum_{k=1}^{m} c_{k}(i x)^{\alpha_{k}} \tag{7.22}
\end{equation*}
$$

Now we consider the equations of the forms (1.3) and (1.4) with constant coefficients $c_{k} \in \mathbb{R}$

$$
\begin{gather*}
c_{0} y(x)+\sum_{k=1}^{m} c_{k}\left(\mathcal{J}_{0+}^{\alpha_{k}} y\right)(x)=f(x) \quad\left(x>0 ; c_{k} \in \mathbb{C}, k=0,1, \ldots, m\right)  \tag{7.23}\\
c_{0} y(x)+\sum_{k=1}^{m} c_{k}\left({ }^{H} D_{0+}^{\alpha_{k}} y\right)(x)=f(x) \quad\left(x>0 ; c_{k} \in \mathbb{C}, k=0,1, \ldots, m\right), \tag{7.24}
\end{gather*}
$$

involving the Hadamard fractional integrals and derivatives $\mathcal{J}_{0+}^{\alpha_{k}} y$ and ${ }^{H} D_{0+}^{\alpha_{k}} y$ defined by (1.12) and (1.11) on the half axis $\mathbb{R}_{+}$, and the equations

$$
\begin{equation*}
c_{0} y(x)+\sum_{k=1}^{m} c_{k}\left(\mathcal{J}_{-}^{\alpha_{k}} y\right)(x)=f(x) \quad\left(x>0 ; c_{k} \in \mathbb{C}, k=0,1, \ldots, m\right) \tag{7.25}
\end{equation*}
$$

$$
\begin{equation*}
c_{0} y(x)+\sum_{k=1}^{m} c_{k}\left({ }^{H} D_{-}^{\alpha_{k}} y\right)(x)=f(x) \quad\left(x>0 ; c_{k} \in \mathbb{C}, k=0,1, \ldots, m\right) \tag{7.26}
\end{equation*}
$$

with the Hadamard fractional integrals and derivatives $\mathcal{J}_{-}^{\alpha_{k}} y$ and ${ }^{H} D_{-}^{\alpha} y$ defined for complex $\alpha \in \mathbb{C}(\operatorname{Re} \alpha>0)$ by

$$
\begin{equation*}
\left(\mathcal{J}_{-}^{\alpha} y\right)(x)=\frac{1}{\Gamma(\alpha)} \int_{x}^{\infty}\left(\log \frac{t}{x}\right)^{\alpha-1} \frac{y(t) d t}{t} \quad(x>0) \tag{7.27}
\end{equation*}
$$

and

$$
\begin{equation*}
\left({ }^{H} D_{-}^{\alpha} y\right)(x)=(-\delta)^{n}\left(\mathcal{J}_{-}^{n-\alpha} y\right)(x) \quad(x>0 ; n=[\operatorname{Re} \alpha]+1) \tag{7.28}
\end{equation*}
$$

respectively; see the paper by Butzer, Kilbas and Trujillo [67].
It is known [67] that for suitable functions $y(x)$ the Mellin transforms of $\mathcal{J}_{0+}^{\alpha} y$, ${ }^{H} D_{0+}^{\alpha} y$ and $\mathcal{J}_{-}^{\alpha} y,{ }^{H} D_{-}^{\alpha} y$ are given by

$$
\begin{gather*}
\left(\mathcal{M} \mathcal{J}_{0+}^{\alpha} y\right)(s)=(-s)^{-\alpha}(\mathcal{M} y)(s), \quad\left(\mathcal{M}^{H} D_{0+}^{\alpha} y\right)(s)=(-s)^{\alpha}(\mathcal{M} y)(s)  \tag{7.29}\\
\left(\mathcal{M J}_{-}^{\alpha} y\right)(s)=s^{-\alpha}(\mathcal{M} y)(s), \quad\left(\mathcal{M}^{H} D_{-}^{\alpha} y\right)(s)=s^{\alpha}(\mathcal{M} y)(s) \tag{7.30}
\end{gather*}
$$

Applying the Mellin transform (7.3) to both sides of (7.23) and (7.24), taking (7.29) into account and using the inverse Mellin transform (7.4), we obtain particular solutions of the equations (7.23) and (7.24) in respective forms

$$
\begin{equation*}
y(x)=\left(\mathcal{M}^{-1}\left[\frac{(\mathcal{M} f)(s)}{c_{0}+\sum_{k=1}^{m} c_{k}(-s)^{-\alpha_{k}}}\right]\right)(x) \tag{7.31}
\end{equation*}
$$

and

$$
\begin{equation*}
y(x)=\left(\mathcal{M}^{-1}\left[\frac{(\mathcal{M} f)(s)}{c_{0}+\sum_{k=1}^{m} c_{k}(-s)^{\alpha_{k}}}\right]\right)(x) \tag{7.32}
\end{equation*}
$$

Using the Mellin convolution formula

$$
\begin{equation*}
\left(\mathcal{M}\left(\int_{0}^{\infty} k\left(\frac{x}{t}\right) f(t) \frac{d t}{t}\right)\right)(p)=(\mathcal{M} k)(p)(\mathcal{M} f)(p) \tag{7.33}
\end{equation*}
$$

by analogy with (7.15) and (7.22) we can introduce the Mellin fractional analogies of the Green function

$$
\begin{align*}
& G_{\alpha_{1}, \ldots, \alpha_{m}}^{3}(x)=\left(\mathcal{M}^{-1}\left[\frac{1}{P_{\alpha}^{3}(x)}\right]\right)(x),  \tag{7.34}\\
& P_{\alpha}^{3}(x)=c_{0}+\sum_{k=1}^{m} c_{k}(-x)^{-\alpha_{k}}  \tag{7.35}\\
& G_{\alpha_{1}, \ldots, \alpha_{m}}^{4}(x)=\left(\mathcal{M}^{-1}\left[\frac{1}{P_{\alpha}^{4}(x)}\right]\right)(x), \quad P_{\alpha}^{4}(x)=c_{0}+\sum_{k=1}^{m} c_{k}(-x)^{\alpha_{k}}
\end{align*}
$$

and represent solutions (7.31) and (7.32) in the respective forms

$$
\begin{align*}
& y(x)=\int_{0}^{\infty} G_{\alpha_{1}, \ldots, \alpha_{m}}^{3}\left(\frac{x}{t}\right) f(t) \frac{d t}{t}  \tag{7.36}\\
& y(x)=\int_{0}^{\infty} G_{\alpha_{1}, \ldots, \alpha_{m}}^{4}\left(\frac{x}{t}\right) f(t) \frac{d t}{t} \tag{7.37}
\end{align*}
$$

Similarly, on the basis (7.30) and (7.33) explicit solutions of the equations (7.25) and (7.26) are deduced in respective forms

$$
\begin{gather*}
y(x)=\int_{0}^{\infty} G_{\alpha_{1}, \ldots, \alpha_{m}}^{5}\left(\frac{x}{t}\right) f(t) \frac{d t}{t}  \tag{7.38}\\
G_{\alpha_{1}, \ldots, \alpha_{m}}^{5}(x)=\left(\mathcal{M}^{-1}\left[\frac{1}{P_{\alpha}^{5}(x)}\right]\right)(x), \quad P_{\alpha}^{5}(x)=c_{0}+\sum_{k=1}^{m} c_{k} s^{-\alpha_{k}} \tag{7.39}
\end{gather*}
$$

and

$$
\begin{gather*}
y(x)=\int_{0}^{\infty} G_{\alpha_{1}, \ldots, \alpha_{m}}^{6}\left(\frac{x}{t}\right) f(t) \frac{d t}{t}  \tag{7.40}\\
G_{\alpha_{1}, \ldots, \alpha_{m}}^{6}(x)=\left(\mathcal{M}^{-1}\left[\frac{1}{P_{\alpha}^{6}(x)}\right]\right)(x), \quad P_{\alpha}^{6}(x)=c_{0}+\sum_{k=1}^{m} c_{k} s^{\alpha_{k}} \tag{7.41}
\end{gather*}
$$

## 8. Multi-dimensional integral and partial fractional differential equations: integral transforms method

The Fourier integral transform method, presented in Section 7 to obtaining explicit solutions of one-dimensional fractional integral and differential equations (7.7)-(7.8) and (7.18)-(7.19), can be also applied to find explicit solutions of multi-dimensional linear integral and differential equation of the form (1.3) and (1.4) with constant coefficients $c_{k} \in \mathbb{R}$ :

$$
\begin{align*}
& c_{0} y(x)+\sum_{k=1}^{m} c_{k}\left(\mathbf{I}^{\alpha_{k}} y\right)(x)=f(x) \quad\left(x \in \mathbb{R}^{n} ; c_{k} \in \mathbb{C}, k=0,1, \ldots, m\right),  \tag{8.1}\\
& c_{0} y(x)+\sum_{k=1}^{m} c_{k}\left(\mathbf{D}^{\alpha_{k}} y\right)(x)=f(x) \quad\left(x \in \mathbb{R}^{n} ; c_{k} \in \mathbb{C}, k=0,1, \ldots, m\right), \tag{8.2}
\end{align*}
$$

involving the Riesz fractional integrals $\mathbf{I}^{\alpha_{k}} y$ and derivatives $\mathbf{D}^{\alpha_{k}} y$ defined in (1.17) and (1.18). Such an equation can be solved by using the multi-dimensional direct and inverse Fourier transforms $\mathcal{F}$ and $\mathcal{F}^{-1}$ :

$$
\begin{gather*}
(\mathcal{F} \varphi)(x)=\int_{\mathbb{R}^{n}} e^{i x \cdot t} \varphi(t) d t \quad\left(x \in \mathbb{R}^{n}\right)  \tag{8.3}\\
\left(\mathcal{F}^{-1} g\right)(x)=\frac{1}{(2 \pi)^{n}} \int_{\mathbb{R}^{n}} e^{-i x \cdot t} g(t) d t \quad\left(x \in \mathbb{R}^{n}\right) \tag{8.4}
\end{gather*}
$$

One may find the theory of multi-dimensional Fourier transforms in the books by Stein and Weiss [68] and by Nikol'skii [69].

The transforms (8.3) and (8.4) are inverse to each other for suitable functions $\varphi, g$ :

$$
\begin{equation*}
\mathcal{F}^{-1} \mathcal{F} \varphi=\varphi, \quad \mathcal{F} \mathcal{F}^{-1} g=g \tag{8.5}
\end{equation*}
$$

According to (1.17) and (1.18), there hold the formulas

$$
\begin{equation*}
\left(\mathcal{F} \mathbf{I}^{\alpha} y\right)(x)=|x|^{-\alpha}(\mathcal{F} y)(x), \quad\left(\mathcal{F} \mathbf{D}^{\alpha} y\right)(x)=|x|^{\alpha}(\mathcal{F} y)(x) . \tag{8.6}
\end{equation*}
$$

for suitable functions $y$.

Applying the Fourier transform (8.3) to both sides of (8.1) and (8.2), taking (8.6) into account and using the inverse Fourier transform (8.4), we obtain explicit solutions of the equations (8.1) and (8.2) in respective forms

$$
\begin{align*}
y(x) & =\left(\mathcal{F}^{-1}\left[\frac{(\mathcal{F} f)(t)}{c_{0}+\sum_{k=1}^{m} c_{k}|t|^{-\alpha_{k}}}\right]\right)(x) .  \tag{8.7}\\
y(x) & =\left(\mathcal{F}^{-1}\left[\frac{(\mathcal{F} f)(t)}{c_{0}+\sum_{k=1}^{m} c_{k}|t|^{\alpha_{k}}}\right]\right)(x) \tag{8.8}
\end{align*}
$$

Using the Fourier convolution formula

$$
\begin{equation*}
\left(\mathcal{F}\left(\int_{\mathbb{R}^{n}} k(x-t) f(t) d t\right)\right)(x)=(\mathcal{F} k)(x)(\mathcal{F} f)(x) \tag{8.9}
\end{equation*}
$$

by analogy with (7.15) and (7.22) we can introduce the Fourier multi-dimensional analogue of the Green function

$$
\begin{equation*}
G_{\alpha_{1}, \ldots, \alpha_{m}}(x)=\left(\mathcal{F}^{-1}\left[\frac{1}{P_{\alpha}(t)}\right]\right)(x), \quad P_{\alpha}(x)=c_{0}+\sum_{k=1}^{m} c_{k}|x|^{\alpha_{k}} \tag{8.10}
\end{equation*}
$$

and rewrite the solutions (8.7) and (8.8) in respective forms of the Fourier convolution of $G_{\alpha_{1}, \ldots, \alpha_{m}}(1 / x)$ and $f(x)$ :

$$
\begin{equation*}
y(x)=\int_{\mathbb{R}^{n}} G_{\alpha_{1}, \ldots, \alpha_{m}}\left(\frac{1}{x-t}\right) f(t) d t \tag{8.11}
\end{equation*}
$$

and of $G_{\alpha_{1}, \ldots, \alpha_{m}}(x)$ and $f(x)$ :

$$
\begin{equation*}
y(x)=\int_{\mathbb{R}^{n}} G_{\alpha_{1}, \ldots, \alpha_{m}}(x-t) f(t) d t \tag{8.12}
\end{equation*}
$$

One may find results and bibliography of papers devoted to investigation of multidimensional fractional integral equations involving the Riesz potential (1.19) and more general constructions in the monograph by Samko [7]. A survey of results for partial differential equations of fractional order and more general abstract equations were presented in the paper by Kilbas and Trujillo [14].

Here we characterize a series of papers where the Laplace, Fourier and Mellin transforms were applied to investigate the so-called fractional diffusion equations, and in most of them formal explicit solutions of certain boundary and initial problems for the considered equations were obtained.

Wyss [70] studied the fractional differential equation

$$
\begin{equation*}
\frac{x_{+}^{-\alpha-1}}{\Gamma(\alpha)} * u(x, t)=\lambda^{2} \frac{\partial^{2} u(x, t)}{\partial x^{2}} \quad(x>0 ; 0<\alpha \leq 1, \lambda>0) \tag{8.13}
\end{equation*}
$$

where $x_{+}^{-\alpha-1} / \Gamma(\alpha)$ is understood as distribution in a special space of generalized function. He investigated two problems for this equation

$$
\begin{equation*}
u(x, 0)=0, \quad u(0, T)=b \quad(t>0) \tag{8.14}
\end{equation*}
$$

when $b=0$ and $b=-1$. He sought a solution $u(x, t)$ of these problems in the form

$$
\begin{equation*}
u(x, t)=f(y), \quad y=t^{-\alpha / 2} x \tag{8.15}
\end{equation*}
$$

and reduced (8.13)-(8.14) to the one-dimensional problem

$$
\begin{equation*}
\lambda^{2} \frac{d^{2} f(y)}{d y^{2}}=\left(I_{-; 2 / \alpha, 1}^{-\alpha} f\right)(y) \quad(y>0), \quad f(0)=0, \quad f(\infty)=b \tag{8.16}
\end{equation*}
$$

where

$$
\begin{equation*}
\left(I_{-; \sigma, \eta}^{\beta} f\right)(y)=\frac{\sigma y^{\sigma \eta}}{\Gamma(\beta)} \int_{y}^{\infty} \frac{t^{\sigma(1-\alpha-\eta)-1} f(t) d t}{\left(t^{\sigma}-y^{\sigma}\right)^{1-\beta}} \quad(\beta>0, \sigma>0, \eta \in \mathbb{C}) \tag{8.17}
\end{equation*}
$$

is the so-called Erdelyi - Kober-type fractional integral - see [3, (18.7)]. Applying to the equation in (8.16) the direct and inverse Mellin transforms (7.3) and (7.4) and taking into account the initial conditions in (8.16), Wyss obtained the following solutions of the above two problems:

$$
u(x, t)=\pi^{-1 / 2} H_{2,3}^{2,1}\left[\begin{array}{l|c}
\frac{x}{2 \lambda} t^{-\alpha / 2} & (1,1),(1, \alpha / 2)  \tag{8.18}\\
(1 / 2,1 / 2),(1,1 / 2),(0,1)
\end{array}\right]
$$

and

$$
u(x, t)=-\pi^{-1 / 2} H_{2,3}^{3,0}\left[\begin{array}{l|c}
\left.\frac{x}{2 \lambda} t^{-\alpha / 2} \left\lvert\, \begin{array}{c}
(1,1),(1, \alpha / 2) \\
(0,1),(1 / 2,1 / 2),(1,1 / 2)
\end{array}\right.\right] \tag{8.19}
\end{array}\right.
$$

respectively. These solutions are given in terms of the so-called $H$-function, which for integers $m, n, p, q \in \mathbb{N}_{0}(0 \leq m \leq q, 0 \leq n \leq p)$, for complex $a_{i}, b_{j} \in \mathbb{C}$ and positive $\alpha_{i}, \beta_{j}>0(1 \leq i \leq p ; 1 \leq j \leq q)$ is defined by

$$
H_{p, q}^{m, n}\left[z \left\lvert\, \begin{array}{c}
\left(a_{i}, \alpha_{i}\right)_{1, p}  \tag{8.20}\\
\left(b_{j}, \beta_{j}\right)_{1, q}
\end{array}\right.\right]=\frac{1}{2 \pi i} \int_{C} \mathcal{H}_{p, q}^{m, n}\left[\left.\begin{array}{c}
\left(a_{i}, \alpha_{i}\right)_{1, p} \\
\left(b_{j}, \beta_{j}\right)_{1, q}
\end{array} \right\rvert\, s\right] z^{-s} d s
$$

where

$$
\mathcal{H}(s) \equiv \mathcal{H}_{p, q}^{m, n}\left[\left.\begin{array}{c}
\left(a_{i}, \alpha_{i}\right)_{1, p}  \tag{8.21}\\
\left(b_{j}, \beta_{j}\right)_{1, q}
\end{array} \right\rvert\, s\right]=\frac{\prod_{j=1}^{m} \Gamma\left(b_{j}+\beta_{j} s\right) \prod_{i=1}^{n} \Gamma\left(1-a_{i}-\alpha_{i} s\right)}{\prod_{i=n+1}^{p} \Gamma\left(a_{i}+\alpha_{i} s\right) \prod_{j=m+1}^{q} \Gamma\left(1-b_{j}-\beta_{j} s\right)}
$$

the contour $C$ being specially chosen and an empty product, if it occurs, being taken to be one. Note that the $H$-function contains most of elementary and special functions; its theory can be found in the books by Mathai and Saxena [71, Chapter 2], Srivastava, Gupta and Goyal [72, Chapter 1], Prudnikov, Brychkov and Marichev [73, Section 8.3] and Kilbas and Saigo [74, Chapters 1 and 2].

Schneider and Wyss [75] considered the equation

$$
\begin{equation*}
u(x, t)=\sum_{k=0}^{m-1} f_{k}(x) t^{k}+\frac{1}{\Gamma(\alpha)} \int_{0}^{t} \frac{\Delta u(x, s) d s}{(t-s)^{1-\alpha}} \quad(m-1<\alpha \leq m, m=1,2) \tag{8.22}
\end{equation*}
$$

where $x \in \mathbb{R}^{n}(n \geq 1), t>0, f_{k}(x)$ are the initial data, i. e.,

$$
\begin{equation*}
\left.\left(\frac{\partial}{\partial t}\right)^{k} u(x, t)\right|_{t=0}=f_{k}(x) \quad(0 \leq k \leq m-1, m=1,2) \tag{8.23}
\end{equation*}
$$

and $\Delta$ is the Laplacian (1.16). (8.22) presents the fractional diffusion and the wave equation when $m=1,0<\alpha \leq 1$ and $m=2,1<\alpha \leq 2$, respectively. Applying to
(8.22) the Laplace transform (6.1) with respect to $t$ and the inverse Mellin transform (7.4), they obtained the solution of (8.22) in the form:

$$
\begin{equation*}
u(x, t)=\sum_{k=0}^{m-1} \int_{\mathbb{R}^{n}} G_{k}^{\alpha}(|x-y|, t) f_{k}(y) d y \tag{8.24}
\end{equation*}
$$

where $y \in \mathbb{R}^{n},|x-y|=\left[\sum_{k=1}^{n}\left(x_{k}-y_{k}\right)^{2}\right]^{1 / 2}$, and the analogies of the Green functions $G_{k}^{\alpha}(r, t) \quad(0 \leq k \leq m-1 m=1,2)$ are expressed via the $H$-function (8.20) by

$$
G_{k}^{\alpha}(r, t)=\frac{\pi^{-n / 2}}{2 r^{n}}\left(\frac{r}{2}\right)^{2 k / \alpha} H_{1,2}^{2,0}\left[\frac{r}{2} t^{-\alpha / 2} \left\lvert\, \begin{array}{c}
(1, \alpha / 2)  \tag{8.25}\\
(n / 2-k / \alpha, 1 / 2),(1-k / \alpha, 1 / 2)
\end{array}\right.\right]
$$

Schneider and Wyss also obtained an explicit solution of the fractional diffusion equation (8.22) with $0<\alpha \leq 1$ in the half space $D=\mathbb{R}^{n-1} \times R_{+}$with the boundary $\partial D=\mathbb{R}^{n-1} \times\{0\}$, supplemented by a special boundary condition. Note that in the case $0<\alpha \leq 1$ Schneider [76] gave a more elegant solution of the equation (8.22) with $m=1$ :

$$
\begin{equation*}
u(x, t)=f(x)+\frac{1}{\Gamma(\alpha)} \int_{0}^{t} \frac{\Delta u(x, s) d s}{(t-s)^{1-\alpha}} \quad(0<\alpha \leq 1), \quad u(x, 0)=f(x) \tag{8.26}
\end{equation*}
$$

by application of the Fourier transform (8.3) with respect to $x$ and the Laplace transform (6.1) with respect to $t$ and their corresponding inversion transforms (8.4) and (6.2).

Fujita [77] studied a two-dimensional equation (8.22) in the case $1<\alpha<2$

$$
\begin{equation*}
u(\mathbf{x}, t)=f(\mathbf{x})+\frac{1}{\Gamma(\alpha)} \int_{0}^{t} \frac{\Delta u(\mathbf{x}, s) d s}{(t-s)^{1-\alpha}} \quad(1<\alpha<2 ; x \in \mathbb{R}, t>0) \tag{8.27}
\end{equation*}
$$

in the space $C([0, \infty) ; S(\mathbb{R}))$ consisting of $S(\mathbb{R})$-valued continuous functions on $[0, \infty)$, where $S(\mathbb{R})$ is the space of rapidly decreasing functions of Schwartz. Using the Fourier transform, he obtained its solution $u(x, t)$ :

$$
\begin{equation*}
u(x, t)=\frac{1}{\alpha} \int_{-\infty}^{\infty} P_{\alpha}(|y|, t) f(t) d t \tag{8.28}
\end{equation*}
$$

where

$$
\begin{equation*}
P_{\alpha}(x, t)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \exp \left[-t|s|^{2 / \alpha} e^{-\gamma \pi \operatorname{sign}(s) i / 2}\right] e^{-i x s} d s, \gamma=2-\frac{2}{\alpha} \tag{8.29}
\end{equation*}
$$

Fujita also investigated propeties of the fundamental solution of (8.27).
Fujita [78] obtained the solution of the equation

$$
\begin{gather*}
u(x, t)=f(x)+\frac{t^{\alpha / 2}}{\Gamma(1+\alpha / 2)} g(x)+\frac{1}{\Gamma(\alpha)} \int_{0}^{t} \frac{\Delta u(x, s) d s}{(t-s)^{1-\alpha}}  \tag{8.30}\\
(1 \leq \alpha \leq 2 ; x \in \mathbb{R}, t>0)
\end{gather*}
$$

in the form

$$
\begin{equation*}
u(x, t)=\frac{E}{2}\left[f\left(x+Y_{\alpha}(t)\right)+f\left(x-Y_{\alpha}(t)\right)+\int_{x-Y_{\alpha}(t)}^{x+Y_{\alpha}(t)} g(t) d t\right] \tag{8.31}
\end{equation*}
$$

where $Y_{\alpha}(t)$ is a continuous, non-decreasing and nonnegative stochastic process with Mittag - Leffler distribution of order $\alpha / 2$, and $E$ stands for the expectation. Using the Fourier transform (7.1) and probability methods Fujita [79] proved energy inequalities for the integro-differential equations (8.27) and (8.30) which correspond to the energy inequality for the wave equation.

The equation (8.13) belongs to the so-called fractional diffusion equations deduced by Nigmatullin [80] (1984), [81] (1986). In the simplest case of a one-dimensional diffusion such an equation is given by

$$
\begin{equation*}
\left({ }^{R L} \mathcal{D}_{0+, t}^{\alpha} u\right)(x, t)=\lambda^{2} \frac{\partial^{2} u(x, t)}{\partial x^{2}} \quad(\alpha>0, \lambda>0) \tag{8.32}
\end{equation*}
$$

where $\left({ }^{R L} \mathcal{D}_{0+, t}^{\alpha} u\right)(x, t)$ is the partial Riemann-Liouville fractional derivative (1.20). When $\alpha=1$ and $\alpha=2$, (8.32) coincides with the well-known heat (diffusion) and wave equations

$$
\begin{equation*}
\frac{\partial u(x, t)}{\partial t}=\lambda^{2} \frac{\partial^{2} u(x, t)}{\partial x^{2}}, \quad \frac{\partial^{2} u(x, t)}{\partial t^{2}}=\lambda^{2} \frac{\partial^{2} u(x, t)}{\partial x^{2}} \quad(\alpha>0, \lambda>0) \tag{8.33}
\end{equation*}
$$

Mainardi [82-84] have studied the equation (8.32) with $x \in \mathbb{R}$ and $t>0$ for $0<\alpha \leq 2$ under natural initial conditions. In [82] he investigated the Cauchy problem for such an equation, used the method suggested by Schneider [76] and applied the Laplace transform with respect to $t$ and the Fourier transform with respect to $x$ to find the explicit solution $u(x, t)$ of this problem in the form

$$
\begin{equation*}
u(x, t)=\int_{-\infty}^{\infty} G^{\alpha}(x-\tau, t) f(\tau) d \tau \tag{8.34}
\end{equation*}
$$

where a fractional analogue of the Green function $G^{\alpha}(x, t)$ is expressed in terms of the integral of the Mittag - Leffler function (2.9) and of a special function of Wright defined for $\alpha>-1$ and complex $\beta \in \mathbb{C}$ by

$$
\begin{equation*}
\phi(\alpha, \beta ; z)=\sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+\beta)} \frac{z^{k}}{k!} . \tag{8.35}
\end{equation*}
$$

In [83, 84] Mainardi has used the same approach to find the fundamental solutions of the Cauchy problem and of the so-called signalling problem for the equation (8.32) with $0<\alpha \leq 2$ in terms of the Wright function (8.35).

Gorenflo and Mainardi [85] considered the fractional diffusion equation (8.32) with $0<\alpha \leq 2$ and $\lambda=1$ in the quarter-plane $\mathbb{R}_{++}=\left\{(x, t) \in \mathbb{R}^{2}: x>0, t>0\right\}$ under certain initial and boundary conditions. They applied the Laplace transform to obtain the explicit solution of these problems in the form (8.34), where a fractional analogy of the Green function $G^{\alpha}(x, t)$ is the inverse Laplace transform with respect to $t$ of $\exp \left(-x t^{\alpha / 2}\right)$ and $-t^{-\alpha / 2} \exp \left(-x t^{\alpha / 2}\right)$, respectively.

Kochubei [86] has considered the diffusion equation

$$
\begin{equation*}
\left({ }^{C} D_{t}^{(\alpha)} u\right)(x, t)-(\Delta u)(x, t)=f(x, t) \tag{8.36}
\end{equation*}
$$

for $x \in \mathbb{R}^{n}$ and $t \in[0, T]$, with the Caputo regularized partial derivative of order $0<\alpha<1$ :

$$
\begin{equation*}
\left({ }^{c} D_{t}^{(\alpha)} u\right)(x, t)=\frac{1}{\Gamma(1-\alpha)}\left[\frac{\partial}{\partial \tau} \int_{0}^{t} \frac{u(x, \tau)}{(t-\tau)^{\alpha}} d \tau-\frac{u(x, 0)}{t^{\alpha}}\right] . \tag{8.37}
\end{equation*}
$$

He proved a uniqueness theorem for a bounded solution of equation (8.36) with the initial condition

$$
\begin{equation*}
u(x, 0)=u_{0}(x) \quad\left(x \in \mathbb{R}^{n}\right) \tag{8.38}
\end{equation*}
$$

Kochubei also found the fundamental solution of the Cauchy problem (8.36), (8.38). The fundamental solution of the Cauchy problem for more general than (8.36) equation, in which $\Delta$ is replaced by a uniformly elliptic operator, was constructed in terms of the $H$-function (8.20) by Eidelman and Kochubei [87].

In conclusion we indicate that the method based on the direct and inverse Laplace and Fourier transforms, being applied with respect to $x \in \mathbb{R}$ and $t>0$, was applied by Kilbas, Pierantozzi, Trujillo and Vazquez [88, 89] to find explicit solutions of the Cauchy problem

$$
\begin{gather*}
\left({ }^{C} D_{t}^{\alpha} u\right)(t, x)=\lambda\left(D_{x}^{\beta} u\right)(t, x)  \tag{8.39}\\
(t>0,-\infty<x<\infty ; 0<\alpha \leq 1, \beta>0, \lambda \neq 0) \\
\lim _{x \rightarrow \pm \infty} u(t, x)=0, \quad u(0+, x)=g(x) \tag{8.40}
\end{gather*}
$$

with the Caputo partial fractional derivative (1.21), and of the Cauchy-type problem

$$
\begin{gather*}
\left({ }^{R L} D_{t}^{\alpha} u\right)(t, x)=\lambda\left(D_{x}^{\beta} u\right)(t, x)  \tag{8.41}\\
(t>0,-\infty<x<\infty ; 0<\alpha \leq 1, \beta>0, \lambda \neq 0) \\
\lim _{x \rightarrow \pm \infty} u(t, x)=0, \quad\left({ }^{R L} D_{t}^{\alpha-1} u\right)(0+, x)=g(x) \tag{8.42}
\end{gather*}
$$

with the Riemann - Liouville partial fractional derivative (1.20). The fundamental solutions of the above problems were also constructed in [88] and [89].

## 9. Problems and new trends of research

In Sections 2-3 we have presented methods and results to investigation of fractional differential equations based on their reduction to Volterra integral equations, in Sections $4-5$ and $6-8$ we have discussed methods and results to investigation of fractional integral and differential equations based on operational and compositional methods, and on integral transforms of Laplace, Fourier and Mellin. Some of these methods similar to that known in the theory of ordinary differential equations. Now we characterize some problems in these directions which can lead to new possible trends of research.

The main difficulties in the method of reduction of Cauchy-type and Cauchy problems for fractional differential equations to the corresponding Volterra integral equations concern the proof of the equivalence between solutions of these two constructions in certain function spaces. This problem is closely connected with mapping properties of the operators of fractional differentiation and integration in the spaces considered. Such properties are well developed for the Riemann-Liouville fractional integrals and derivatives (1.6) and (1.7) in the spaces of integrable and continuous functions, and so at present the existence and uniqueness of a solution $y(x)$ in these spaces are proved for the simplest (model) nonlinear Cauchy-type problem (2.18), (2.19) and (2.23), (2.19) and system of these problems, for nonlinear Cauchy problems (3.4), (3.7) and (3.13),
(3.14). It seems that the existence and uniqueness results in these function spaces can be proved for Cauchy-type and Cauchy problems for fractional differential equations of another type.

Mapping properties of fractional integration and differentiation operators in function spaces are suitable if we can characterize their images, and such a characterization is given by a homeomorphism of the spaces considered with respect to fractional calculus operators. Mapping properties of the Riemann-Liouville fractional integrals (1.6) are well developed in Hölder spaces $H^{\lambda}([a, b])$ and in $L_{p}(a, b)(1 \leq p<\infty)$ spaces on a finite interval $[a, b]$ of the real line $\mathbb{R}$ and in the weighted spaces of these functions with power weight [ 3 , Sections $3-4]$. Therefore we hope that these results can be applied to study the existence and uniqueness of the Cauchy-type problems for the equations (2.18), (2.23), (3.4), (3.7), (3.13) and more general fractional differential equations in Hölder- and $L_{p}(a, b)$-spaces.

Mapping properties of the Liouville fractional integration operator $I_{0+}^{\alpha}, I_{+}^{\alpha} \equiv I_{-\infty+}^{\alpha}$ (defined by (1.6) with $a=0$ and (7.9)) and $I_{-}^{\alpha}$ (defined by (5.19)), are also known in weighted spaces of Hölder and $p$-summable functions on a half-axis $\mathbb{R}_{+}$and on the whole axis $\mathbb{R}$; see $[3$, Sections 5, 9]. These results can be applied to study Cauchytype and Cauchy problems to fractional differential equations with the Liouville and Caputo fractional derivatives. Applied problems can lead to studying fractional differential equations with Riemann-Liouville, Liouville, Caputo, Hadamard and other fractional derivatives in special function spaces. In this case it is necessary to develop mapping properties of fractional calculus operators in such new spaces, which will lead to new problems in fractional calculus.

It should be noted that if there is the equivalence of Cauchy-type problems and corresponding Volterra integral equations, the known methods in the theory of Volterra integral equations can be applied to investigate the initial value probelms in various function spaces. In particular, the above arguments can be used to study Cauchytype and Cauchy problems for linear differential equations of fractional order. To find explicit solutions of these problems and fractional differential equations, we can apply the method discussed in Sections 2 and 3.

Section 4 shows that the operational calculus method allows us to solve nonhomogeneous linear fractional differential equations with constant coefficients. But their solutions are usually obtained in certain sufficiently completed spaces of functions. Thus the problem here is a justification of the solutions obtained in more simple function spaces. Another problem is in studying the properties of special functions generalizing the Mittag - Leffler function (2.9).

An approach presented in Section 5 and based on compositions of fractional integration and differentiation operators with elementary and special functions, allows us to solve new classes of linear fractional integral and differential equations with nonconstant coefficients. The results were obtained on the basis of compositions of fractional integration and differentiation operators with special functions of Mittag-Leffler and Bessel types. We hope that it is possible to find new composition formulas between fractional calculus operators and various special functions which will lead to explicit solutions of new classes of fractional integral and differential equations.

The method based on Laplace, Fourier and Mellin transforms can be used more suitable to solve linear fractional integral and differential equations with constant coefficients. In these cases we also have the problem concerning function spaces of solutions. As we have seen in Sections 6 and 7, formally the one-dimensional Laplace and Fourier and Mellin transforms yield the explicit solutions of the non-homogeneous linear fractional integral and differential equation (6.4)-(6.5), (7.7)-(7.8), (7.18)-(7.19), (7.23)-(7.24) and (7.25)-(7.26), respectively. Since the Laplace, Fourier and Mellin
transforms (6.1) and (7.3) are defined on the half-axis $\mathbb{R}_{+}$, while the Fourier transform (7.1) on the whole real line $\mathbb{R}$, we also need to investigate function spaces with respect to these integral transforms. The same concern solutions of multi-dimensional integral and differential equations (8.1)-(8.2).

Another problem is a representation of the explicit solutions (6.9)-(6.10), (7.12)(7.13), (7.31)-(7.32) and (8.7)-(8.8) in more simple forms. In some cases these solutions can be represented via the fractional analogies of Green functions (6.20), (7.15), (7.22), (7.34), (7.35), (7.39), (7.41) and (8.10) as convolutions of the form (6.21)-(6.22), (7.16)(7.17), (7.20)-(7.21), (7.36)-(7.37), (7.38), (7.40) and (8.11)-(8.12). The known explicit representations of such analogies are expressed in terms of the Mittag - Leffler function (2.9), its modifications and generalizations, and the $H$-function (8.20), and the Wright function (8.35). We hope that there are other equations whose explicit solutions can be given via the above functions, their generalizations and modifications. It seems that in some cases such fractional analogies of Green function can be expressed in terms of special functions of hypergeometric and Bessel type.

It should be noted that results presented in Section 8 and concerning Cauchy problems to some fractional partial differential equations, were obtained without proof of the equivalence between the initial problems and corresponding multi-dimensional integral equations. This equivalence depend on mapping properties of multi-dimensional fractional calculus operators in some spaces of functions. Therefore the problem to prove such an equivalence is still open, and it seems that solution of this problem can lead to investigation of more general and new problems.

The above method of integral transforms to solve in closed form some fractional partial differential equations with constant coefficients was suitable because of the simplest domains for equations considered: half-space or the whole space. In more general case such an approach is not suitable, and while considering fractional partial differential equations on a domain in $\mathbb{R}^{n}$ another integral transforms must be used. Probably, the Radon transform or its modifications are suitable for such an investigation.

It seems that another integral transforms such as the Hankel transform, the Meijer transform and other transforms with special function kernels can be also used to deduce explicit solutions of some classes of ordinary and partial fractional differential equations.

It was noted in Section 8 that the simplest partial diffusion equation (8.32) coincides with classical heat and wave equations in limit cases $\alpha=1$ and $\alpha=2$, respectively. This fact leads us to the hypothesis that partial fractional differential equations have more various properties then the usual ones, and that the properties of the latter can be deduced from the properties of the former. In this connection, from our point of view, it is interesting to construct general theory of partial differential equations of fractional order which generalize classical theory of partial differential equations of elliptic, hyperbolic and parabolic types. It seems that in this way we can obtain new properties of partial fractional differential equations which are impossible in classical cases.

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## Резюме

A.A. Килбас. Новые направления в теории дробных интегральных и дифференциальных уравнений.

В статье рассматриваются одномерные и многомерные интегральные уравнения, обыкновенные дифференциальные уравнения и дифференциальные уравнения в частных производных типа Римана - Лиувилля, Лиувилля, Капуто, Адамара и Рисса. Обсуждается метод основанный на редукции задачи Коши для одномерного нелинейного дробного дифференциального уравнения к интегральному уравнению Вольтерра. Описывается единый подход к нахождению решения в замкнутой форме для ряда классов одномерных и многомерных интегральных уравнений, и дифференциальных уравнений в частных производных дробного порядка. Этот подход основан на операторном исчислении и интегральных преобразованиях Лапласа, Фурье и Меллина. Обсуждаются перспективы данного направления исследований.

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[^0]
[^0]:    Килбас Анатолий Александрович - доктор физико-математических наук, профессор Белорусского государственного университета.

