Multi-Agent Temporak Logics with Multi-valuations

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ABSTRACT: The semantical basis, i. e., relational models, are used for modelling of computational processes and analysis of databases with incomplete information, for instance, with information forgotten in the past. In the situation we consider, the agents' accessibility relations may have lacunas; agents may have no access to some potentially known and stored information.

Yet innovative point is that in the relational models we consider various valuations V_i for agents' knowledge and a global valuation based on these valuations. Besides, agents' logical operations inside formulas may be nested, as a consequence they may interfere; that is, we consider not autonomous but cooperating agents. Satisfiability and decidability issues are discussed.

We find algorithms solving satisfiability problem and hence we obtain the decidability of the decidability problem. Open problems are discussed.

We start from logics based (in a sense) Linear Temporal Logic;
But based at Non-Transitive Time

DEFINITIONS, NOTATION

Language

(i) Boolean logical operations $\wedge, \vee, \rightarrow, \neg$.

(ii) Binary temporal operations $\mathbf{U}_l$ (until for the each agent l, where $l \in Ag$ and Ag is a finite set of all agents) and

(iii) Unary operations next: $\mathbf{N}_l$ for $i \in Ag$.

Let the set of agents Ag from the definition of formulas has k different agents', $-$ $Ag := [1, k]$.

Definition. *An* multi-agent non-transitive frame *is a tuple*

$$\mathcal{F}^{nt}_{ma} = \langle W^{nt}_{ma}, (\bigcup_{\xi \in In, j \in [1,k]} \langle R_{\xi,j} \rangle), Nxt, \rangle, \text{ such that}$$

- $W^{nt}_{ma} := \bigcup_{\xi \in In \subset N} It[\xi] = N$, *where* N *is the set of all natural numbers; for any* $\xi \in In$, $d(\xi) \in In, d(\xi) > \xi$ *and* $It[\xi]$ *is the closed interval of all natural numbers situated between* ξ *and* $d(\xi)$*:* $It[\xi] := [\xi, d(\xi)]$*;*

- $\forall \xi_1, \xi_2 \in In$, $\xi_1 \neq \xi_2 \Rightarrow (\xi_1, d(\xi_1)) \cap (\xi_2, d(\xi_2)) = \emptyset$;

- *any* $R_{\xi,j}$ *is the restriction of the standard linear order* ($\leq$) *in the interval* $It[\xi]$ *on a subset* $Dom_{\xi,j} \subseteq It(\xi)$*;*

- Nxt *is the standard next relation on* N*:* $[n \ Nxt \ m]$ *iff* $m = n + 1$*; we will write* $Nxt(a) = b$ *to denote that* $b = a + 1$ *.*

Definition. *A multi-valued, multi-agent non-transitive model* $\mathcal{MA}^{nt}_{ma}$ *is a pair* $\langle \mathcal{F}^{nt}_{ma}, \{V_l \mid \ l \in Ag\}\rangle$*, where*

(i) $\mathcal{F}^{nt}_{ma}$ *is a multi-agent non-transitive frame;*

(ii) Any V_l *is an agents' valuation for the agent* l *of a fixed for* $\mathcal{MA}^{nt}_{ma}$ *set* P *of propositional letters in this frame, that is, for any letter* p, $V_l(p) \subseteq |\mathcal{F}^{nt}_{ma}|$

We model a feature of cooperation between agents by introduction a summarizing (global) valuation V_0 which is defined as follows

$$(\mathcal{M}\mathcal{A}_{nt}^{mv}, a) \Vdash_{V_0} p \;\; \Leftrightarrow \quad ||\{l \mid (\mathcal{M}\mathcal{A}_{ma}^{nt}, a) \Vdash_{V_l} p\}|| > TH,$$

where TH (a threshold) is a fixed number bigger than $k/2+1$.

The choice of the threshold can differ from the pointed one and include, for example, weights for competence of the agent's, etc. We here have many ways how to define TH more, i.e. should it be the same for all models or to be its own for each model.

But if we fix and decide which way to use, the rest will not be effected by our choice. So, we fix an agreement about TH and have the set of all such models with all possible valuations.

How to compute truth values:

Definition. *For any* $a \in \mathcal{MA}^{nt}_{ma}$ *and any* $j \in Ag := [1, k]$*:*

$$(\mathcal{MA}^{nt}_{ma}, a) \Vdash_{V_j} \neg\varphi \quad \Leftrightarrow \quad (\mathcal{MA}^{nt}_{ma}, a) \nVdash_{V_j} \varphi;$$

$$(\mathcal{MA}^{nt}_{ma}, a) \Vdash_{V_j} (\varphi \ \wedge \ \psi) \quad \Leftrightarrow \ ((\mathcal{MA}^{nt}_{ma}, a) \Vdash_{V_j} \varphi) \wedge ((\mathcal{MA}^{nt}_{ma}, a) \Vdash_{V_j} \psi);$$

$$(\mathcal{MA}^{nt}_{ma}, a) \Vdash_{V_j} (\varphi \ \vee \ \psi) \quad \Leftrightarrow \ ((\mathcal{MA}^{nt}_{ma}, a) \Vdash_{V_j} \varphi) \vee ((\mathcal{MA}^{nt}_{ma}, a) \Vdash_{V_j} \psi);$$

$$(\mathcal{MA}^{nt}_{ma}, a) \Vdash_{V_j} (\varphi \ \rightarrow \ \psi) \quad \Leftrightarrow \ ((\mathcal{MA}^{nt}_{ma}, a) \Vdash_{V_j} \psi) \vee ((\mathcal{MA}^{nt}_{ma}, a) \nVdash_{V_j} \varphi);$$

For all formulas $\varphi \mathbf{U}_l \psi$ and any V_j we define the truth values as follows

(note that for any a, $a \in It(\xi)$ with the maximal ξ and):

$$(\mathcal{M}\mathcal{A}_{ma}^{nt}, a) \Vdash_{V_j} (\varphi \ \mathbf{U}_l \ \psi) \quad \Leftrightarrow \quad (a \in It(\xi)) \ \&$$

$$\exists b[(aR_{\xi,j}b) \wedge ((\mathcal{M}\mathcal{A}_{ma}^{nt}, b) \Vdash_{V_l} \psi) \wedge$$

$$\forall c[(c \in Dom_{\xi,j} \ \& \ aR_{\xi,j}c \ \& \ c < b) \Rightarrow (\mathcal{M}\mathcal{A}_{ma}^{nt}, c) \Vdash_{V_l} \varphi]];$$

$$(\mathcal{M}\mathcal{A}_{ma}^{nt}, a) \Vdash_{V_j} \mathbf{N}_l \varphi \quad \Leftrightarrow \quad [(a \ Nxt \ b) \Rightarrow (\mathcal{M}\mathcal{A}_{ma}^{nt}, b) \Vdash_{V_l} \varphi].$$

EXAMPLES

(1) The formula $\Diamond_1 p \wedge \neg\Diamond_2 p$ being true w.r.t. V_1 says that the accessibility relation for the agent 2 has a hole, lacuna, which nonetheless is accessible for the agent 1.

(2) Total opposition for in all first interval of time:

$\varphi_{op} := [\Box_1 p \to \Box_2 \neg p] \wedge [\Box_2 p \to \Box_1 \neg p]$. This formula being evaluated via global valuation V_0 says that these both agents are totally opposite in their opinion for stable facts at all states accessible for them.

(3) Agree in all visible time but since then are in complete opposition:

$\Box_0[((\Box_1 p \to \Box_2 p) \wedge (\Box_2 p \to \Box_1 p)] \wedge \Diamond_0 \mathbf{N}_0 \varphi_{op}$.

(4) Revolt: $\Diamond_1 \top \wedge \Box_1[p \to \Box_2 p] \wedge \Box_1[\neg p \to \Box_2[\neg p] \wedge \Diamond_1 \mathbf{N}_1[p \wedge \Diamond_2 \neg p]$ (being evaluated either by V_1 or V_0). The agent 1 totally dominate another one in current time and enforce 2 to think that if the thing p is true now than for second agent 2 it is true always (total domination), but in next time interval this is not a case: there is a state where 1 think p is true, but the second agent 2 sees a state were p is not true.

(5) Total recall: $\Diamond_1 p \wedge \Box_1(p \to \Diamond_1[\neg p \wedge \neg \Box_1 \neg p]) \wedge \Diamond_1 \Diamond_1 \Box_1 p$. This formula ($w.r.t. V_1$) says that the agent 1 always swapping its opinion about truth of p from true to false and vise versa or lose p at all at the initial interval of time, but then - somewhere at next time interval its decides p is always true.

The logic $MA^{Int}_{Lin}(TH)$ is the set of all formulas which are valid in any model $\mathcal{MA}^{nt}_{ma}$ for all states and valuations.

Truncated Models

Definition. *For any model $\mathcal{MA}_{ma}^{nt}$, a truncated model $\mathcal{MA}_{ma}^{nt}(m)$ for any $m \in N$ is the model based at the frame with the base set*

$|\mathcal{MA}_{ma}^{nt}| \ \setminus \{x \mid x \in N, X > d^{m+2}(0)\}$ *and having the following relations and valuations.*

(i) Relations Nxt are the same as been, but we re-define Nxt at $d^{m+2}(0)$ assuming $Nxt(d^{m+2}(0)) = d^{m+2}(0)$, here d is the distance function of the frame of $\mathcal{MA}_{ma}^{nt}$.

(ii) Besides, we transfer to this base set all agents' accessibility relations, but assume that $d^{m+2}(0)$ does not belong to the domains of all agents' accessibility relations.

We may transfer the rules for computation truth values of formulas to truncated models without any amendments.

Lemma 1. *Assume that a model* $\mathcal{MA}^{nt}_{ma}$ *based at a frame* $\mathcal{F}^{nt}_{ma}$ *is given and a formula* α *with temporal degree* n *is satisfied at this model at the state* 0 *by a valuation* V_l. *Then* α *is satisfied at* 0 *by the valuation* V_l *at the truncated model* $\mathcal{MA}^{nt}_{ma}(n+1)$.

Lemma 2. *If a formula* α *is refuted in a model* $\mathcal{MA}^{nt}_{ma}(m)$ *at the world* 0 *by a valuation* V_l, *then* α *may be refuted in a model based at a standard frame* $\mathcal{F}^{mv}_{nt}$.

Definition. *Rule* $\mathbf{r}$ *is true (or valid) on* $\mathcal{MA}^{nt}_{ma}(m)$ *iff*

$$[\forall_{V_l}\forall a\ ((\mathcal{MA}^{nt}_{ma}(m),a)\Vdash_{V_l}\bigwedge_{1\leq i\leq s}\varphi_i)]\quad\Rightarrow$$

$$[\forall V_l\forall a\ ((\mathcal{MA}^{nt}_{ma}(m),a)\Vdash_{V_l}\psi)].$$

If $[\forall_{V_l}\forall a\,((\mathcal{MA}^{nt}_{ma}(m),a)\Vdash_{V_l}\bigwedge_{1\leq i\leq s}\varphi_i)]$, *but* $\exists V_l\exists a\,((\mathcal{MA}^{nt}_{ma}(m),a)\nVdash_{V_l}\psi)$ *we say* $\mathbf{r}$ *is refuted in* $\mathcal{MA}^{nt}_{ma}(m)$ *by* V_l *and we denote it by* $\mathcal{MA}^{nt}_{ma}\nVdash_{V_l}\mathbf{r}$.

Lemma 3. *If there is an algorithm verifying for any given rule and any given model* $\mathcal{MA}^{nt}_{ma}(m)$ *if this rule may be refuted in this model then there exists an algorithm verifying if any given formula is satisfiable.*

Definition. *For any given rule* $\mathbf{r}$*, a rule* $\mathbf{r_{nf}}$ *in the reduced normal form is a reduced normal form of* $\mathbf{r}$ *iff*

(i) $\mathbf{r_{nf}}$ *has all variable-letters from* $\mathbf{r}$ *and maybe some more own ones;*

(ii) For any truncated model $\mathcal{MA}^{nt}_{ma}(m)$ *the rule* $\mathbf{r}$ *may be refuted in*

$\mathcal{MA}^{nt}_{ma}(m)$ *if and only if the rule* $\mathbf{r_{nf}}$ *may be refuted in this model;*

Theorem 4. *There exists an algorithm running in (single) exponential time, which, for any rule* $\mathbf{r}$*, constructs some its reduced form* $\mathbf{r_{nf}}$*. The variables of* $\mathbf{r_{nf}}$ *are all variables of* $\mathbf{r}$ *and the set of new variables denoting all sub-formulas of* $\mathbf{r}$*.*

Lemma 5. *If a rule in a reduced normal form* $\mathbf{r_{nf}}$ *is refuted in a truncated model* $\mathcal{MA}^{nt}_{ma}(g)$ *for some* g *then* $\mathbf{r_{nf}}$ *can be refuted in some such model where* $\forall \xi \in In$, $d(\xi) - \xi \leq dis(r) \times v + 2$, *where* $dis(r)$ *is the number of disjuncts in* $\mathbf{r_{nf}}$ *and* v *is the number of valuations in* $\mathcal{MA}^{nt}_{ma}(g)$.

Theorem 5. *The satisfiability problem for the logic* $MA^{Int}_{Lin}(TH)$ *is decidable: there is an algorithm, described in the series of cited lemmas, which verify satisfiability.*

Non-linear Logics

Definition. *An k-multi-modal frame is a tuple* $\mathcal{F}_k = \langle W, R_1, \ldots, R_k, \rangle$, *where* W *is a set (of worlds/states) and all* R_i *are binary relations on* W.

Definition. *An k-multi-modal frame with the objective accessibility relation* R_0 *is a* $k+1$*-multi-modal frame*

$\mathcal{F}_k = \langle W, R_0, R_1, \ldots, R_k, \rangle$ *such that* , $\forall i$, $R_i \subseteq R_0$.

We mean here that the relation R_0 is the objective one, which does not depend on peseption/knowledge of the agents (objects) i. Therefore we suggest $R_i \subseteq R_0$;

Definition. *An k-multi-valued, model $\mathcal{M}_k$ is a pair*

$\langle \mathcal{F}_k, V_1, \ldots, V_k \rangle$, *where*

(i) $\mathcal{F}_k$ is a k-multi-modal frame;

(ii) Any V_l is a valuation of a fixed for $\mathcal{M}_k$ set P of propositional letters in this frame,

that is, for any letter $p \in P$, $V_l(p) \subseteq |\mathcal{F}_k|$; we will use notation $(\mathcal{M}_k, a) \Vdash_{V_l} p$ iff $a \in V_l(p)$.

We may extend the valuations from propositional letters to all formulas as follows:

Definition *For any* $a \in \mathcal{M}_k$ *and any* V_j*:*

$$(\mathcal{M}_k, a) \Vdash_{V_j} \neg\varphi \quad \Leftrightarrow \quad (\mathcal{M}_k, a) \nVdash_{V_j} \varphi;$$

$$(\mathcal{M}_k, a) \Vdash_{V_j} (\varphi \ \wedge \ \psi) \quad \Leftrightarrow \ ((\mathcal{M}_k, a) \Vdash_{V_j} \varphi) \wedge ((\mathcal{M}_k, a) \Vdash_{V_j} \psi);$$

$$(\mathcal{M}_k, a) \Vdash_{V_j} (\varphi \ \vee \ \psi) \quad \Leftrightarrow \ ((\mathcal{M}_k, a) \Vdash_{V_j} \varphi) \vee ((\mathcal{M}_k, a) \Vdash_{V_j} \psi);$$

$$(\mathcal{M}_k, a) \Vdash_{V_j} (\varphi \ \rightarrow \ \psi) \quad \Leftrightarrow \ ((\mathcal{M}_k, a) \Vdash_{V_j} \psi) \vee ((\mathcal{M}_k, a) \nVdash_{V_j} \varphi);$$

The part above is a standard one, but for all formulas $\Box_i\psi$ *and* $\Diamond_i\psi$ *and any* V_j *the rules are following:*

$$(\mathcal{M}_k, a) \Vdash_{V_j} \Box_i\varphi \quad \Leftrightarrow \quad (\forall b,\ aR_ib \Rightarrow (\mathcal{M}_k, b) \Vdash_{V_j} \varphi);$$

$$(\mathcal{M}_k, a) \Vdash_{V_j} \Diamond_i\varphi \quad \Leftrightarrow \quad (\exists b,\ aR_ib \Rightarrow (\mathcal{M}_k, b) \Vdash_{V_j} \varphi).$$

Let $\mathcal{K}$ be any class of k -multi-modal frames.

We may, following the classical scheme, define the multi-modal logic $\mathcal{L}(\mathcal{K})$ of this class as all formulas which are true w.r.t. any valuation at all worlds of any multi-valued model $\mathcal{M}$ based at any multi-modal frame from $\mathcal{K}$ (though it seems, in multi-agent environment, the general logic itself would not be of main point of attraction - satisfiability looks more important, but nonetheless, we may model in logics desirable laws about dependencies and hierarchy of the accessibility relations, etc., we will comment it later).

Lemma *If a formula φ is j-satisfied in a model $\mathcal{M}$ based on a frame from $\mathcal{K}_f$ by a valuation V_j, that is for some a, $(\mathcal{M}_k, a) \Vdash_{V_j} \varphi$ than φ may be satisfied in a finite model $\mathcal{M}_{fin}$ based at a frame from $\mathcal{K}_f$ by a valuation V_j, where $||\mathcal{M}_{fin}|| \leq 2^{2^{||Sub\varphi|| \times k}}$.*

Proof – a modified filtration technique

Lemma *If the model $\mathcal{M}$ has objective accessibility relation than the relation R_0 in the filtrated model $\mathcal{M}_{\equiv}$ is also objective.*

Using these Lemmas we get

Theorem. *The satisfiability problem for the class $\mathcal{K}_f$ is decidable.*

Now we would like to extend this result to other classes of frames but not only the class $\mathcal{K}_f$ of all frames.

Theorem. *Let $\mathcal{K}_r$, $\mathcal{K}_t$, $\mathcal{K}_{r,t}$ be respectively the sets of all reflexive, transitive, and reflexive and transitive frames without R_0. Then the satisfiability problem for any of these classes is decidable.*

Notice that we cannot yet prove the case with objective accessibility relations R_0, so the question is open.

Temporal Multi-Agent Logics with Overlap

Definition. *An* multi-agent non-transitive frame *is a tuple*

$$\mathcal{F} := \langle N, \forall a \in N, R_a, Nxt, \rangle, \text{ such that}$$

For all $a \in N$, R_a is the linear order on the interval [a, a+k] for some k

The definitions of multi-agent valuations, models, rules of computations for compound formulas as earlier. Similarly we formulate multi-agent logic L_{OVL}^{MA} and satisfiability problem.

Fresh result:

Theorem. *The satisfiability problem for L_{OVL}^{MA} is decidable.*

Inference Rules, Valid and Admissible

We will consider a variation of $\mathcal{LTL}_{NT}$ — its extension, generated by models with uniformly bound measure of non-transitivity.

Definition A non-transitive possible-worlds linear frame $\mathcal{F}$ with measure of intransitivity m $(m \in N)$ is a frame:

$$\mathcal{F} := \langle N, \leq, \text{Next}, (\bigcup_{i \in N} \{R_i\}) \rangle,$$

defined as earlier but with intervals of transitivity of length at most m.

Definition. Logic $\mathcal{LTL}_{NT}(m)$ is the set of all formulas which are valid at any model $\mathcal{M}$ with the measure of intransitivity m.

Definition. A rule $r := \varphi(x_1, \dots, x_n), \dots, \varphi_m(x_1, \dots, x_n)/\psi(x_1, \dots, x_n)$ is said to be *admissible* in a logic L if for all formulas $\alpha_1, \dots, \alpha_n$

$$[(\bigwedge_{1 \leq i \leq m} \varphi_i(\alpha_1, \dots, \alpha_n)) \in L] \Longrightarrow [\psi(\alpha_1, \dots, \alpha_n) \in L].$$

Theorem . There is an algorithm verifying admissibility of inference rules in $\mathcal{LTL}_{NT}(m)$, Admissibility problem is decidable.

For general case without uniform bound on intransitivity - open problem.

Branching time

Definition A non-transitive possible-worlds branching frame $\mathcal{F}$ with measure of intransitivity m ($m \in N$) is a frame compound from frames:

$$\mathcal{F} := \langle N, \leq, \text{Next}, (\bigcup_{i \in N} \{R_i\}) \rangle,$$

by branching any state in new similar to $\mathcal{F}$ frame.

The corresponding logic $\mathcal{LTL}_{NT}(br, m)$ to be defined as earlier.

Theorem . There is an algorithm solving admissibility problem in $\mathcal{LTL}_{NT}(br, m)$, Admissibility problem is decidable.

Open problems

Investigation of intransitive multi-valued models has remaining set of open usual problems as for any logic: axiomatization, unifiability and decidability with respect admissible inference rules.

Next interesting open question is extension of our results to branching time logics with PAST.

We did not consider yet extended versions of our logic for the case with future and past. The case within the framework when intervals of time are not discrete but continuous is not investigated yet.

Next open avenue for research is embedding in this framework fuzzy logics - the case when truth values of the formulas at any state are not binary but again multi-valued. This may be some tools borrowed from Lukasiewicz logic or modern fuzzy-logic with continuous intervals of truth values.