Reconstruction of values of an algebraic function via polynomial Hermite-Pade m-system

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Main problem

Let f be a **multivalued** analytic function in $\widehat{\mathbb{C}} \setminus \Sigma$, where $\#\Sigma < \infty$. Denote by $\mathcal{A}(\widehat{\mathbb{C}} \setminus \Sigma)$ the set of such functions f for given Σ and by $\mathcal{A}$ the set of f for all Σ.

Actually, in our main results we will consider only algebraic functions f, i.e. f satisfies the equation $P(z, f(z)) = 0$, where $P(z, w)$ is an irreducible polynomial in z and w.

Fix $z_0 \in \widehat{\mathbb{C}}$. Without loss of generality we set $z_0 = \infty$.
Let a germ f_∞ at ∞ of f be given by its Taylor series:
$f_\infty(z) = \sum_{k=0}^{\infty} \frac{c_k}{z^k}$ (we suppose that f_∞ is holomorphic at ∞).

*How to **constructively** reconstruct values of f via this series for f_∞?*
We want to reconstruct "as many values as possible" in "a largest possible domain".

P.Henrici, 1966 "Naively speaking, a procedure may be called constructive if it yields the desired mathematical object as the limit of a single sequence of rational functions of the data of the problem."

Padé polynomials

Let $f_\infty = \sum_{k=0}^{\infty} \frac{c_k}{z^k}$ be a holomorphic germ at ∞.

Definition

Padé polynomials (of order n) constructed from f_∞ at the point $\infty \in \widehat{\mathbb{C}}$ are the polynomials $p_{n,i}$, $i = 0, 1$, such that $\deg p_{n,i} \le n$, $i = 0, 1$, *at least one $p_{n,i} \not\equiv 0$ and*

$$p_{n,0}(z) + p_{n,1}(z) f_\infty(z) = O\left(\frac{1}{z^{n+1}}\right) \text{ as } z \to \infty. \qquad (1)$$

Actually, the condition (1) is a system of n linear homogeneous equations for $n+1$ unknown coefficients of the polynomial $p_{n,1}$. Indeed, let $p_{n,1}(z) = \sum_{k=0}^{n} a_k z^k$. Then

$$p_{n,1}(z) f_\infty(z) = \sum_{k=0}^{n} \alpha_k z^k + \sum_{k=1}^{n} \left(\sum_{l=0}^{n} a_l c_{l+k} \right) \frac{1}{z^k} + O\left(\frac{1}{z^{n+1}}\right).$$

Hence the Padé polynomials always exist, but, in general, are not unique.

Padé polynomials

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Actually, the condition (1) is a system of n linear homogeneous equations for $n+1$ unknown coefficients of the polynomial $p_{n,1}$. Hence the Padé polynomials always exist, but, in general, are not unique.

The fraction $-\frac{p_{n,0}}{p_{n,1}}$ is called nth Padé approximant. It is always unique.

Stahl's theorem, preliminaries, capacity

Capacity. Let $K \subset \mathbb{C}$ be a non-polar compact set. Then there exists Green function $g_{\widehat{\mathbb{C}}\setminus K}(\cdot,\infty)$ of $\widehat{\mathbb{C}} \setminus K$, i. e.
$g_{\widehat{\mathbb{C}}\setminus K}(z,\infty) = 0$ for $z \in K$,
$g_{\widehat{\mathbb{C}}\setminus K}(z,\infty) \in \mathrm{Harm}(\mathbb{C} \setminus K)$, and
$g_{\widehat{\mathbb{C}}\setminus K}(z,\infty) = \log|z| + \gamma + O(1/z), z \to \infty.$

$\operatorname{cap} K := e^{-\gamma}$ is the (logarithmic) capacity of K.

Convergence in capacity. A function sequence $\varphi_n(z)$ converges to a function $\varphi(z)$ in capacity on a set D if for any $\varepsilon > 0$

$$\lim_{n\to\infty} \operatorname{cap}\{z \in D : |\varphi_n(z) - \varphi(z)| > \varepsilon\} = 0.$$

Notation: $\varphi_n(z) \xrightarrow{\text{cap}} \varphi(z)$ on D.

Stahl's theorem, preliminaries, operator $\mathrm{dd^c}$

Operator $\mathrm{dd^c}$ on $\widehat{\mathbb{C}}$. On smooth functions $\varphi \in C^\infty(\widehat{\mathbb{C}})$ it acts in the local coordinate $\zeta = x + iy$ as
$\mathrm{dd^c}\,\varphi := (\varphi_{xx} + \varphi_{yy})dxdy = \Delta\varphi dxdy \in \mathrm{Meas}(\widehat{\mathbb{C}})$.
In general case the operator $\mathrm{dd^c}$ is defined on currents by duality: for a current T and a test function ϕ, we have
$\mathrm{dd^c}\,T(\phi) = T(\mathrm{dd^c}\,\phi)$.

Examples

1) If φ is a subharmonic function, then $\mathrm{dd^c}\,\varphi \in \mathrm{Meas}_+(\widehat{\mathbb{C}})$.
2) For rational functions r, we have Poincaré-Lelong formula:

$$\frac{1}{2\pi}\,\mathrm{dd^c}\log|r| = \sum_{a_j : r(a_j)=0} \delta_{a_j} - \sum_{b_j : r(b_j)=\infty} \delta_{b_j},$$

where δ_x is δ-measure at x.
3) For a non-polar compact set $K \subset \mathbb{C}$, we have
$\mathrm{dd^c}\,g_{\widehat{\mathbb{C}}\setminus K}(\cdot,\infty) = 2\pi(\lambda_K - \delta_\infty)$, where λ_K is the equilibrium measure of ∂K.

Stahl's theorem, preliminaries, admissible compact sets

Let $f \in \mathcal{A}(\widehat{\mathbb{C}} \setminus \Sigma)$ and f_∞ be a germ of f at ∞.

Denote by $\mathcal{K}_{f_\infty}$ the set of compact sets $K \subset \mathbb{C}$ such that f_∞ extends in $\mathbb{C} \setminus K$ as a single-valued meromorphic function and the domain $\mathbb{C} \setminus K$ is maximal with respect to this extension.
"Maximal with respect to this extension" means that we can not extend f_∞ in any domain $D \supsetneq \mathbb{C} \setminus K$.

Stahl's theorem, 1985

Let $f \in \mathcal{A}$ that is equivalent to $f \in \mathcal{A}(\widehat{\mathbb{C}} \setminus \Sigma)$ for some finite set Σ. Let f_∞ be a germ of f at ∞. Recall that

$$p_{n,0}(z) + p_{n,1}(z) f_\infty(z) = O\left(\frac{1}{z^{n+1}}\right) \text{ as } z \to \infty.$$

Theorem (Stahl, 1985)

1) There exists an unique compact set $S \in \mathcal{K}_{f_\infty}$ *such that* $\operatorname{cap} S = \min_{K \in \mathcal{K}_{f_\infty}} \operatorname{cap} K$. *$S$ is called Stahl's compact set.*

It is known that $S = \overline{\bigcup_{j=0}^{J} s_j}$, *where s_j are analytic arcs.*

2) $\dfrac{1}{n} \mathrm{dd^c} \log |p_{n,j}| \xrightarrow{*} \mathrm{dd^c}\, g_{\widehat{\mathbb{C}} \setminus S}(\cdot, \infty)$ *in* $\mathrm{Meas}(\widehat{\mathbb{C}})$.

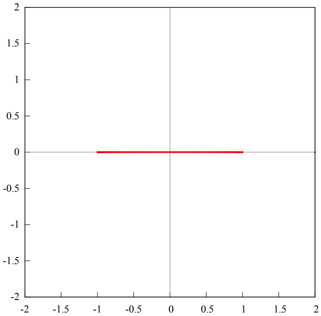
3) $-\dfrac{p_{n,0}}{p_{n,1}} \xrightarrow{\text{cap}} f_\infty$ *on any compact set* $G \subset \mathbb{C} \setminus S$.

4) $\left| f_\infty + \dfrac{p_{n,0}}{p_{n,1}} \right|^{1/n} \xrightarrow{\text{cap}} e^{-2 g_{\widehat{\mathbb{C}} \setminus S}(\cdot, \infty)}$ *on any compact set* $G \subset \mathbb{C} \setminus S$.

Stahl's compact set for function with two branch points

If f has only two branch points, $a, b \in \mathbb{C}$, then Stahl's compact set S is the interval $[a, b]$.

Example, $f(z) = \sqrt[3]{\dfrac{z-1}{z+1}}$



Blue points are the zeroes of the Padé polynomial $p_{300,0}$ for f_∞.
Red points are the zeroes of the Padé polynomial $p_{300,1}$ for f_∞.

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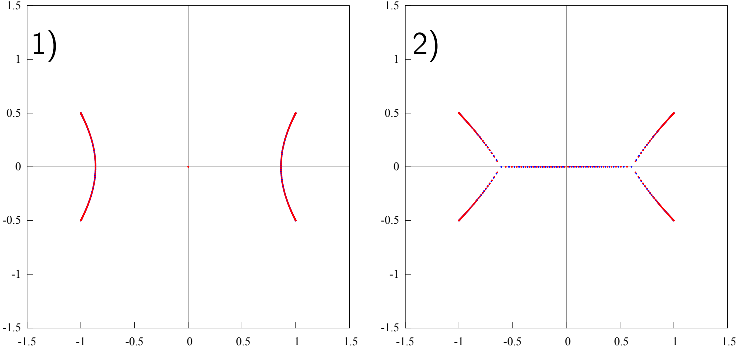


$$1) f(z) = \sqrt{\frac{z-(1+i/2)}{z} \cdot \frac{z-(1-i/2)}{z} \cdot \frac{z-(-1+i/2)}{z} \cdot \frac{z-(-1-i/2)}{z}},$$

$$2) f(z) = \sqrt[4]{\frac{z-(1+i/2)}{z} \cdot \frac{z-(1-i/2)}{z} \cdot \frac{z-(-1+i/2)}{z} \cdot \frac{z-(-1-i/2)}{z}}.$$

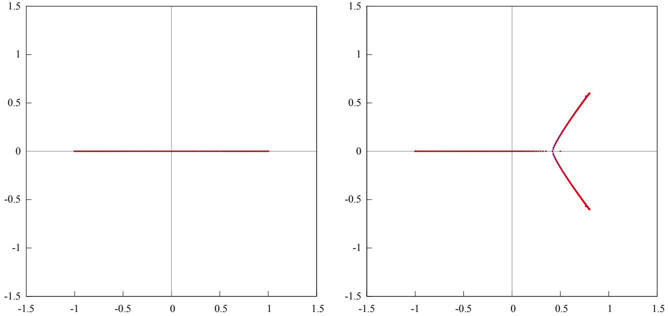
Stahl's compact set depends on f_∞

Let $f(z)$ be the following algebraic function:

$$f^3 + 3(\xi^2 + \sqrt{2\sqrt{3}-3})f + 2i\xi(-\xi^2 - \sqrt{3}\sqrt{2\sqrt{3}-3}) = 0,$$

where $\xi(z) = \dfrac{2-z}{2z-1}$. Then the set of branch points

$\Sigma = \left\{\pm 1, \dfrac{4 \pm 3i}{5}\right\}$ and all these points are of the second order.



Zeroes of Padé polynomials for two different germs f_∞ of f.

Hermite–Padé polynomials of types I and II

Fix $m \in \mathbb{N}$. Let f_∞ be a holomorphic germ at ∞.

Definition

The Hermite–Padé polynomials of type I (of order n) for $[1, f_\infty, f_\infty^2, \dots, f_\infty^m]$ at the point $\infty \in \widehat{\mathbb{C}}$ are the polynomials $Q_{n,i}$, $0 \le i \le m$, such that $\deg Q_{n,i} \le n$, at least one $Q_{n,i} \not\equiv 0$ and

$$\sum_{j=0}^{m} Q_{n,j}(z) f_\infty^j(z) = O\left(\frac{1}{z^{m(n+1)}}\right) \quad \text{as } z \to \infty.$$

Definition

The Hermite–Padé polynomials of type II (of order n) for $[1, f_\infty, f_\infty^2, \dots, f_\infty^m]$ at ∞ are the polynomials $q_{n,i}$, $0 \le i \le m$, such that $\deg q_{n,i} \le mn$, at least one $q_{n,i} \not\equiv 0$, and, for all $1 \le j \le m$,

$$q_{n,0}(z) f_\infty^j(z) - q_{n,j}(z) = O\left(z^{-(n+1)}\right) \quad \text{as } z \to \infty.$$

For $m = 1$, in both cases we have Padé polynomials:

$$p_{n,0}(z) + p_{n,1}(z) f_\infty(z) = O\left(z^{-(n+1)}\right) \text{ as } z \to \infty.$$

Our case

In what follows, f is an algebraic function of degree $m+1$,
i. e. f satisfies the equation $P(z, f(z)) = 0$, where $P(z, w)$ is an irreducible polynomial in z and w and $\deg_w P(z, w) = m + 1$.
Let $\mathfrak{R}$ be the (compact) Riemann surface of f.
Then f is lifted on $\mathfrak{R}$ as a meromorphic function.

For points of $\mathfrak{R}$ that lie over z we will use the notation $\boldsymbol{z}$.
Denote by π the natural projection $\boldsymbol{z} \mapsto z$, i.e. $\pi(\boldsymbol{z}) = z$.
For generic points we have $\boldsymbol{z} = (z, w)$, where $P(z, w) = 0$, and $\pi(z, w) = z$.

So, we have the following.
$\mathfrak{R}$ is a compact Riemann surface.
$\pi : \mathfrak{R} \to \widehat{\mathbb{C}}$ is an (m+1)-sheeted branched covering of $\widehat{\mathbb{C}}$.
Σ is the set of critical values of π.
f is a meromorphic function on $\mathfrak{R}$.
f_∞ is the germ of f at some point $\boldsymbol{\infty}^{(0)} \in \pi^{-1}(\infty)$, which is not critical for π.

We will consider Hermite–Padé polynomials for $[1, f_\infty, f_\infty^2, \dots, f_\infty^m]$.

Nuttall partition of $\mathfrak{R}$ into sheets

Let $u(\boldsymbol{z})$ be harmonic function on $\mathfrak{R} \setminus \pi^{-1}(\infty)$ with the following singularities at $\pi^{-1}(\infty)$:

$$u(\boldsymbol{z}) = -m \log|z| + O(1), \quad \boldsymbol{z} \to \boldsymbol{\infty}^{(0)},$$
$$u(\boldsymbol{z}) = \log|z| + O(1), \quad \boldsymbol{z} \to \pi^{-1}(\infty) \setminus \boldsymbol{\infty}^{(0)}.$$

Let $z \in \mathbb{C}$ and $\pi^{-1}(z) = \{\boldsymbol{z}^{(0)}, \boldsymbol{z}^{(1)}, \dots, \boldsymbol{z}^{(j)}, \dots, \boldsymbol{z}^{(m)}\}$, where the points $\boldsymbol{z}^{(j)}$ are ordered in accordance with nondecreasing of the values of u at them:

$$u(\boldsymbol{z}^{(0)}) \le u(\boldsymbol{z}^{(1)}) \le \dots \le u(\boldsymbol{z}^{(j)}) \le \dots \le u(\boldsymbol{z}^{(m)}).$$

For $z = \infty$, we consider $u(\boldsymbol{z}) - \log|z|$ instead of u.

$$\mathfrak{R}^{(0)} := \{\boldsymbol{z}^{(0)} \in \mathfrak{R} : u(\boldsymbol{z}^{(0)}) < u(\boldsymbol{z}^{(1)})\};$$
$$\mathfrak{R}^{(j)} := \{\boldsymbol{z}^{(j)} \in \mathfrak{R} : u(\boldsymbol{z}^{(j-1)}) < u(\boldsymbol{z}^{(j)}) < u(\boldsymbol{z}^{(j+1)})\}, \quad j = 1, \dots, m-1$$
$$\mathfrak{R}^{(m)} := \{\boldsymbol{z}^{(m)} \in \mathfrak{R} : u(\boldsymbol{z}^{(m-1)}) < u(\boldsymbol{z}^{(m)})\}.$$

Note that always $\boldsymbol{\infty}^{(0)} \in \mathfrak{R}^{(0)}$.

Nuttall partition of $\mathfrak{R}$ into sheets

$$u(\boldsymbol{z}^{(0)}) \le u(\boldsymbol{z}^{(1)}) \le \cdots \le u(\boldsymbol{z}^{(j)}) \le \cdots \le u(\boldsymbol{z}^{(m)}).$$

$$\begin{aligned}
\mathfrak{R}^{(0)} &:= \{\boldsymbol{z}^{(0)} \in \mathfrak{R} : u(\boldsymbol{z}^{(0)}) < u(\boldsymbol{z}^{(1)})\};\\
\mathfrak{R}^{(j)} &:= \{\boldsymbol{z}^{(j)} \in \mathfrak{R} : u(\boldsymbol{z}^{(j-1)}) < u(\boldsymbol{z}^{(j)}) < u(\boldsymbol{z}^{(j+1)})\}, \quad j = 1, \ldots, m-1\\
\mathfrak{R}^{(m)} &:= \{\boldsymbol{z}^{(m)} \in \mathfrak{R} : u(\boldsymbol{z}^{(m-1)}) < u(\boldsymbol{z}^{(m)})\}.
\end{aligned}$$

For $z \in \widehat{\mathbb{C}}$ we **define** functions $u_j(z) := u(\boldsymbol{z}^{(j)})$, $j = 0, \ldots, m$. So

$$u_0(z) \le u_1(z) \le \cdots \le u_j(z) \le \cdots \le u_m(z).$$

Set

$$\begin{aligned}
F_j &:= \{z \in \widehat{\mathbb{C}} : u_{j-1}(z) = u_j(z)\}, \quad j = 1, \ldots, m,\\
F &:= \cup_{j=1}^{m} F_j.
\end{aligned}$$

Lemma. $F_j = \overline{\bigcup_{l=0}^{L_j} \alpha_j}$, *where* α_j *are analytic arcs.*

It is clear that $\pi(\partial\mathfrak{R}^{(j)}) = F_j \cup F_{j+1}$ for $j = 1, \ldots, m-1$, $\pi(\partial\mathfrak{R}^{(0)}) = F_1$ and $\pi(\partial\mathfrak{R}^{(m)}) = F_m$.

Theorem for Hermite–Padé polynomials of type II

The Hermite–Padé polynomials of type II (of order n) for $[1, f_\infty, f_\infty^2, \dots, f_\infty^m]$ at ∞ are the polynomials $q_{n,i}$, $0 \le i \le m$, such that $\deg q_{n,i} \le mn$, at least one $q_{n,i} \not\equiv 0$, and, for all $1 \le j \le m$,

$$q_{n,0}(z) f_\infty^j(z) - q_{n,j}(z) = O\left(z^{-(n+1)}\right) \quad \text{as } z \to \infty$$

Theorem (Nuttall, 1984)

1) $\dfrac{1}{n}\,\mathrm{dd^c} \log|q_{n,j}| \xrightarrow{\;*\;} -\,\mathrm{dd^c}\, u_0$ *in* $\mathrm{Meas}(\widehat{\mathbb{C}})$.

2) $\dfrac{q_{n,1}}{q_{n,0}} \xrightarrow{\text{cap}} f(\boldsymbol{z}^{(0)})$ *on any compact set* $G \subset \mathbb{C} \setminus F_1$.

3) For any compact set $G \subset \mathbb{C} \setminus F_1$ *and any* $\varepsilon > 0$, *as* $n \to \infty$,

$$\operatorname{cap}\left\{ z \in G : \left| \frac{q_{n,1}}{q_{n,0}} - f(\boldsymbol{z}^{(0)}) \right|^{1/n} \cdot e^{u_1(z) - u_0(z)} \geqslant 1 + \varepsilon \right\} \to 0.$$

Theorem for HP type II and Stahl's theorem

Theorem (Nuttall, 1984)

1) $\frac{1}{n}\,\mathrm{dd^c}\log|q_{n,j}| \xrightarrow{*} -\mathrm{dd^c}\,u_0$ *in* $\mathrm{Meas}(\widehat{\mathbb{C}})$.

2) $\frac{q_{n,1}}{q_{n,0}} \xrightarrow{\text{cap}} f(\boldsymbol{z}^{(0)})$ *on any compact set* $G \subset \mathbb{C}\setminus F_1$.

3) For any compact set $G \subset \mathbb{C}\setminus F_1$ *and any* $\varepsilon > 0$, *as* $n \to \infty$,

$$\operatorname{cap}\left\{ z \in G : \left|\frac{q_{n,1}}{q_{n,0}} - f(\boldsymbol{z}^{(0)})\right|^{1/n} \cdot e^{u_1(z)-u_0(z)} \geqslant 1+\varepsilon \right\} \to 0.$$

Theorem (Stahl, 1985)

2) $\frac{1}{n}\,\mathrm{dd^c}\log|p_{n,j}| \xrightarrow{*} \mathrm{dd^c}\,g_{\widehat{\mathbb{C}}\setminus S}(\cdot,\infty)$ *in* $\mathrm{Meas}(\widehat{\mathbb{C}})$.

3) $-\frac{p_{n,0}}{p_{n,1}} \xrightarrow{\text{cap}} f_\infty$ *on any compact set* $G \subset \mathbb{C}\setminus S$.

4) $\left|f_\infty + \frac{p_{n,0}}{p_{n,1}}\right|^{1/n} \xrightarrow{\text{cap}} e^{-2g_{\widehat{\mathbb{C}}\setminus S}(\cdot,\infty)}$ *on any compact set* $G \subset \mathbb{C}\setminus S$.

Theorem for Hermite–Padé polynomials of type I

The Hermite–Padé polynomials of type I (of order n) for $[1, f_\infty, f_\infty^2, \dots, f_\infty^m]$ at the point $\infty \in \widehat{\mathbb{C}}$ are the polynomials $Q_{n,i}$, $0 \le i \le m$, such that $\deg Q_{n,i} \le n$, at least one $Q_{n,i} \not\equiv 0$ and

$$\sum_{j=0}^{m} Q_{n,j}(z) f_\infty^j(z) = O\left(\frac{1}{z^{m(n+1)}}\right) \quad \text{as } z \to \infty.$$

Theorem (K., Chirka, Palvelev, Suetin, 2017)

1) $\dfrac{1}{n}\,\mathrm{dd^c}\log|Q_{n,j}| \xrightarrow{*} \mathrm{dd^c}\, u_m = -\,\mathrm{dd^c}(u_0 + \dots + u_{m-1})$ *in* $\operatorname{Meas}(\widehat{\mathbb{C}})$*.*

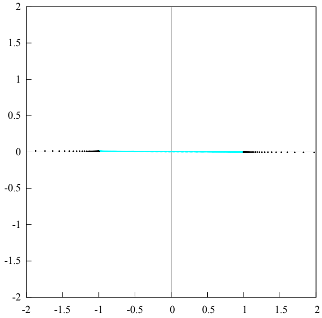
2) $\dfrac{Q_{n,m-1}}{Q_{n,m}} \xrightarrow{\text{cap}} -(f(\boldsymbol{z}^{(0)}) + \dots + f(\boldsymbol{z}^{(m-1)}))$ *on any compact set* $G \subset \mathbb{C} \setminus F_m$*.*

3) For any compact set $G \subset \mathbb{C} \setminus F_m$ *and any* $\varepsilon > 0$*, as* $n \to \infty$*,*

$$\operatorname{cap}\left\{ z \in G : \left| \frac{Q_{n,m-1}}{Q_{n,m}} + \sum_{s=0}^{m-1} f(\boldsymbol{z}^{(s)}) \right|^{1/n} \cdot e^{u_m(z) - u_{m-1}(z)} \geqslant 1 + \varepsilon \right\} \to 0.$$

Example: $\infty \in F_m$, $m = 2$

$$f(z) = \sqrt[3]{\frac{z-1}{z+1}}$$

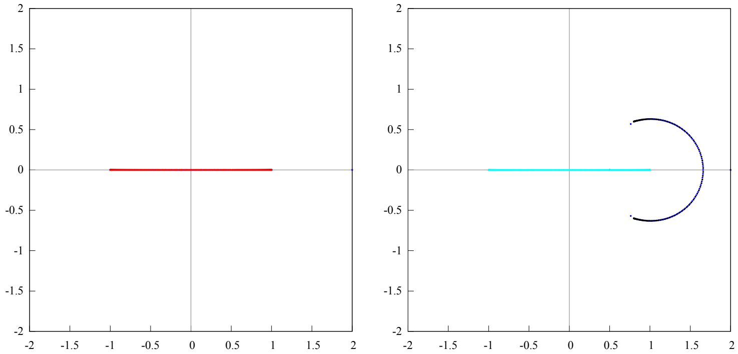


F_2: Black points are the zeroes of Hermite–Padé polynomial of type I $Q_{200,2}$ for f_∞.
F_1: Light blue points are the zeroes of Hermite–Padé polynomial of type II $q_{200,2}$ for f_∞.

f_∞ is some germ of f:

$$f^3 + 3(\xi^2 + \sqrt{2\sqrt{3}-3})f + 2i\xi(-\xi^2 - \sqrt{3}\sqrt{2\sqrt{3}-3}) = 0,$$

where $\xi(z) = \dfrac{2-z}{2z-1}$. The set $\Sigma = \left\{\pm 1, \dfrac{4 \pm 3i}{5}\right\}$.



S: Red points are the zeroes of Padé polynomial $P_{200,1}$.

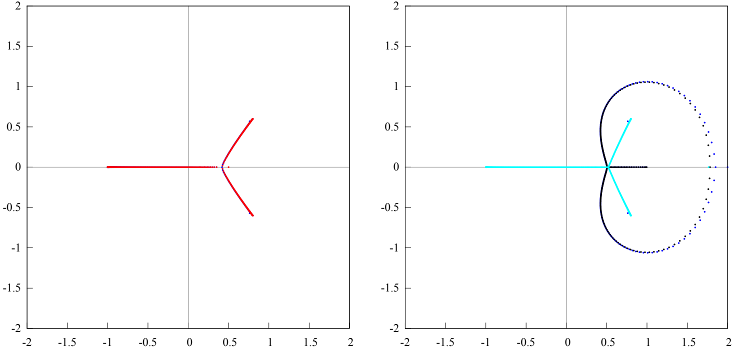
F_2: Blue and black points are the zeroes of HP-I $Q_{200,0}$ and $Q_{200,2}$.

F_1: Light blue points are the zeroes of HP-II $q_{200,0}$.

f_∞ is another germ of f:

$$f^3 + 3(\xi^2 + \sqrt{2\sqrt{3}-3})f + 2i\xi(-\xi^2 - \sqrt{3}\sqrt{2\sqrt{3}-3}) = 0,$$

where $\xi(z) = \dfrac{2-z}{2z-1}$. The set $\Sigma = \left\{\pm 1, \dfrac{4 \pm 3i}{5}\right\}$.

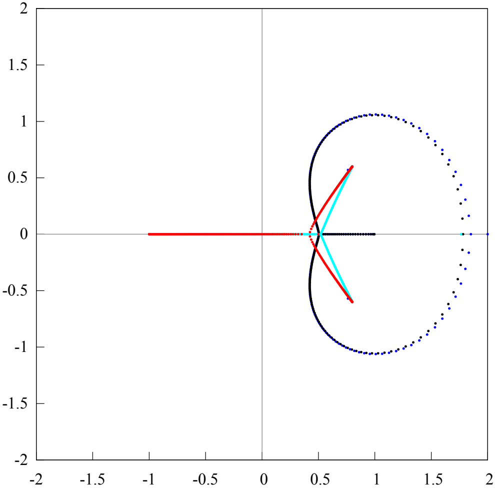


S: Red points are the zeroes of Padé polynomial $P_{200,1}$.

F_2: Blue and black points are the zeroes of HP-I $Q_{200,0}$ and $Q_{200,2}$.

F_1: Light blue points are the zeroes of HP-II $q_{200,0}$.

$$f^3 + 3(\xi^2 + \sqrt{2\sqrt{3} - 3})f + 2i\xi(-\xi^2 - \sqrt{3}\sqrt{2\sqrt{3} - 3}) = 0$$



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F_2: Blue and black points are the zeroes of HP-I $Q_{200,0}$ and $Q_{200,2}$.

F_1: Light blue points are the zeroes of HP-II $q_{200,0}$.

$m = 2$ and Shafer approximations

For $m = 2$, we have the following:
$\frac{q_{n,1}}{q_{n,0}} \xrightarrow{\text{cap}} f(\boldsymbol{z}^{(0)})$ on any compact set $G \subset \mathbb{C} \setminus F_1$;
$\frac{Q_{n,1}}{Q_{n,2}} \xrightarrow{\text{cap}} -(f(\boldsymbol{z}^{(0)}) + f(\boldsymbol{z}^{(1)}))$ on any compact set $G \subset \mathbb{C} \setminus F_2$.
Actually, from [K., Chirka, Palvelev, Suetin, 2017] it follows that $\frac{Q_{n,0}}{Q_{n,2}} \xrightarrow{\text{cap}} f(\boldsymbol{z}^{(0)})f(\boldsymbol{z}^{(1)})$ on any compact set $G \subset \mathbb{C} \setminus F_2$. So, the equation

$$Q_{n,2}(z)w^2 + Q_{n,1}(z)w + Q_{n,0}(z) = 0$$

asymptotically reconstructs the quadratic equation satisfied by $f(z^{(0)})$, $f(z^{(1)})$. Its solutions

$$\left(-Q_{n,1}(z) + \sqrt{Q_{n,1}^2(z) - 4Q_{n,0}(z)Q_{n,2}(z)}\right)/2Q_{n,2}(z)$$

are called Shafer approximations.
We know nothing about their convergence.
Moreover, Shafer approximations are not constructive, unlike $q_{n,1}/q_{n,0}$ and $Q_{n,1}/Q_{n,2}$.

What do we have for arbitrary m?

We know that

$\frac{q_{n,1}}{q_{n,0}} \xrightarrow{\text{cap}} f(\boldsymbol{z}^{(0)})$ on any compact set $G \subset \mathbb{C} \setminus F_1$,
$\frac{Q_{n,m-1}}{Q_{n,m}} \xrightarrow{\text{cap}} -(f(\boldsymbol{z}^{(0)}) + \cdots + f(\boldsymbol{z}^{(m-1)}))$ on any compact set $G \subset \mathbb{C} \setminus F_m$.

In 2018 S. Suetin suggested that for each $k = 1, \ldots, m$ we should find polynomials such that their ratios reconstruct the sum of values of f on first k Nuttall sheets of the Riemann surface, instead of the value on kth sheet itself.

Polynomial Hermite–Padé m-system

Definition

*Fix a natural number $k \in \{1, \dots, m\}$. k**th polynomials of the Hermite–Padé** m**-system** (of order n), for $[1, f_\infty, \dots, f_\infty^m]$ at ∞ are C_{m+1}^k polynomials $P_{n;i_1,\dots,i_k}$, $0 \le i_1 < i_2 < \dots < i_k \le m$, such that $\deg P_{n;i_1,\dots,i_k} \le (m+1-k)n$, at least one $P_{n;i_1,\dots,i_k} \not\equiv 0$, and, for each index set $0 < j_1 < \dots < j_k \le m$,*

$$P_{n;j_1\dots,j_k}(z) + \sum_{s=1}^{k} (-1)^s P_{n;0,j_1,\dots,j_{s-1},j_{s+1},\dots,j_k}(z) f_\infty^{j_s}(z) = O\left(\frac{1}{z^{kn+1}}\right) \tag{2}$$

as $z \to \infty$.

Actually, the condition (2) is a system of $(n(m+1)+1)C_m^k$ linear homogeneous equations for $n(m+1)C_m^k + C_{m+1}^k$ unknown coefficients of the polynomials $P_{n;i_1,\dots,i_k}$.
Hence the polynomials of the Hermite–Padé m-system always exist, but, in general, are not unique.

Theorem for kth polynomials of the Hermite–Padé m-system

Theorem (2021)

Fix $k = 1, \dots, m$. Let the projection $\pi\colon \mathfrak{R} \to \widehat{\mathbb{C}}$ be such that all its critical points are of the first order and that for each $z \in \widehat{\mathbb{C}}$ there is at most one critical point of π in the set $\pi^{-1}(z)$. Then

1) $\dfrac{1}{n}\,\mathrm{dd^c}\log|P_{n;j_1\dots,j_k}| \xrightarrow{*} -\mathrm{dd^c}(u_0 + \dots + u_{k-1})$ *in* $\mathrm{Meas}(\widehat{\mathbb{C}})$;

2) $\dfrac{P_{n;0,\dots,k-2,k}}{P_{n;0,\dots,k-1}} \xrightarrow{\mathrm{cap}} f(\boldsymbol{z}^{(0)}) + \dots + f(\boldsymbol{z}^{(k-1)})$ *on any compact set* $G \subset \mathbb{C} \setminus F_k$;

3) For any compact set $G \subset \mathbb{C} \setminus F_k$ and any $\varepsilon > 0$, as $n \to \infty$,

$$\mathrm{cap}\left\{ z \in G : \left| \frac{P_{n;0,\dots,k-2,k}}{P_{n;0,\dots,k-1}} - \sum_{s=0}^{k-1} f(\boldsymbol{z}^{(s)}) \right|^{1/n} e^{u_k(z)-u_{k-1}(z)} \geqslant 1+\varepsilon \right\} \to 0$$

So, successively evaluating kth polynomials of the Hermite–Padé m-system for all $k = 1, \dots, m$, we constructively reconstruct the values of f on all Nuttall sheets of $\mathfrak{R}$, except the last one $\mathfrak{R}^{(m)}$, outside the set $\pi^{-1}(F)$, where $F := \cup_{j=1}^m F_j$.

Connection with HP polynomials of type II

Put $k = 1$.
1st polynomials of the Hermite–Padé m-system:
$\deg P_{n;i_1} \leqslant mn$, $0 \leqslant i_1 \leqslant m$, and for each $0 < j_1 \leqslant m$

$$P_{n;j_1}(z) - P_{n;0}(z) f_\infty^{j_1}(z) = O\left(\frac{1}{z^{n+1}}\right) \text{ as } z \to \infty.$$

The Hermite–Padé polynomials of type II:
$\deg q_{n,i} \leq nm$, $i = 0, \ldots, m$, and $\forall j = 1, \ldots m$

$$q_{n,0}(z) f_\infty^{j}(z) - q_{n,j}(z) = O\left(\frac{1}{z^{n+1}}\right) \text{ as } z \to \infty.$$

So, the Hermite–Padé polynomials of type II and 1st polynomials of the Hermite–Padé m-system precisely coincide.

Connection with HP polynomials of type I

Put $k = m$.
mth polynomials of the Hermite–Padé m-system:
$\deg P_{n;i_1,\dots,i_k} \leqslant n$, $0 \leqslant i_1 < i_2 < \cdots < i_m \leqslant m$, and for
$0 < 1 < \cdots < m \leqslant m$, as $z \to \infty$,

$$P_{n;1,\dots,m}(z)+\sum_{s=1}^{m}(-1)^s P_{n;0,1,\dots,s-1,s+1,\dots,m}(z) f_\infty^s(z) = O\left(\frac{1}{z^{mn+1}}\right).$$

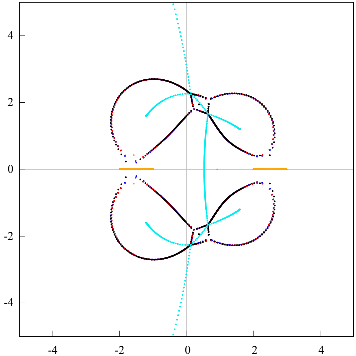
The Hermite–Padé polynomials of type I:
$\deg Q_{n,i} \leq n, i = 0, \dots, m$, and

$$Q_{n,0}(z) + \sum_{j=1}^{m} Q_{n,j}(z) f_\infty^j(z) = O\left(\frac{1}{z^{m(n+1)}}\right) \quad \text{as } z \to \infty.$$

So, polynomials $(-1)^j Q_{n;j}$ are particular case of mth polynomials of the Hermite–Padé m-system.

f_∞ is some germ of f:

$$f(z) = \left(\prod_{j=1}^{2} (A_j - \varphi_{\Delta_1}(z)) \Big/ \prod_{k=1}^{2} (B_k - \varphi_{\Delta_2}(z)) \right)^{1/2}.$$



F_1: Orange points are the zeroes of HP-II $q_{300,0}$.
F_3: Blue, red and black points are the zeroes of HP-I $Q_{300,0}$, $Q_{300,1}$ and $Q_{300,2}$.
F_2: Light blue points are the zeroes of the 2nd polynomial of HP 4-system $P_{300;0,1}$.

Thank you for your attention!