# ON THE LOCUS OF $p$-CHARACTERS DEFINING SIMPLE REDUCED ENVELOPING ALGEBRAS 

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#### Abstract

We confirm in two cases the conjecture stating that the reduced enveloping algebra $U_{\xi}(\mathfrak{g})$ of a restricted Lie algebra $\mathfrak{g}$ is simple if and only if the alternating bilinear form associated with the given $p$-character $\xi \in \mathfrak{g}^{*}$ is nondegenerate.


Key words: restricted Lie algebras, solvable Lie algebras, Frobenius Lie algebras, reduced enveloping algebras.

In the representation theory of a finite dimensional $p$-Lie algebra $\mathfrak{g}$ over an algebraically closed field $k$ of characteristic $p>0$ one is naturally led to consider the family of reduced enveloping algebras $U_{\xi}(\mathfrak{g})$ associated with linear functions $\xi \in \mathfrak{g}^{*}$ (see [1]). The algebra $U_{\xi}(\mathfrak{g})$ is defined as the factor algebra of the universal enveloping algebra $U(\mathfrak{g})$ by its ideal generated by central elements $x^{p}-x^{[p]}-\xi(x)^{p} \cdot 1$ with $x \in \mathfrak{g}$, and $\xi$ is called the $p$-character of any $\mathfrak{g}$-module which can be realized as a module over $U_{\xi}(\mathfrak{g})$. There is a certain, still far from fully understood, relation between generic properties of the family of reduced enveloping algebras and generic properties of the family of stabilizers of linear functions. The stabilizer $\mathfrak{z}(\xi)$ of $\xi \in \mathfrak{g}^{*}$ coincides with the radical of the alternating bilinear form $\beta_{\xi}: \mathfrak{g} \times \mathfrak{g} \rightarrow k$ defined by the rule

$$
\beta_{\xi}(x, y)=\xi([x, y]) \quad \text { for } x, y \in \mathfrak{g} .
$$

The Lie algebra $\mathfrak{g}$ is called Frobenius if $\beta_{\xi}$ is nondegenerate for at least one $\xi$.
In general one cannot determine the type of one particular algebra $U_{\xi}(\mathfrak{g})$ just knowing $\mathfrak{z}(\xi)$. It is quite interesting and surprising that sometimes this can be done. In [2] it was conjectured that $U_{\xi}(\mathfrak{g})$ is simple if and only if $\mathfrak{z}(\xi)=0$, that is, if and only if $\beta_{\xi}$ is nondegenerate. The purpose of the present article is to verify this conjecture in two cases. When $\mathfrak{g}$ is solvable and $p>2$ we do this using the description of irreducible $\mathfrak{g}$-modules due to Strade [3]. We have to make more careful selections of subalgebras from which irreducible $\mathfrak{g}$-modules are obtained by induction. The second case occurs when $\mathfrak{g}$ is Frobenius and all adjoint derivations of $\mathfrak{g}$ lie in the Lie algebra of the automorphism group. Here we apply geometric arguments to the extension of the family of reduced enveloping algebras constructed in [4].

An example at the end of the paper shows that semisimplicity of the algebra $U_{\xi}(\mathfrak{g})$ cannot be recognized in terms of $\mathfrak{z}(\xi)$ by means of a possible generalization of the above conjecture.

## 1. Solvable Lie algebras

It is assumed in this section that $\mathfrak{g}$ is solvable and $p>2$. Recall that a polarization of $\mathfrak{g}$ at $\xi \in \mathfrak{g}^{*}$ is a Lie subalgebra which is simultaneously a maximal totally isotropic subspace with respect to the alternating bilinear form $\beta_{\xi}$ [5].

Denote by $\mathcal{P}$ the set of all triples $(\mathfrak{p}, \mathfrak{a}, \lambda)$ such that $\mathfrak{a} \subset \mathfrak{p} \subset \mathfrak{g}$ are vector subspaces, $\lambda \in \mathfrak{a}^{*}$ is a linear function and there exists a chain of subspaces

$$
\begin{equation*}
0=\mathfrak{a}_{0} \subset \mathfrak{a}_{1} \subset \ldots \subset \mathfrak{a}_{n}=\mathfrak{a} \subset \mathfrak{p}=\mathfrak{p}_{n} \subset \ldots \subset \mathfrak{p}_{1} \subset \mathfrak{p}_{0}=\mathfrak{g} \tag{1}
\end{equation*}
$$

with the property that

$$
\begin{equation*}
\left[\mathfrak{p}_{i-1}, \mathfrak{a}_{i}\right] \subset \mathfrak{a}_{i} \quad \text { and } \quad \mathfrak{p}_{i}=\left\{x \in \mathfrak{p}_{i-1} \mid \lambda\left(\left[x, \mathfrak{a}_{i}\right]\right)=0\right\} \tag{2}
\end{equation*}
$$

for all $i=1, \ldots, n$. As one checks by induction on $i$, each $\mathfrak{p}_{i}$ is a $p$-subalgebra of $\mathfrak{g}$, and $\mathfrak{a}_{i}$ is an ideal of $\mathfrak{p}_{i-1}$. In particular, $\mathfrak{p}$ is a $p$-subalgebra of $\mathfrak{g}$, and $\mathfrak{a}$ is an ideal of $\mathfrak{p}$. Furthermore, $\lambda$ vanishes on $[\mathfrak{p}, \mathfrak{a}]$ and, therefore, also on $[\mathfrak{a}, \mathfrak{a}]$.

Lemma 1. Suppose that $(\mathfrak{p}, \mathfrak{a}, \lambda) \in \mathcal{P}$. If $\xi \in \mathfrak{g}^{*}$ is a linear function such that $\lambda(x)^{p}-\lambda\left(x^{[p]}\right)=\xi(x)^{p}$ for all $x \in \mathfrak{a}$ and $W$ is an irreducible $U_{\xi}(\mathfrak{p})$-module such that $x w=\lambda(x) w$ for all $x \in \mathfrak{a}$ and $w \in W$, then the induced $\mathfrak{g}$-module $U_{\xi}(\mathfrak{g}) \otimes_{U_{\xi}(\mathfrak{p})} W$ is irreducible.

Here $U_{\xi}(\mathfrak{p})$ stands for the reduced enveloping algebra of $\mathfrak{p}$ corresponding to the restriction of $\xi$ to $\mathfrak{p}$. The proof is obtained by a repeated application of the characteristic $p$ analog of Blattner's irreducibility criterion [6, Theorem 3].

We will need additional conditions on triples. Denote by $\mathcal{P}^{\prime}$ the set of all triples $(\mathfrak{p}, \mathfrak{a}, \lambda)$ such that $\mathfrak{a} \subset \mathfrak{p} \subset \mathfrak{g}$ are vector subspaces, $\lambda \in \mathfrak{a}^{*}$ is a linear function, and there exists a chain of subspaces

$$
\begin{equation*}
0=\mathfrak{a}_{0} \subset \mathfrak{a}_{1} \subset \ldots \subset \mathfrak{a}_{n}=\mathfrak{a} \subset \mathfrak{p} \subset \widetilde{\mathfrak{p}}_{n} \subset \ldots \subset \widetilde{\mathfrak{p}}_{1} \subset \widetilde{\mathfrak{p}}_{0}=\mathfrak{g} \tag{3}
\end{equation*}
$$

with the property that

$$
\begin{gather*}
{\left[\widetilde{\mathfrak{p}}_{i-1}, \mathfrak{a}_{i}\right] \subset \mathfrak{a}_{i},}  \tag{4}\\
\widetilde{\mathfrak{p}}_{i}=\left\{x \in \widetilde{\mathfrak{p}}_{i-1} \mid \lambda\left(\left[x, \mathfrak{a}_{i}^{\prime}\right]\right)=0\right\}, \quad \text { where } \quad \mathfrak{a}_{i}^{\prime}=\left\{y \in \mathfrak{a}_{i} \mid \lambda(y)=0\right\},  \tag{5}\\
\mathfrak{p}=\left\{x \in \widetilde{\mathfrak{p}}_{n} \mid \lambda([x, \mathfrak{a}])=0\right\} \tag{6}
\end{gather*}
$$

for all $i=1, \ldots, n$. We will say that chain (3) is $(\mathfrak{p}, \mathfrak{a}, \lambda)$-admissible in this case.
Lemma 2. In a $(\mathfrak{p}, \mathfrak{a}, \lambda)$-admissible chain each $\widetilde{\mathfrak{p}}_{i}$ is a p-subalgebra, $\mathfrak{a}_{i}$ is an ideal of $\widetilde{\mathfrak{p}}_{i-1}$, and $\mathfrak{a}_{i}^{\prime}$ is an ideal of $\widetilde{\mathfrak{p}}_{i}$. Furthermore, $\mathfrak{p}$ is an ideal of $\widetilde{\mathfrak{p}}_{n}$.

Proof. Since $\left[\widetilde{\mathfrak{p}}_{i}, \mathfrak{a}_{i}\right] \subset \mathfrak{a}_{i}$ by (4) and $\lambda$ vanishes on $\left[\mathfrak{p}_{i}, \mathfrak{a}_{i}^{\prime}\right]$ by (5), we deduce that $\left[\widetilde{p}_{i}, \mathfrak{a}_{i}^{\prime}\right] \subset \mathfrak{a}_{i}^{\prime}$. Since the normalizer of $\mathfrak{a}_{i}^{\prime}$ in $\mathfrak{g}$ is a $p$-subalgebra, an induction on $i$ shows that so too is $\widetilde{\mathfrak{p}}_{i}$. Now $[\mathfrak{p}, \mathfrak{a}] \subset \mathfrak{a}$ and $\lambda$ vanishes on $[\mathfrak{p}, \mathfrak{a}]$ by (4) and (6), whence $[\mathfrak{p}, \mathfrak{a}] \subset \mathfrak{a}_{n}^{\prime}$. It follows $\left[\left[\widetilde{\mathfrak{p}}_{n}, \mathfrak{p}\right], \mathfrak{a}\right] \subset\left[\widetilde{\mathfrak{p}}_{n}, \mathfrak{a}_{n}^{\prime}\right]+[\mathfrak{p}, \mathfrak{a}] \subset \mathfrak{a}_{n}^{\prime}$, and so $\left[\widetilde{\mathfrak{p}}_{n}, \mathfrak{p}\right] \subset \mathfrak{p}$.

Lemma 3. It holds $\mathcal{P}^{\prime} \subset \mathcal{P}$.
Proof. Let $(\mathfrak{p}, \mathfrak{a}, \lambda) \in \mathcal{P}^{\prime}$. Consider a $(\mathfrak{p}, \mathfrak{a}, \lambda)$-admissible chain (3) and for each $i$ define $\mathfrak{p}_{i}=\left\{x \in \widetilde{\mathfrak{p}}_{i} \mid \lambda\left(\left[x, \mathfrak{a}_{i}\right]\right)=0\right\}$. We obtain then a chain (1) with $\mathfrak{p}_{i} \subset \widetilde{\mathfrak{p}}_{i}$, and it is checked straightforwardly that (2) is fulfilled. Thus $(\mathfrak{p}, \mathfrak{a}, \lambda) \in \mathcal{P}$.

Lemma 4. Suppose that $\mathfrak{a}$ is a one-dimensional ideal of a solvable Lie algebra $\mathfrak{h}$, and $\mathfrak{b}$ is an ideal of $\mathfrak{h}$, minimal with respect to the property that $\mathfrak{a} \subset \mathfrak{b}, \mathfrak{a} \neq \mathfrak{b}$ and $[\mathfrak{a}, \mathfrak{b}]=0$. Then $\mathfrak{b}$ is abelian.

Proof. Put $\mathfrak{c}=\{x \in \mathfrak{b} \mid[x, \mathfrak{b}]=0\}$. Then $\mathfrak{c}$ is an ideal of $\mathfrak{h}$ and $\mathfrak{a} \subset \mathfrak{c} \subset \mathfrak{b}$. By the minimality of $\mathfrak{b}$ we have either $\mathfrak{c}=\mathfrak{b}$ or $\mathfrak{c}=\mathfrak{a}$. In the first case $[\mathfrak{b}, \mathfrak{b}]=0$, and we are done. Suppose that $\mathfrak{c}=\mathfrak{a}$. Then the multiplication in $\mathfrak{b}$ induces a nondegenerate alternating bilinear form $\mathfrak{b} / \mathfrak{a} \times \mathfrak{b} / \mathfrak{a} \rightarrow \mathfrak{a}$. In particular, $\mathfrak{b} / \mathfrak{a}$ has even dimension. On the other hand, $\mathfrak{b} / \mathfrak{a}$ is an irreducible $\mathfrak{h}$-module by the minimality of $\mathfrak{b}$, and therefore $\operatorname{dim} \mathfrak{b} / \mathfrak{a}$ is a power of $p$, hence odd, by [3, Satz 3]. We arrive at a contradiction.

Lemma 5. Suppose that $(\mathfrak{p}, \mathfrak{a}, \lambda) \in \mathcal{P}^{\prime}$. If $\mathfrak{a} \neq \mathfrak{p}$, then there exists a vector subspace $\mathfrak{b} \subset \mathfrak{p}$ such that $\mathfrak{a}$ is contained in $\mathfrak{b}$ properly, $[\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{a}^{\prime}=\operatorname{ker} \lambda$, and for every linear function $\mu \in \mathfrak{b}^{*}$ extending $\lambda$ there exists $\mathfrak{q}$ satisfying $(\mathfrak{q}, \mathfrak{b}, \mu) \in \mathcal{P}^{\prime}$.

Proof. Consider a ( $\mathfrak{p}, \mathfrak{a}, \lambda$ )-admissible chain (3). By Lemma $2 \mathfrak{a}$ and $\mathfrak{p}$ are ideals of $\widetilde{\mathfrak{p}}_{n}$. Let us choose an ideal $\mathfrak{b}$ of $\widetilde{\mathfrak{p}}_{n}$ such that $\mathfrak{a} \subset \mathfrak{b} \subset \mathfrak{p}, \mathfrak{a} \neq \mathfrak{b}$, and $\mathfrak{b}$ is minimal with respect to these properties. Then $[\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{a}$ since $\tilde{\mathfrak{p}}_{n}$ is solvable and $[\mathfrak{b}, \mathfrak{a}] \subset \mathfrak{a}^{\prime}$ by (6). If $\mathfrak{a} \neq \mathfrak{a}^{\prime}$, then $\operatorname{dim} \mathfrak{a} / \mathfrak{a}^{\prime}=1$. Lemma 4 applied to the Lie algebra $\widetilde{\mathfrak{p}}_{n} / \mathfrak{a}^{\prime}$ and its one-dimensional ideal $\mathfrak{a} / \mathfrak{a}^{\prime}$ shows that $\mathfrak{b} / \mathfrak{a}^{\prime}$ is abelian in this case. Thus we have $[\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{a}^{\prime}$ in any case. If $\mu \in \mathfrak{b}^{*}$ extends $\lambda$, then put

$$
\widetilde{\mathfrak{p}}_{n+1}=\left\{x \in \widetilde{\mathfrak{p}}_{n} \mid \mu\left(\left[x, \mathfrak{b}^{\prime}\right]\right)=0\right\} \quad \text { and } \quad \mathfrak{q}=\left\{x \in \widetilde{\mathfrak{p}}_{n+1} \mid \mu([x, \mathfrak{b}])=0\right\},
$$

where $\mathfrak{b}^{\prime}=\{y \in \mathfrak{b} \mid \mu(y)=0\}$. Note that $\mathfrak{b} \subset \mathfrak{q}$ since $\mu$ is zero on $[\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{a}^{\prime}$. Obviously $\mathfrak{q} \subset \widetilde{\mathfrak{p}}_{n+1} \subset \widetilde{\mathfrak{p}}_{n}$. Setting $\mathfrak{a}_{n+1}=\mathfrak{b}$, we obtain an extension of (3) to a $(\mathfrak{q}, \mathfrak{b}, \mu)$-admissible chain. Thus $(\mathfrak{q}, \mathfrak{b}, \mu) \in \mathcal{P}^{\prime}$.

We say that $(\mathfrak{p}, \mathfrak{a}, \lambda) \in \mathcal{P}$ is maximal if $\mathfrak{p}=\mathfrak{a}$. Denote by $\mathcal{P}_{\max } \subset \mathcal{P}$ the subset of all maximal triples and put $\mathcal{P}_{\text {max }}^{\prime}=\mathcal{P}_{\max } \cap \mathcal{P}^{\prime}$. All conclusions of the next proposition with $\mathcal{P}$ in place of $\mathcal{P}^{\prime}$ were obtained by Strade [3] in a somewhat different language.

Proposition 1. (i) Given $\xi \in \mathfrak{g}^{*}$, there exists $(\mathfrak{p}, \mathfrak{a}, \lambda) \in \mathcal{P}_{\text {max }}^{\prime}$ such that $\lambda=\left.\xi\right|_{\mathfrak{p}}$. In this case $\mathfrak{p}$ is a polarization of $\mathfrak{g}$ at $\xi$.
(ii) Given an irreducible $\mathfrak{g}$-module $V$, there exists $(\mathfrak{p}, \mathfrak{a}, \lambda) \in \mathcal{P}_{\max }^{\prime}$ such that the subspace $V_{\lambda}=\{v \in V \mid x v=\lambda(x) v$ for all $x \in \mathfrak{a}\}$ is nonzero.

Proof. Denote by $\mathcal{P}_{\xi}^{\prime} \subset \mathcal{P}^{\prime}$ the subset of those triples $(\mathfrak{p}, \mathfrak{a}, \lambda)$ for which $\lambda=\left.\xi\right|_{\mathfrak{p}}$. This subset is nonempty as we may take $\mathfrak{a}=0, \mathfrak{p}=\mathfrak{g}$. Suppose that $(\mathfrak{p}, \mathfrak{a}, \lambda) \in \mathcal{P}_{\xi}^{\prime}$ and $\mathfrak{p} \neq \mathfrak{a}$. Find $\mathfrak{b}$ as in Lemma 5 and set $\mu=\left.\xi\right|_{\mathfrak{b}}$. There exists $(\mathfrak{q}, \mathfrak{b}, \mu) \in \mathcal{P}^{\prime}$ which belongs to $\mathcal{P}_{\xi}^{\prime}$ by the choice of $\mu$. We have here $\operatorname{dim} \mathfrak{b}>\operatorname{dim} \mathfrak{a}$. This argument shows that $\mathcal{P}_{\xi}^{\prime} \cap \mathcal{P}_{\text {max }}^{\prime}$ is nonvoid. Indeed, it suffices to pick out $(\mathfrak{p}, \mathfrak{a}, \lambda) \in \mathcal{P}_{\xi}^{\prime}$ for which $\operatorname{dim} \mathfrak{a}$ is maximal possible. By Lemma $3(\mathfrak{p}, \mathfrak{a}, \lambda) \in \mathcal{P}$. There exists then a chain (1) satisfying (2). It follows by induction on $i$ that $\mathfrak{p}_{i}=\left\{x \in \mathfrak{g} \mid \xi\left(\left[x, \mathfrak{a}_{i}\right]\right)=0\right\}$. Hence $\mathfrak{p}=\mathfrak{a}$ is a maximal totally isotropic subspace of $\mathfrak{g}$ with respect to $\beta_{\xi}$.

Denote by $\mathcal{P}_{V}^{\prime} \subset \mathcal{P}^{\prime}$ the subset of those triples $(\mathfrak{p}, \mathfrak{a}, \lambda)$ for which $V_{\lambda} \neq 0$. The triple $(\mathfrak{g}, 0,0)$ is again in $\mathcal{P}_{V}^{\prime}$. Suppose that $(\mathfrak{p}, \mathfrak{a}, \lambda) \in \mathcal{P}_{V}^{\prime}$ and $\mathfrak{p} \neq \mathfrak{a}$. Let $\mathfrak{b}$ be as in Lemma 5 . Since $[\mathfrak{b}, \mathfrak{a}] \subset \mathfrak{a}^{\prime}$, the subspace $V_{\lambda}$ is stable under $\mathfrak{b}$. Hence the abelian Lie algebra $\mathfrak{b} / \mathfrak{a}^{\prime}$ operates in $V_{\lambda}$. It follows that $V_{\lambda}$ contains a one-dimensional $\mathfrak{b}$-submodule, say $k v$. The equality $x v=\mu(x) v$ defines a linear function $\mu \in \mathfrak{b}^{*}$ which extends $\lambda$. We have $v \in V_{\mu}$ by the construction. Lemma 5 provides a triple $(\mathfrak{q}, \mathfrak{b}, \mu) \in \mathcal{P}^{\prime}$ which belongs to $\mathcal{P}_{V}^{\prime}$. The intersection $\mathcal{P}_{V}^{\prime} \cap \mathcal{P}_{\text {max }}^{\prime}$ is therefore nonvoid, similarly as in case (i).

Proposition 2. Suppose that $\xi \in \mathfrak{g}^{*}$ and $(\mathfrak{p}, \mathfrak{a}, \lambda) \in \mathcal{P}_{\max }^{\prime}$ with $\lambda=\left.\xi\right|_{\mathfrak{p}}$. If $\xi$ vanishes on $\mathfrak{z}(\xi)$, then $\xi\left(\mathfrak{p}^{[p]}\right)=0$. In this case the one-dimensional $\mathfrak{p}$-module $k_{\lambda}$ on which $\mathfrak{p}$ operates via $\lambda$ has $p$-character $\lambda$, and so $U_{\xi}(\mathfrak{g}) \otimes_{U_{\lambda}(\mathfrak{p})} k_{\lambda}$ is an irreducible $\mathfrak{g}$-module of dimension $p^{\frac{1}{2}(\operatorname{dim} \mathfrak{g}-\operatorname{dim} \mathfrak{z}(\xi))}$.

Proof. For each subspace $\mathfrak{h} \subset \mathfrak{g}$ denote by $\mathfrak{h}^{\perp} \subset \mathfrak{g}$ its orthogonal complement with respect to $\beta_{\xi}$. One has then $\left(\mathfrak{h}^{\perp}\right)^{\perp}=\mathfrak{h}+\mathfrak{z}(\xi)$. Consider a $(\mathfrak{p}, \mathfrak{a}, \lambda)$-admissible chain (3). Put $\widetilde{\mathfrak{p}}=\widetilde{\mathfrak{p}}_{n}$ and $\mathfrak{p}^{\prime}=\mathfrak{a}_{n}^{\prime}$. Note that $\mathfrak{a}_{i-1}^{\prime} \subset \mathfrak{a}_{i}^{\prime}$ for all $i=1, \ldots, n$. It follows then from (5) by induction on $i$ that $\widetilde{\mathfrak{p}}_{i}=\mathfrak{a}_{i}^{\prime \perp}$ for each $i$. For $i=n$ we obtain $\mathfrak{p}^{\prime \perp}=\widetilde{\mathfrak{p}}$. Hence $\widetilde{\mathfrak{p}}^{\perp}=\mathfrak{p}^{\prime}+\mathfrak{z}(\xi)$. Note that $\mathfrak{z}(\xi) \subset \mathfrak{p}$ since $\mathfrak{p}$ is a maximal totally isotropic subspace of $\mathfrak{g}$ with respect to $\beta_{\xi}$. Under the hypotheses of Proposition $2 \mathfrak{z}(\xi) \subset \mathfrak{p} \cap \operatorname{ker} \xi=\mathfrak{p}^{\prime}$. Thus $\widetilde{\mathfrak{p}}^{\perp}=\mathfrak{p}^{\prime}$.

Observe that $\left[\mathfrak{p}, \mathfrak{p}^{[p]}\right] \subset[\mathfrak{p}, \mathfrak{p}]$ since $\mathfrak{p}$ is an ideal of $\widetilde{\mathfrak{p}}$ by Lemma 2. Hence $\xi$ vanishes on $\left[\widetilde{p}, \mathfrak{p}^{[p]}\right]$, and so $\mathfrak{p}^{[p]} \subset \widetilde{\mathfrak{p}}^{\perp}=\mathfrak{p}^{\prime}$. This shows that $\xi\left(\mathfrak{p}^{[p]}\right)=0$. The claim about irreducibility follows from Lemma 1, and the dimension formula follows from the equality $\operatorname{dim} \mathfrak{g}-\operatorname{dim} \mathfrak{p}=\frac{1}{2}(\operatorname{dim} \mathfrak{g}-\operatorname{dim} \mathfrak{z}(\xi))$.

Proposition 3. Suppose that $\xi \in \mathfrak{g}^{*}$ and $(\mathfrak{p}, \mathfrak{a}, \lambda) \in \mathcal{P}_{\max }$ with $\lambda=\left.\xi\right|_{\mathfrak{p}}$. Then every maximal torus of $\mathfrak{z}(\xi)$ is a maximal torus of $\mathfrak{p}$.

Proof. Consider a chain (1) satisfying (2). We have $\mathfrak{a}_{i}^{\perp}=\mathfrak{p}_{i}$, and therefore $\mathfrak{p}_{i}^{\perp}=$ $=\mathfrak{z}(\xi)+\mathfrak{a}_{i}$. As $\mathfrak{a}_{i}$ is an ideal of $\mathfrak{p}_{i-1}$, we get $\left[\mathfrak{p}_{i-1}, \mathfrak{a}_{i}^{[p]}\right] \subset\left[\mathfrak{a}_{i}, \mathfrak{a}_{i}\right]$ for $i>0$, which is contained in the kernel of $\xi$. This shows that $\mathfrak{a}_{i}^{[p]} \subset \mathfrak{p}_{i-1}^{\perp}=\mathfrak{z}(\xi)+\mathfrak{a}_{i-1}$.

Denote by $\mathfrak{b}_{i}$ the $[p]$-closure of $\mathfrak{a}_{i}$. Then $\mathfrak{b}_{i}$ is an ideal of $\mathfrak{p}$ since so is $\mathfrak{a}_{i}$. Hence $\mathfrak{z}(\xi)+\mathfrak{b}_{i}$ is a $p$-subalgebra for each $i$, and it follows that $\mathfrak{b}_{i}^{[p]} \subset \mathfrak{z}(\xi)+\mathfrak{b}_{i-1}$.

Suppose that $\mathfrak{t}$ is a maximal torus of $\mathfrak{z}(\xi)$ and $s \in \mathfrak{p}$ is a [p]-semisimple element which centralizes $\mathfrak{t}$. We will prove that $s \in \mathfrak{t}+\mathfrak{b}_{i}$ by the downward induction on $i=0, \ldots, n$. For $i=n$ the assertion is clear since $\mathfrak{t}+\mathfrak{b}_{n}=\mathfrak{p}$. Suppose that $s \in \mathfrak{t}+\mathfrak{b}_{i}$ for some $i>0$. Then $s=t+x$, where $t \in \mathfrak{t}, x \in \mathfrak{b}_{i}$ and $[t, x]=0$. By the above $s^{[p]}=t^{[p]}+x^{[p]} \in \mathfrak{z}(\xi)+\mathfrak{b}_{i-1}$. Since $s$ is a linear combination of elements $s^{\left[p^{r}\right]}$ with $r>0$, we get $s \in \mathfrak{z}(\xi)+\mathfrak{b}_{i-1}$. The $p$-Lie algebra $\mathfrak{h}_{i}=\left(\mathfrak{z}(\xi)+\mathfrak{b}_{i-1}\right) / \mathfrak{b}_{i-1}$ is a homomorphic image of $\mathfrak{z}(\xi)$, and therefore the image of $\mathfrak{t}$ in $\mathfrak{h}_{i}$ is a maximal torus of $\mathfrak{h}_{i}$ by [7, Theorem 2.16]. It follows that $s \in \mathfrak{t}+\mathfrak{b}_{i-1}$, providing the induction step. We can now conclude that $s \in \mathfrak{t}+\mathfrak{b}_{0}=\mathfrak{t}$, and the proof is complete.

Corollary 1. If $\mathfrak{z}(\xi)$ is $[p]$-nilpotent, then so too is $\mathfrak{p}$.
We come to the main result of this section:
Theorem 1. Let $\mathfrak{g}$ be a solvable finite dimensional $p$-Lie algebra over an algebraically closed field of characteristic $p>2$, and let $\xi \in \mathfrak{g}^{*}$.
(i) The algebra $U_{\xi}(\mathfrak{g})$ is simple if and only if $\beta_{\xi}$ is nondegenerate.
(ii) If $\beta_{\xi}$ is nondegenerate, then $\xi$ admits a $[p]$-nilpotent polarization $\mathfrak{p}$ such that $\xi\left(\mathfrak{p}^{[p]}\right)=0$, and the single irreducible $U_{\xi}(\mathfrak{g})$-module is induced from the one-dimensional $U_{\xi}(\mathfrak{p})$-module on which $\mathfrak{p}$ operates via $\xi$.

Proof. Suppose that $\beta_{\xi}$ is nondegenerate so that $\mathfrak{z}(\xi)=0$. By Proposition 1 there exists $(\mathfrak{p}, \mathfrak{p}, \lambda) \in \mathcal{P}_{\max }^{\prime}$ such that $\lambda=\left.\xi\right|_{\mathfrak{p}}$. Then $\mathfrak{p}$ is $[p]$-nilpotent by Corollary 1 . By Proposition $2 U_{\xi}(\mathfrak{g}) \otimes_{U_{\lambda}(\mathfrak{p})} k_{\lambda}$ is an irreducible $\mathfrak{g}$-module of dimension $p^{\frac{1}{2} \operatorname{dim} \mathfrak{g}}$. Since $U_{\xi}(\mathfrak{g})$ is of dimension $p^{\operatorname{dim} \mathfrak{g}}$, it has to be simple. This proves (ii) and also one implication in (i).

Suppose now that $U_{\xi}(\mathfrak{g})$ is simple, and let $V$ be its irreducible module. In view of Proposition 1, there exists $(\mathfrak{p}, \mathfrak{p}, \lambda) \in \mathcal{P}_{\max }^{\prime}$ such that $V_{\lambda} \neq 0$. Let $0 \neq v \in V_{\lambda}$ so that $k v \subset V_{\lambda}$ is a one-dimensional irreducible $U_{\xi}(\mathfrak{p})$-submodule. By Lemma 1 the $\mathfrak{g}$-module $U_{\xi}(\mathfrak{g}) \otimes_{U_{\xi}(\mathfrak{p})} k v$ is irreducible, hence of dimension $p^{\frac{1}{2} \operatorname{dim} \mathfrak{g}}$. Therefore $\operatorname{dim} \mathfrak{p}=$ $=\frac{1}{2} \operatorname{dim} \mathfrak{g}$. Let $\eta \in \mathfrak{g}^{*}$ be any linear function such that $\left.\eta\right|_{\mathfrak{p}}=\lambda$. By Proposition $1 \mathfrak{p}$ is a maximal totally isotropic subspace of $\mathfrak{g}$ with respect to $\beta_{\eta}$. The well-known formula $\operatorname{dim} \mathfrak{g}+\operatorname{dim} \mathfrak{z}(\eta)=2 \operatorname{dim} \mathfrak{p}$ now yields $\mathfrak{z}(\eta)=0$. By Proposition 2 applied to the linear function $\eta$ in place of $\xi$ the $p$-character of the $\mathfrak{p}$-module $k v$ equals $\lambda$. Hence $\lambda=\left.\xi\right|_{\mathfrak{p}}$. We may thus use $\eta=\xi$ in the argument above to conclude that $\mathfrak{z}(\xi)=0$. The proof is complete.

## 2. Frobenius Lie algebras with exponentiable adjoint derivations

Let $\mathfrak{g}$ be an arbitrary finite dimensional $p$-Lie algebra over the ground algebraically closed field $k$. We want to compare two sets

$$
\mathcal{X}=\left\{\xi \in \mathfrak{g}^{*} \mid U_{\xi}(\mathfrak{g}) \text { is simple }\right\}, \quad \mathcal{Y}=\left\{\xi \in \mathfrak{g}^{*} \mid \beta_{\xi} \text { is nondegenerate }\right\} .
$$

Lemma 6. There exists a homogeneous polynomial function $f$ on the vector space $V=\mathfrak{g}^{*} \oplus k$ such that

$$
\mathcal{X}=\left\{\xi \in \mathfrak{g}^{*} \mid f(\xi, 1) \neq 0\right\}, \quad \mathcal{Y}=\left\{\xi \in \mathfrak{g}^{*} \mid f(\xi, 0) \neq 0\right\}
$$

Proof. Let $n=\operatorname{dimg}$. We will exploit the algebraic family of $p^{n}$-dimensional associative algebras $U_{\xi, \lambda}=U_{\xi, \lambda}(\mathfrak{g})$ parameterized by points $(\xi, \lambda) \in V$ (see [4]). The algebra $U_{\xi, \lambda}$ contains $\mathfrak{g}$ as a generating subspace and has defining relations

$$
x y-y x=\lambda[x, y], \quad x^{p}=\lambda^{p-1} x^{[p]}+\xi(x)^{p} \cdot 1 \quad(x, y \in \mathfrak{g}) .
$$

In particular, two special cases of these algebras are $U_{\xi, 1} \cong U_{\xi}(\mathfrak{g})$ and $U_{\xi, 0} \cong S_{\xi}(\mathfrak{g})$, the factor algebra of the symmetric algebra $S(\mathfrak{g})$ by its ideal generated by all elements $x^{p}-\xi(x)^{p} \cdot 1$ with $x \in \mathfrak{g}$.

There is a $p$-representation $\operatorname{ad}_{\xi, \lambda}: \mathfrak{g} \rightarrow \operatorname{Der} U_{\xi, \lambda}$ such that $\operatorname{ad}_{\xi, \lambda}(x)(y)=[x, y]$ for $x, y \in \mathfrak{g}$. In this way $U_{\xi, \lambda}$ may be regarded as a module algebra over the restricted universal enveloping algebra $U_{0}(\mathfrak{g})$ and as a module over the smash product algebra $R_{\xi, \lambda}=U_{\xi, \lambda} \# U_{0}(\mathfrak{g})$. Let

$$
\varphi_{\xi, \lambda}: R_{\xi, \lambda} \rightarrow T_{\xi, \lambda}=\operatorname{End}_{k} U_{\xi, \lambda}
$$

denote the corresponding representation. Note that $\operatorname{dim} R_{\xi, \lambda}=\operatorname{dim} T_{\xi, \lambda}=p^{2 n}$. Hence the map $\varphi_{\xi, \lambda}$ is bijective if and only if $U_{\xi, \lambda}$ is a simple $R_{\xi, \lambda}$-module. Now the $R_{\xi, \lambda}-$ submodules of $U_{\xi, \lambda}$ are precisely those left ideals that are stable under the action $\mathrm{ad}_{\xi, \lambda}$. When $\lambda \neq 0$ such left ideals are precisely the two-sided ideals, and the simplicity of $U_{\xi, \lambda}$ as a $R_{\xi, \lambda}$-module is equivalent to the simplicity as an algebra. In particular,

$$
\mathcal{X}=\left\{\xi \in \mathfrak{g}^{*} \mid \varphi_{\xi, 1} \text { is bijective }\right\}
$$

On the other hand, according to [4, Proposition 3.4] the algebra $S_{\xi}(\mathfrak{g})$ has a unique maximal $\mathfrak{g}$-invariant ideal $I$, and the codimension of this ideal is $p^{\operatorname{codim}_{\mathfrak{g}}(\xi)}$. In order that $S_{\xi}(\mathfrak{g})$ be a simple $R_{\xi, 0}$-module, it is necessary and sufficient that $I=0$, which amounts to $\mathfrak{z}(\xi)=0$, that is, to $\xi \in \mathcal{Y}$. It follows that

$$
\mathcal{Y}=\left\{\xi \in \mathfrak{g}^{*} \mid \varphi_{\xi, 0} \text { is bijective }\right\}
$$

It remains to show that the bijectivity of $\varphi_{\xi, \lambda}$ can be expressed by means of the condition $f(\xi, \lambda) \neq 0$ for a suitable homogeneous polynomial function $f$ on $V$. We may view $R_{\xi, \lambda}$ and $T_{\xi, \lambda}$ as fibers of two algebraic vector bundles $R$ and $T$ over $V$. Let $e_{1}, \ldots, e_{n}$ be any basis for $\mathfrak{g}$. The monomials $e_{1}^{a_{1}} \cdots e_{n}^{a_{n}}$ with $0 \leq a_{i}<p$ form a basis for each $U_{\xi, \lambda}$. These monomials give rise to a basis for each $R_{\xi, \lambda}$ and a basis for each $T_{\xi, \lambda}$, yielding trivializations of $R$ and $T$. The entries of the matrix of $\varphi_{\xi, \lambda}$ in the above bases are polynomial functions in $(\xi, \lambda)$. Taking $f(\xi, \lambda)$ to be the determinant of this matrix, we see that $\varphi_{\xi, \lambda}$ is bijective if and only if $f(\xi, \lambda) \neq 0$.

As explained in [4], for each $0 \neq t \in k$ there is a $\mathfrak{g}$-equivariant algebra isomorphism $\theta_{t}: U_{\xi, \lambda} \rightarrow U_{t \xi, t \lambda}(\mathfrak{g})$. Hence the algebra $U_{\xi, \lambda}$ has no nontrivial $\mathfrak{g}$-invariant ideals if and only if so does $U_{t \xi, t \lambda}(\mathfrak{g})$. In other words, bijectivity of $\varphi_{\xi, \lambda}$ is equivalent to bijectivity of $\varphi_{t \xi, t \lambda}$. It follows that the zero locus of the polynomial function $f$ is a conical subset of $V$, whence $f$ is homogeneous.

Remark. It is possible to compute the degree of the polynomial function $f$ in Lemma 6 proceeding as follows. The isomorphisms $\theta_{t}$ induce actions of the onedimensional torus $\mathbb{G}_{m}$ on $R$ and $\frac{T}{R}$. Taking quotients modulo these actions we pass to a morphism of vector bundles $\bar{R} \rightarrow \bar{T}$ over the projective space $\mathbb{P}(V)$ associated with $V$. Let also $\bar{U}=U / \mathbb{G}_{m}$, where $U$ is the vector bundle over $V \backslash\{0\}$ with fibers $U_{\xi, \lambda}$. Each line bundle over $\mathbb{P}(V)$ is isomorphic to some $L(s)$, defined as the quotient of $(V \backslash\{0\}) \times k$ by the action of $\mathbb{G}_{m}$ such that $t \cdot(v, c)=\left(t v, t^{s} c\right)$, where $s \in \mathbb{Z}$. The scalar multiples of any monomial $e_{1}^{a_{1}} \cdots e_{n}^{a_{n}}$ produce a $\mathbb{G}_{m}$-stable line subbundle of $U$. This leads to a decomposition

$$
\bar{U} \cong \bigoplus_{\left\{\left(a_{1}, \ldots, a_{n}\right) \mid 0 \leq a_{i}<p\right\}} L\left(-a_{1}-\cdots-a_{n}\right)
$$

The bundle $\bar{R}$ is isomorphic to a direct sum of $p^{n}$ copies of $\bar{U}$, while $\bar{T} \cong \bar{U} \otimes \bar{U}^{*}$. As a result, $\bigwedge^{p^{2 n}} \bar{R} \cong L(-d)$, where

$$
d=p^{n} \cdot \sum_{\left\{\left(a_{1}, \ldots, a_{n}\right) \mid 0 \leq a_{i}<p\right\}}\left(a_{1}+\cdots+a_{n}\right)=\frac{n p^{2 n}(p-1)}{2}
$$

while $\bigwedge^{p^{2 n}} \bar{T} \cong L(0)$ is trivial. Now $f$ can be identified with a section of the line bundle $\operatorname{Hom}(L(-d), L(0)) \cong L(d)$. This means that $\operatorname{deg} f=d$.

Corollary 2. If $\mathfrak{g}$ is Frobenius, that is, $\mathcal{Y} \neq \varnothing$, then $f \neq 0$, and therefore $\mathcal{X} \neq \varnothing$.
Whether $\mathcal{X} \neq \varnothing$ implies $\mathcal{Y} \neq \varnothing$ is a special case of the still open Kac-Weisfeiler conjecture from [8].

Proposition 4. If $\mathfrak{g}$ is Frobenius and $\mathcal{Y} \subset \mathcal{X}$, then $\mathcal{X}=\mathcal{Y}$.
Proof. By Lemma 6 the complements $\mathcal{X}^{c}=\mathfrak{g}^{*} \backslash \mathcal{X}$ and $\mathcal{Y}^{c}=\mathfrak{g}^{*} \backslash \mathcal{Y}$ are hypersurfaces in $\mathfrak{g}^{*}$. The inclusion $\mathcal{Y} \subset \mathcal{X}$ entails $\mathcal{X}^{c} \subset \mathcal{Y}^{c}$. Therefore each irreducible component of $\mathcal{X}^{c}$ is an irreducible component of $\mathcal{Y}^{c}$. Since $\mathcal{Y}^{c}$ is a conical subset of $\mathfrak{g}^{*}$, so too is each irreducible component of $\mathcal{Y}^{c}$. It follows that $\mathcal{X}^{c}$ is a conical subset as well. Hence the polynomial function $\xi \mapsto f(\xi, 1)$ defining $\mathcal{X}^{c}$ is homogeneous. We can write

$$
f(\xi, \lambda)=\sum_{i=0}^{d} f_{i}(\xi) \lambda^{i}
$$

where each $f_{i}$ is a homogeneous polynomial function of degree $d-i$ on $\mathfrak{g}^{*}$. Since $\mathfrak{g}$ is Frobenius, we have $\mathcal{Y} \neq \varnothing$, whence $f_{0} \neq 0$. But then we must have $f_{i}=0$ for all $i>0$, that is, $f(\xi, \lambda)$ does not depend on $\lambda$.

Theorem 2. Let $\mathfrak{g}$ be a Frobenius $p$-Lie algebra with the automorphism group $G$. Suppose that ad $\mathfrak{g} \subset \operatorname{Lie} G$. Then $\mathcal{X}=\mathcal{Y}$.

Proof. Both $\mathcal{X}$ and $\mathcal{Y}$ are stable under the coadjoint action of $G$. For any $\xi \in \mathcal{Y}$ the nondegeneracy of $\beta_{\xi}$ yields $\mathfrak{g} \cdot \xi=\mathfrak{g}^{*}$. Hence the tangent space at $\xi$ to the $G$-orbit $G \xi$ coincides with $\mathfrak{g}^{*}$, and therefore $G \xi$ is open in $\mathfrak{g}^{*}$. Since any two nonempty open subsets of $\mathfrak{g}^{*}$ have nonempty intersection, we conclude that $\mathcal{Y}$ is a single $G$-orbit. As $\mathcal{X}$ is also nonempty and open in $\mathfrak{g}^{*}$, we get $\mathcal{X} \bigcap \mathcal{Y} \neq \varnothing$, whence $\mathcal{Y} \subset \mathcal{X}$. Now Proposition 4 applies.

## 3. The semisimple locus: an example

Let us now look at a different pair of subsets of $\mathfrak{g}^{*}$ :

$$
\mathcal{X}=\left\{\xi \in \mathfrak{g}^{*} \mid U_{\xi}(\mathfrak{g}) \text { is semisimple }\right\}, \quad \mathcal{Y}=\left\{\xi \in \mathfrak{g}^{*} \mid \mathfrak{z}(\xi) \text { is toral }\right\} .
$$

It was proved in $\left[4\right.$, Section 4] that both of them are open in $\mathfrak{g}^{*}$ and that $\mathcal{Y} \neq \varnothing$ implies $\mathcal{X} \neq \varnothing$. Moreover, the stabilizers $\mathfrak{z}(\xi)$ of all linear functions $\xi \in \mathcal{Y}$ have equal dimensions. If $s$ denotes their common dimension, then for each $\xi \in \mathcal{X}$ the semisimple algebra $U_{\xi}(\mathfrak{g})$ has precisely $p^{s}$ nonisomorphic simple modules, all of equal dimension.

One may ask what are those $p$-Lie algebras for which $\mathcal{X}=\mathcal{Y}$. For instance, if $\mathfrak{g}$ is the Lie algebra of a simply connected semisimple algebraic group $G$ and $p$ is good for the root system of $G$, then $\mathcal{X}$ consists precisely of the regular semisimple linear functions [9, Corollary 3.6] so that the equality $\mathcal{X}=\mathcal{Y}$ does hold. In this section, we provide examples of nilpotent $p$-Lie algebras for which $\mathcal{X} \neq \mathcal{Y}$.

Consider a $p$-Lie algebra $\mathfrak{g}$ whose center $\mathfrak{t}$ is a toral subalgebra of codimension 2 in $\mathfrak{g}$ and $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{t}$. Let $u, v \in \mathfrak{g}$ span a subspace complementary to $\mathfrak{t}$ in $\mathfrak{g}$. There is an element $0 \neq t \in \mathfrak{t}$ such that $[u, v]=t$. Then $[\mathfrak{g}, \mathfrak{g}]=k t$ is a one-dimensional subspace.

Since $\mathfrak{g}$ is nilpotent, it has a largest toral subalgebra. Clearly this subalgebra coincides with $\mathfrak{t}$. Now $\mathfrak{t} \subset \mathfrak{z}(\xi)$ for all $\xi \in \mathfrak{g}^{*}$. Hence $\mathfrak{z}(\xi)$ is toral if and only if $\mathfrak{z}(\xi)=\mathfrak{t}$. If $\mathfrak{z}(\xi) \neq \mathfrak{t}$, then $\mathfrak{z}(\xi)=\mathfrak{g}$, which occurs precisely when $\xi$ vanishes on $[\mathfrak{g}, \mathfrak{g}]$. It follows that

$$
\mathcal{Y}=\left\{\xi \in \mathfrak{g}^{*} \mid \xi(t) \neq 0\right\} .
$$

Denote by $\mathfrak{t}^{*(1)}$ the vector space of all $p$-semilinear maps $\mathfrak{t} \rightarrow k$, that is, $\mathfrak{t}^{*(1)}$ is the Frobenius twist of the dual space $\mathfrak{t}^{*}$. The map $\wp: \mathfrak{t}^{*} \rightarrow \mathfrak{t}^{*(1)}$ defined by the rule

$$
\wp(\lambda)(x)=\lambda(x)^{p}-\lambda\left(x^{[p]}\right) \quad \text { for } \lambda \in \mathfrak{t}^{*} \text { and } x \in \mathfrak{t}
$$

is a finite surjective morphism of algebraic varieties. There is also a bijective morphism $\mathfrak{t}^{*} \rightarrow \mathfrak{t}^{*(1)}$ given by $\lambda \mapsto \lambda^{p}$, where $\lambda^{p}(x)=\lambda(x)^{p}$.

With any simple $\mathfrak{g}$-module $V$ one can associate a linear function $\lambda \in \mathfrak{t}^{*}$ such that each element $x \in \mathfrak{t}$ acts in $V$ as a scalar multiplication by $\lambda(x)$. If $\xi$ is the $p$ character of $V$, then $\wp(\lambda)=\left.\xi^{p}\right|_{\mathfrak{t}}$. Conversely, for any pair $\lambda \in \mathfrak{t}^{*}$ and $\xi \in \mathfrak{g}^{*}$ satisfying the previous equality there is precisely one simple $U_{\xi}(\mathfrak{g})$-module $V$ which has $\lambda$ as the associated function. If $\lambda(t)=0$, then $[\mathfrak{g}, \mathfrak{g}]$ annihilates $V$, whence $\operatorname{dim} V=1$. Otherwise $V$ is induced from a one-dimensional representation of any abelian subalgebra of codimension 1 in $\mathfrak{g}$ so that $\operatorname{dim} V=p$. Since all fibers of the map $\wp$ have cardinality $N=p^{\operatorname{dim} \mathfrak{t}}$, for each $\xi \in \mathfrak{g}^{*}$ there are precisely $N$ nonisomorphic simple $U_{\xi}(\mathfrak{g})$-modules. In order that $U_{\xi}(\mathfrak{g})$ be semisimple, it is necessary and sufficient that its dimension $p^{\operatorname{dim} \mathfrak{g}}$ be equal to $\sum(\operatorname{dim} V)^{2}$, the sum over all those modules. This happens precisely when all simple $U_{\xi}(\mathfrak{g})$-modules have dimension $p$. We conclude that

$$
\mathcal{X}=\left\{\xi \in \mathfrak{g}^{*} \mid \lambda(t) \neq 0 \text { for each } \lambda \in \wp^{-1}\left(\left.\xi^{p}\right|_{\mathfrak{t}}\right)\right\} .
$$

Suppose now that $t$ is such that $t^{[p]} \notin k t$. Then neither $\mathcal{X} \subset \mathcal{Y}$ nor $\mathcal{Y} \subset \mathcal{X}$. To see this let $\lambda$ and $\xi$ be as above. If $\lambda(t)=0$, but $\lambda\left(t^{[p]}\right) \neq 0$, then the equality $\lambda(t)^{p}-\lambda\left(t^{[p]}\right)=\xi(t)^{p}$ yields $\xi(t) \neq 0$. In this case $\xi \in \mathcal{Y}$, but $\xi \notin \mathcal{X}$. Now the subspace

$$
S=\left\{\lambda \in \mathfrak{t}^{*} \mid \lambda(t)=\lambda\left(t^{[p]}\right)=0\right\}
$$

has codimension 2 in $\mathfrak{t}^{*}$. Hence $\wp(S)$ is a closed subvariety of codimension 2 in $\mathfrak{t}^{*(1)}$, and it follows that there exists $\xi \in \mathfrak{g}^{*}$ such that $\xi(t)=0$, but $\left.\xi^{p}\right|_{\mathfrak{t}} \notin \wp(S)$. In this case $\xi \notin \mathcal{Y}$, but $\xi \in \mathcal{X}$.

This work was Supported by the Russian Foundation for Basic Research (Grant No. 10-01-00431) and the Presidential Grant for Support of Leading Scientific Schools (Grant No. 5383.2012.1).

## Резюме

C.M. Скрябин. О локусе $p$-характеров, определяющих простые редуцированные обертывающие алгебры.

В двух случаях подтверждена гипотеза, утверждающая, что редуцированная обёртываюшая алгебра $U_{\xi}(\mathfrak{g})$ ограниченной алгебры Ли $\mathfrak{g}$ является простой тогда и только тогда, когда альтернирующая билинейная форма, ассоциированная с заданным $p$-характером $\xi \in \mathfrak{g}^{*}$, невырождена.

Ключевые слова: ограниченные алгебры Ли, разрешимые алгебры Ли, фробениусовы алгебры Ли, редуцированные обертывающие алгебры.

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Поступила в редакцию
03.02.12

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