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On Irrationality Measure of $\arctan\frac{1}{3}$

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Abstract—We investigate the arithmetic properties of the value $\arctan\frac{1}{3}$. We elaborate special integral construction with the property of symmetry for evaluating irrationality measure of this number. We research linear form, generated by this integral, and prove a new result for extent of the irrationality of $\arctan\frac{1}{3}$, which improves the previous one.

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INTRODUCTION

For all irrational number we can evaluate the quality of its approximation by rational numbers with the help of characteristic called irrationality measure. For any irrational number γ the *irrationality measure* (*index*) $\mu(\gamma)$ is defined as lower boundary of numbers μ such that for all $\varepsilon > 0$ there exists a number $q_0(\varepsilon) > 0$ thus for all whole numbers p, q with $q \geq q_0(\varepsilon)$ the inequality $\left|\gamma - \frac{p}{q}\right| \geq q^{-\mu-\varepsilon}$ is true. Exact value of this index is known only for some numbers, in most cases as soon as estimations from above are known. To give a more precise estimations is actual problem of number theory in present day.

For obtaining estimations we construct rational approximation of concerned values. Usually for this it is applying different single-valued and multiple real and complex integrals which give functional approximation of researched values. The better approximation is succeeded to construct, the better estimation can be obtained. At last decades in this field of researches many considerable results are obtained, for example, estimations for $\ln 2$, $\ln 3$ and π are improved. At present for investigation the integral constructions having symmetry property relatively replacement of some parameters actively are used.

First integral construction with the symmetry property is applied by V.Kh. Salikhov in [1] and estimation for $\ln 3$ is improved. Later on with help of symmetric integrals different type the new results for values of logarithmic function are obtained. In 2014 Q. Wu and L. Wang improved estimation for $\ln 3$ applying a new type of symmetric integral construction [2]. At present their idea is developed and is applied for other numbers. In present work the similar integral having symmetry property will be used also.

1. THE MAIN CONSTRUCTION

Irrationality measure of values of function $\arctan x$ is estimated repeatedly by the general methods. First results have been received at 1987 by M. Huttner [3]. He proved the general theorem about the estimations of irrationality measure of values of hypergeometric Gauss function the following kind $F_2^1\left(1, \frac{1}{k}, 1 + \frac{1}{k}|\varepsilon x^k\right)$, $k \in \mathbb{N}$, $k \geq 2$, $\varepsilon = \pm 1$. Later in papers of A. Heimonen, T. Matala-Aho, K. Vaananen [4, 5] the method using approximation of hypergeometric function by Jacobi polynomials has been constructed. This method gives the estimations for values of the kind $F_2^1\left(1, \frac{1}{k}, 1 + \frac{1}{k}|z\right)$,

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