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Lattices Generated by Modular Elements

M. P. Shushpanov[1*]

[1]*Ural Federal University, ul. Mira 19, Ekaterinburg, 620002 Russia*

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Abstract—We prove that the lattice generated by three modular elements is modular. For every $n > 3$ we construct a nonmodular lattice L generated by n elements each of which is modular in L.

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An element a of the lattice $\langle L; \wedge, \vee \rangle$ is said to be *modular* if

$$\forall x, y \in L : x \leq y \rightarrow (a \vee x) \wedge y = (a \wedge y) \vee x.$$

The study of the modular elements is inspired by the following fact: The normal subgroups (ideals) are modular elements in the subgroup (subring) lattice but the converse is not true. It is known that the sublattice generated by any collection of normal subgroups (ideals) is modular. Then it seems natural to find out if the sublattice generated by the lattice modular elements is modular. The answer to this question lies in Theorems 1 and 2.

Let $p(x, y, z)$ be a lattice polynomial on the variables x, y, and z, i.e., $p(x, y, z)$ is an element of some free lattice of rank 3. We denote by $p^{\#}(x, y, z)$ the polynomial which is the result of the operations $\vee$ and $\wedge$ interchange in $p(x, y, z)$. We denote by $t(x, y, z)$ the polynomial $(x \vee y) \wedge (y \vee z) \wedge (z \vee x)$, and by $s(x, y, z)$ the polynomial $t^{\#}(x, y, z)$. It seems clear that $t(x, y, z)$ remains the same lattice element under any permutation of x, y, and z. The same holds true for $s(x, y, z)$. We then denote the polynomial $(x \vee (y \wedge z)) \wedge (y \vee z)$ by $f(x, y, z)$.

Lemma. *Let a and b be modular elements of the lattice L, and x be an arbitrary element of this lattice. Then $t(a, b, x) = (a \wedge (b \vee x)) \vee (b \wedge (x \vee a))$ and $a \vee t(a, b, x) = (a \vee b) \wedge (x \vee a)$.*

Proof. Modularity of a and the inequality $b \leq b \vee x$ put together imply $t(a, b, x) = (a \vee b) \wedge (b \vee x) \wedge (x \vee a) = ((a \wedge (b \vee x)) \vee b) \wedge (x \vee a)$. At the same time modularity of b and the inequality $a \wedge (b \vee x) \leq a \leq x \vee a$ allow us to write down $((a \wedge (b \vee x)) \vee b) \wedge (x \vee a) = (a \wedge (b \vee x)) \vee (b \wedge (x \vee a))$. Also $a \vee t(a, b, x) = a \vee (a \wedge (b \vee x)) \vee (b \wedge (x \vee a)) = a \vee (b \wedge (x \vee a)) = (a \vee b) \wedge (x \vee a)$, here the latter inequality is due to modularity of the element b and the inequality $a \leq x \vee a$. □

In order to prove the main result we need the information on governing relations of free modular lattice of the rank 3. So we list these relations in which the letters a, b, and c denote the generating elements of such a lattice:

$$f(a, b, c) = f^{\#}(a, b, c), \tag{1}$$

$$f(b, c, a) = f^{\#}(b, c, a), \tag{2}$$

$$f(c, a, b) = f^{\#}(c, a, b), \tag{3}$$

$$t(a, b, c) = ((a \wedge (b \vee c)) \vee (b \wedge (c \vee a))) \wedge ((b \wedge (c \vee a)) \vee (c \wedge (a \vee b))) \wedge ((c \wedge (a \vee b)) \vee (a \wedge (b \vee c))), \tag{4}$$

[*]E-mail: Mikhail.Shushpanov@gmail.com.