УЧЕНЫЕ ЗАПИСКИ КАЗАНСКОГО УНИВЕРСИТЕТА
Том 153, кн. 3 Физико-математические науки *2011*

UDK 519.63

ON DEFORMED MINISUPERSPACE VARIABLES IN QUANTUM COSMOLOGY

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Abstract

We present several examples in noncommutative quantum cosmology, using the WKB-type approximation with a deformation on the minisuperspace variables. This procedure gives a straightforward algorithm to incorporate noncommutativity to cosmology and inflation.

Key words: noncommutative cosmology, quantum cosmology, quasiclassical approximation.

Introduction

There is a renewed interest in noncommutative theories to explain the appropriate modification of classical General Relativity and, hence, of spacetime symmetries at short-distance scales, that implies modifications at large scales. General quantum mechanics arguments indicate that it is impossible to measure a classical background spacetime at the Planck scale due to the effects of gravitational backreaction [1]. It is therefore tempting to incorporate the dynamical features of spacetime at deeper kinematical level using the standard techniques of noncommutative classical field theory based on the so called Moyal product in which for all calculation purposes (differentiation, integration, etc.) the space time coordinates are treated as ordinary (commutative) variables and noncommutativity enters into play in the way in which fields are multiplied [2]. Using a modified symplectic structure on the space variables in the Hamilton approach, we assume that the minisuperspace variables do not commute, and for this purpose we will modify the Poisson structure. This approach does not modify the Hamiltonian structure in the noncommutative fields. According to the approach used here, the momenta in both spaces are the same, $\mathrm{P}_{\mathrm{q}^{\mu}_{\mathrm{nc}}} = \mathrm{P}_{\mathrm{q}^{\mu}}$; that is, they commute in both spaces.

Another way to extract useful dynamical information is through the WKB semiclassical approximation to the quantum Wheeler – DeWitt equation using the wave function $\Psi = \mathrm{e}^{\mathrm{i}S(\mathrm{q}^{\mu})}$. In this approach, we consider the usual approximation to the derivatives and the corresponding relation between the Einstein – Hamilton – Jacobi (EHJ) equation. It was possible to obtain classical solutions at the master equation found by this procedures. The classical field equations were checked for all solutions using the REDUCE 3.8 algebraic packages.

The main idea in this paper is to find classical commutative and noncommutative quantum solutions.

1. Quantum cosmology and the WKB approximation

Our goal is to present a WKB-type method for noncommutative quantum cosmology. We start by reviewing the quantum cosmological models we are interested in here, and find the classical evolution through a WKB-type approximation. The following models are presented: Kantowski – Sachs cosmology, FRW cosmology with cosmological constant coupled to a scalar field, and a cosmological model within the framework of string theory.

1.1. Kantowski – Sachs cosmology. The first example we are interested in is the Kantowski – Sachs universe. This is one of the simplest anisotropic cosmological models. We are also interested in a wide set of analytical solutions it admits. The Kantowski – Sachs line element is [3]

$$ds^2 = -N^2 dt^2 + \exp\left(2\sqrt{3}\beta\right) dr^2 + \exp\left(-2\sqrt{3}(\beta+\Omega)\right) \left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right). \quad (1)$$

From the general relativity Lagrangian we can construct the canonical momenta,

$$\Pi_\Omega = -\frac{12}{N}\exp\left(-\sqrt{3}\beta - 2\sqrt{3}\Omega\right)\dot{\Omega}, \quad \Pi_\beta = \frac{12}{N}\exp\left(-\sqrt{3}\beta - 2\sqrt{3}\Omega\right)\dot{\beta}. \quad (2)$$

Using canonical quantization and a particular factor ordering, we get the WDW equation. Through the usual identifications $\Pi_\Omega = -i\frac{\partial}{\partial\Omega}$ and $\Pi_\beta = -i\frac{\partial}{\partial\beta}$ we get

$$\left[\frac{\partial^2}{\partial\Omega^2} - \frac{\partial^2}{\partial\beta^2} - 48\exp\left(-2\sqrt{3}\Omega\right)\right]\psi(\Omega,\beta) = 0. \quad (3)$$

The solution to this equation is given by

$$\psi = \exp\left(\pm i\nu\sqrt{3}\beta\right) K_{i\nu}\left(4\exp\left[-\sqrt{3}\Omega\right]\right), \quad (4)$$

where ν is the separation constant and K_{iv} are the modified Bessel functions.

We now proceed to apply the WKB-type method. For this we propose the wave function

$$\Psi(\beta,\Omega) \approx \exp\left[i(S_1(\beta) + S_2(\Omega))\right]. \quad (5)$$

The WKB approximation is reached in the limit

$$\left|\frac{\partial^2 S_1(\beta)}{\partial\beta^2}\right| \ll \left(\frac{\partial S_1(\beta)}{\partial\beta}\right)^2, \quad \left|\frac{\partial^2 S_2(\Omega)}{\partial\Omega^2}\right| \ll \left(\frac{\partial S_2(\Omega)}{\partial\Omega}\right)^2 \quad (6)$$

and gives the Einstein – Hamilton – Jacobi (EHJ) equation

$$-\left(\frac{\partial S_2(\Omega)}{\partial\Omega}\right)^2 + \left(\frac{\partial S_1(\beta)}{\partial\beta}\right)^2 - 48\exp\left(-2\sqrt{3}\Omega\right) = 0. \quad (7)$$

Solving Eq. (7) one gets the functions S_1, S_2 and can find the temporal evolution. First we fix the value of $N(t) = 24\exp\left(-\sqrt{3}\beta - 2\sqrt{3}\Omega\right)$; by using (2) and the definition for the momenta $\Pi_\beta = \frac{dS_1(\beta)}{d\beta}$ and $\Pi_\Omega = \frac{dS_2(\Omega)}{d\Omega}$ we obtain the classical solutions

$$\begin{aligned} \Omega(t) &= \frac{1}{2\sqrt{3}}\ln\left[\frac{48}{P_{\beta_0}^2}\cosh^2\left(2\sqrt{3}P_{\beta_0}(t-t_0)\right)\right], \\ \beta(t) &= \beta_0 + 2P_{\beta_0}(t-t_0), \end{aligned} \quad (8)$$

where β_0 and P_{β_0} are the initial conditions. These solutions are the same as we get by solving the field equations of general relativity.

2. Stringy quantum cosmology

Our final example is related to the graceful exit of pre-big bang cosmology [4]. This model is based on the gravi-dilaton effective action in 1+3 dimensions

$$S = -\frac{\lambda_s}{2}\int d^4x\,\sqrt{-g}\,e^{-\phi}(R + \partial_\mu\phi\,\partial^\nu\phi + V), \tag{9}$$

where λ_s is the fundamental string length, ϕ is the dilaton field with V the possible dilaton potential. Working with an isotropic background, and setting $a(t) = \exp\left(\beta(t)/\sqrt{3}\right)$, after integrating by parts, we get

$$S = -\frac{\lambda_s}{2}\int d\tau\left(\bar{\phi}'^2 - \beta'^2 + Ve^{-2\bar{\phi}}\right). \tag{10}$$

We have used the time parametrization [1] $dt = e^{-\bar{\phi}}\,d\tau$, the gauge $g_{00} = 1$, and introduced $\bar{\phi} = \phi - \ln\int\left(\frac{d^3x}{\lambda_s^3}\right) - \sqrt{3}\beta$. From this action we calculate the canonical momenta, $\Pi_\beta = \lambda_s\beta'$ and $\Pi_{\bar{\phi}} = -\lambda_s\bar{\phi}'$. From the classical Hamiltonian we find the WDW equation:

$$\left[\frac{\partial^2}{\partial\bar{\phi}^2} - \frac{\partial^2}{\partial\beta^2} + \lambda_s^2 V(\bar{\phi},\beta)\,e^{-2\bar{\phi}}\right]\Psi(\bar{\phi},\beta) = 0. \tag{11}$$

In particular for a potential of the form $V(\bar{\phi}) = -V_0\,e^{m\bar{\phi}}$, the quantum solution is

$$\Psi(\bar{\phi},\beta) = \exp\left(\pm - i\frac{m-2}{2}\nu\beta\right)K_{i\nu}\left[\frac{2\lambda_s\sqrt{V_0}}{m-2}\exp\left(\left(\frac{m-2}{2}\right)\bar{\phi}\right)\right]. \tag{12}$$

The classical solutions for the scale factor and the dilaton are

$$\begin{cases} \bar{\phi}(\tau) = \dfrac{1}{m-2}\ln\left[\dfrac{P_{\beta_0}^2}{V_0\lambda_s^2}\,\mathrm{sech}^2\left(\dfrac{P_{\beta_0}}{2\lambda_s}(m-2)(\tau-\tau_0)\right)\right], \\ \beta(\tau) = \beta_0 + \dfrac{P_\beta}{\lambda_s}(\tau-\tau_0). \end{cases} \tag{13}$$

Here $m = 0$ and $m = 4$; the solutions were obtained in [4] and are used in connection with the graceful exit from pre-big bang cosmology in quantum string cosmology.

3. Noncommutative quantum cosmology and the WKB-type approximation

In this section we construct noncommutative quantum cosmology for the examples presented in the previous section and calculate the classical evolution via a WKB-type approximation. To get the classical cosmological solutions would be a very difficult task in any model of noncommutative gravity [5–7] due to the highly nonlinear nature of the field equation. We will follow the original proposals of noncommutative quantum cosmology that was developed in [3]. This will allow us to get the desired classical solutions. The first noncommutative example that we present is the noncommutative KS, and finally stringy noncommutative quantum cosmology. We start by presenting, in quite a general form, the construction of noncommutative quantum cosmology and the WKB-type method to calculate the classical evolution.

Let us start with a generic form for the commutative WDW equation. This is defined in the minisuperspace variables x, y. As mentioned in [3] a noncommutative deformation of the minisuperspace variables is assumed

$$[x, y] = i\theta. \tag{14}$$

[1]The prime denotes differentiation in respect to τ.

This noncommutativity[2] can be formulated in terms of noncommutative minisuperspace functions with the Moyal product of functions

$$f(x,y) \star g(x,y) = f(x,y) \exp\left(i\frac{\theta}{2}\left(\overleftarrow{\partial_x}\overrightarrow{\partial_y} - \overleftarrow{\partial_y}\overrightarrow{\partial_x}\right)\right) g(x,y). \tag{15}$$

Then the noncommutative WDW equation can be written as

$$\left(-\Pi_{\mathrm{x}}^2 + \Pi_{\mathrm{y}}^2 - \mathrm{V}(\mathrm{x},\mathrm{y})\right) \star \Psi(\mathrm{x},\mathrm{y}) = 0. \tag{16}$$

We know from noncommutative quantum mechanics [9, 10] that the symplectic structure is modified changing the commutator algebra. It is possible to return to the original commutative variables and usual commutation relations if we introduce the following change of variables:

$$x \to x + \frac{\theta}{2}\Pi_y \quad \text{and} \quad y \to y - \frac{\theta}{2}\Pi_x. \tag{17}$$

Taking this into account and using the usual substitutions $\Pi_{\mathrm{q}^\mu} = -\mathrm{i}\partial_{\mathrm{q}^\mu}$ we arrive to

$$\left[\frac{\partial^2}{\partial \mathrm{x}^2} - \frac{\partial^2}{\partial \mathrm{y}^2} - \mathrm{V}\left(\mathrm{x} - \mathrm{i}\frac{\theta}{2}\frac{\partial}{\partial \mathrm{y}}, \mathrm{y} + \mathrm{i}\frac{\theta}{2}\frac{\partial}{\partial \mathrm{x}}\right)\right]\Psi(\mathrm{x},\mathrm{y}) = 0. \tag{18}$$

This is the Noncommutative WDW equation (NCWDW) and its solutions give the quantum description of the noncommutative Universe. We can use the NCWDW to find the temporal evolution of our noncommutative cosmology by a WKB-type procedure. For this we propose that the noncommutative wave function has the form $\Psi_{NC}(\beta,\Omega) \approx$ $\approx \exp\left[i(S_{NC1}(\beta) + S_{NC2}(\Omega))\right]$, which in the limit

$$\left|\frac{\partial^2 S_{NC1}(\beta)}{\partial \beta^2}\right| \ll \left(\frac{\partial S_{NC1}(\beta)}{\partial \beta}\right)^2, \quad \left|\frac{\partial^2 S_{NC2}(\Omega)}{\partial \Omega^2}\right| \ll \left(\frac{\partial S_{NC2}(\Omega)}{\partial \Omega}\right)^2, \tag{19}$$

yields the noncommutative Einstein – Hamilton – Jacobi equation (NCEHJ), that gives the solutions to S_{NC1} and S_{NC2}. After the identification $\Pi_{x_{NC}} = -\dfrac{\partial(S_{NC1})}{\partial x}$ and $\Pi_{y_{NC}} = -\dfrac{\partial(S_{NC2})}{\partial y}$ together with the definitions of the canonical momenta and Eq. (17) we can find the time dependent solutions for x and y.

In the rest of this section we will apply this ideas to the examples that have already been presented.

4. Noncommutative Kantowski – Sachs cosmology

Using the method outlined in the preceding paragraphs with respect to Eq. (3) we find the NCWDW equation

$$\left[\frac{\partial^2}{\partial \Omega^2} - \frac{\partial^2}{\partial \beta^2} - 48\exp\left\{-2\sqrt{3}\left(\Omega - i\frac{\theta}{2}\frac{\partial}{\partial \beta}\right)\right\}\right]\Psi(\Omega,\beta) = 0. \tag{20}$$

Then the solution of the NCWDW equation is

$$\Psi(\Omega,\beta) = \exp\left(\pm i\sqrt{3}\nu\beta\right) K_{i\nu}\left(4\exp\left[-\sqrt{3}\Omega \pm \frac{3}{2}\nu\theta\right]\right). \tag{21}$$

[2]This commutation relation implies an uncertainty principle giving an absolute minimal distance in minisuperspace.

Usually the next step is to construct a “Gaussian” wave packet and do the physics with the new wave function. This is not needed for our purposes, as we are interested in the classical solutions by applying the WKB-type method outlined in the previous section. Using Eqs. (5), and (6) we find the solutions for $S_1(\beta)$ and $S_2(\Omega)$ which have the form

$$\begin{aligned} S_1(\beta) &= P_{\beta_0}\beta, \\ S_2(\Omega) &= -\frac{1}{\sqrt{3}}\sqrt{P_{\beta_0}^2 - 48\exp\left(-\sqrt{3}\theta P_{\beta_0}\right)\,\exp\left(-2\sqrt{3}\Omega\right)} + \\ &\quad + \frac{P_{\beta_0}}{\sqrt{3}}\,\mathrm{arctanh}\left[\frac{\sqrt{P_{\beta_0}^2 - 48\exp\left(-\sqrt{3}\theta P_{\beta_0}\right)\exp\left(-2\sqrt{3}\Omega\right)}}{P_{\beta_0}}\right]. \end{aligned} \tag{22}$$

Then the deformation of the momenta provide us with the noncommutative classical solutions

$$\begin{aligned} \Omega(t) &= \frac{1}{2\sqrt{3}}\ln\left[\frac{48}{P_{\beta_0}^2}\cosh^2 2\sqrt{3}P_{\beta_0}(t-t_0)\right] - \frac{\theta}{2}P_{\beta_0}, \\ \beta(t) &= \beta_0 + 2P_{\beta_0}(t-t_0) - \frac{\theta}{2}P_{\beta_0}\tanh^2\left[2\sqrt{3}P_{\beta_0}(t-t_0)\right]. \end{aligned} \tag{23}$$

These solutions have already been obtained in [8], where the authors deform the symplectic structure at a classical level changing the Poisson brackets.

5. Stringy noncommutative quantum cosmology

As in the previous examples, we introduce the noncommutative relation $[\bar{\phi},\beta] = i\theta$, and from the classical Hamiltonian we find the NCWDW equation

$$\left[\frac{\partial^2}{\partial\bar{\phi}^2} - \frac{\partial^2}{\partial\beta^2} - \lambda_s^2 V(\bar{\phi},\beta)\exp\left\{(m-2)(\bar{\phi} - i\frac{\theta}{2}\frac{\partial}{\partial\beta})\right\}\right]\Psi(\bar{\phi},\beta) = 0. \tag{24}$$

The noncommutative wave function is

$$\Psi(\bar{\phi},\beta) = \exp\left(\pm - i\frac{m-2}{2}\nu\beta\right)K_{i\nu}\left[\frac{2\lambda_s\sqrt{V_0}}{m-2}\exp\left\{(m-2)\left(\bar{\phi}\mp\frac{m-2}{4}\theta\nu\right)\right\}\right]. \tag{25}$$

Using the NCWKB-type method the classical solutions for the noncommutative string cosmology are:

$$\begin{aligned} \bar{\phi}(\tau) &= \frac{1}{m-2}\ln\left[\frac{P_{\beta_0}^2}{V_0\lambda_s^2}\mathrm{sech}^2\left(\frac{P_{\beta_0}}{2\lambda_s}(m-2)(\tau-\tau_0)\right)\right] - \frac{\theta}{2}P_{\beta_0}, \\ \beta(\tau) &= \beta_0 + \frac{P_\beta}{\lambda_s}(\tau-\tau_0) + \theta\frac{P_{\beta_0}}{2}\tanh\left[\frac{P_{\beta_0}}{2\lambda_s}(m-2)(\tau-\tau_0)\right]. \end{aligned} \tag{26}$$

The classical evolution for string cosmology can be calculated for $m = 0$ and $m = 4$. An interesting issue concerns the B field that is turned off in the string cosmology model [4] and does not contribute to the effective action. In open string theory, however, noncommutativity arises precisely in the low-energy limit of string theory in the presence of a constant B-field. The θ parameter that we have introduced in the minisuperspace could then be understood as a kind of B-field related with the Neveu – Schwarz B-field.

6. Conclusions

In this work we have presented the NCWKB-type method for noncommutative quantum cosmology and with this procedure found the noncommutative quantum solutions for two noncommutative quantum cosmological models.

By means of the WKB approximation of the corresponding NCWDW equation, one gets the noncommutative generalized Einstein – Hamilton – Jacobi equation (NCEHJ), from which the classical evolution of the noncommutative model is obtained. The examples we have studied here were the Kantowski – Sachs cosmological model and a string quantum cosmological model. In the commutative scenario, the classical solutions found from the WKB-type method are solutions to the corresponding Einsteins field equations. In this approach the effects of noncommutativity are encoded in the potential through the Moyal product of functions. We only need the NCWDW equation and the approximations (6) to get the NCEHJ and, from it, the noncommutative classical behavior can easily be constructed. As already mentioned, in [11, 12] the effects of noncommutativity were studied in connection with inflation, but the noncommutative deformation was only done in the matter sector neglecting the gravity sector. The procedure developed here has the advantage that we can implement noncommutativity in both sectors in a straightforward way and find the classical solutions (i.e., inflationary models). These ideas are being explored and will be reported elsewhere.

We would like to thank M.P. Ryan for enlightening discussions on quantum cosmology. This work was partially supported by PROMEP grants 103.5/08/2919.

Резюме

Э. Мена, О. Обрегон, М. Сабидо, Э. Кано, К. Йи-Ромеро. О деформированных минисуперпространственных переменных в квантовой космологии.

Рассмотрено несколько примеров из некоммутативной квантовой космологии с применением метода квазиклассического приближения с деформацией минисуперпространственных переменных. Данная методика представляет собой прямой алгоритм для включения некоммутативности в космологию и теорию расширения Вселенной.

Ключевые слова: некоммутативная космология, квантовая космология, квазиклассическое приближение.

References

1. *Szabo R.J.* Magnetic background and noncommutative field theory // Int. J. Mod. Phys. A. – 2004. – V. 19. – P. 1837–1861.
2. *Szabo R.J.* Quantum Field Theory on Noncommutative Spaces // Phys. Rep. – 2003. – V. 378. – P. 207–299.
3. *García-Compeán H., Obregón O., Ramírez C.* Noncommutative quantum cosmology // Phys. Rev. Lett. – 2002. – V. 88. – P. 161301-1–161301-4.
4. *Gasperini M., Maharana J., Veneziano G.* Graceful exit in quantum string cosmology // Nucl. Phys. B. – 1996. – V. 472. – P. 349–360.
5. *García-Compeán H., Obregón O., Ramírez C., Sabido M.* Noncommutative self-dual gravity // Phys. Rev. D. – 2003. – V. 68, No 4. – P. 044015-1–044015-8.
6. *Bañados M., Chandia O., Grandi N., Schaposnik F.A., Silva G.A.* Chern – Simons formulation of noncommutative gravity in three dimensions // Phys. Rev. D. – 2001. – V. 64, No 8. – P. 084012-1–084012-8.

7. *Nishino H., Rajpoot S.* Teleparallel Complex Gravity as Foundation for Noncommutative Gravity // Phys. Lett. B. – 2002. – V. 532, No 3–4. – P. 334–344.

8. *Barbosa G.D., Pinto-Neto N.* Noncommutative geometry and cosmology // Phys. Rev. D. – 2004. – V. 70. – P. 103512-1–103512-15.

9. *Gamboa J., Loewe M., Rojas J.C.* Noncommutative quantum mechanics // Phys. Rev. D. – 2001. – V. 64, No 6. – P. 067901-1–067901-3.

10. *Chaichian M., Sheikh-Jabbari M.M., Tureanu. A.* Hydrogen Atom Spectrum and the Lamb Shift in Noncommutative QED // Phys. Rev. Lett. – 2001. – V. 86, No 13. – P. 2716–2719.

11. *Brandenberger R., Ho P.M.* Noncommutative spacetime, stringy spacetime uncertainty principle, and density fluctuations // Phys. Rev. D. – 2020. – V. 66. – P. 023517-1–023517-10.

12. *Huang Q.G., Li M.* Power spectra in spacetime noncommutative inflation // Nucl. Phys. B. – 2005. – V. 713. – P. 219–234.

Поступила в редакцию
15.12.10

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