Turing decidable autostability degrees of almost prime models

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A.I.Maltsev

Autostable models

A. I. Maltsev

constructive and recursive structures

recursive equivalence and autoequivalence

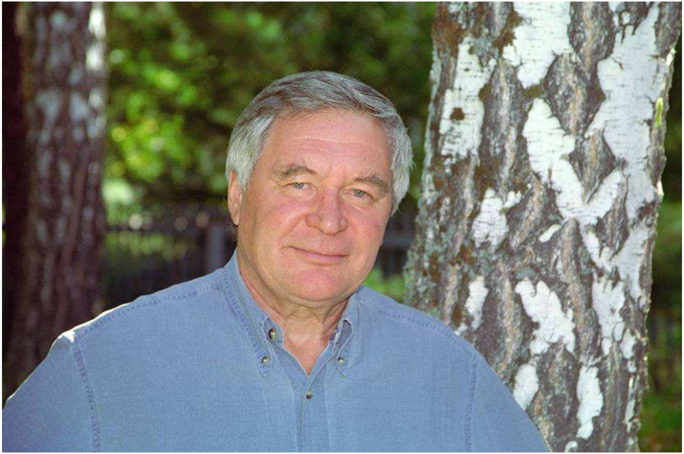
recursive stable models and autostable.(= computable categoricity)

B. L. van der Waerden.
Question of unique of algebraic close of fields

A. Fröhlich and J. C. Shepherdson

Ju. L. Ershov in 1968
strongly constructive model (= decidable)
He proved that any decidable theory has a strong constructive model and started to study the model theory for decidable models.
Existence of computable models and autostability of filds.

Yu.L.Ershov



M. Morley has introduced an equivalent notion of decidable model.
He solved the problem about decidability for countable saturated models.

M.Morley

Gödel numbering of terms and formulas (with computable signature)

With each subset $S \subseteq L_{\mathbb{N}}$ of language $L_{\mathbb{N}}$ we associate the set of all Gödel numbers $\gamma(S)$ of all elements from S.
The set S is called decidable if the set of Gödel numbers of its elements is recursive.
The set S is called computably enumerable if the set of Gödel numbers of its elements is recursively enumerable.

A. I. Maltsev has introduced the notion of autoequivalence, which is weaker.
Let us say that constructivizations ν and μ of the model $\mathfrak{M}$ are *autoequivalent*, if there exist computable function f and automorphism λ of the model $\mathfrak{M}$ such that $\lambda\nu = \mu f$.
The model is called *decidable autostable* (relative to strong constructivization) if for every two strong constructivizations ν_1 and ν_2 of the model $\mathfrak{M}$ there exist automorphism λ of model $\mathfrak{M}$ and computable function f such that $\lambda\nu_1 = \nu_2 f$.

The constructivizations ν and μ of the model $\mathfrak{M}$ are Δ-*autoequivalent* (relative to strong constructivization), if there exist function *f* from Δ and automorphism λ of the model $\mathfrak{M}$ such that $\lambda\nu = \mu f$.
The model is called Δ-*autostable* (relative to strong constructivization) if for every two (strong) constructivizations ν_1 and ν_2 of the model $\mathfrak{M}$ there exist automorphism α of model $\mathfrak{M}$ and function *f* from Δ such that $\alpha\nu_1 = \nu_2 f$.

Index Set

Definition

The **index set** of a structure $\mathcal{A}$ is the set $I(\mathcal{A})$ of all indices of computable (isomorphic) copies of $\mathcal{A}$, where a computable index for a structure $\mathcal{B}$ is a number e, such that $\varphi_e = \chi_{D(\mathcal{B})}$.

Definition

For a class K of structures, closed under isomorphism, the **index set** is the set $I(K)$ of all indices for computable members of K.

$$I(K) = \{e : \exists \mathcal{B} \in K \, \varphi_e = \chi_{D(\mathcal{B})}\}$$

Index Set. Goncharov - Knight

Problem. To study complexity of Index sets of Δ-autostable models for different sets Δ and connection between this Index sets.

Prime models

We call $\mathfrak{M}$ *prime model* of complete theory T, if it can be elementary embedded into every other model of theory T.

If we enrich the model $\mathfrak{M}$ by adding constants to its signature for finite collection A of elements from $\mathfrak{M}$, we call *finite enrichment of model* $\mathfrak{M}$ *by constants* the resulting model and denote it $(\mathfrak{M}, \bar{a})_{a \in A}$, where $\bar{a}$ is finite collection of elements from $\mathfrak{M}$.

Almost prime model

The model $\mathfrak{M}$ is atomic if for every collection of elements $a_1, \dots, a_n \in |\mathfrak{M}|$ there exists formula $\psi(x_1, \dots, x_n)$ such that $\mathfrak{M} \models \psi(a_1, \dots, a_n)$ and for every formula $\varphi(x_1, \dots, x_n)$, if $\mathfrak{M} \models \varphi(a_1, \dots, a_n)$, then

$$\mathfrak{M} \models (\forall x_1) \dots (\forall x_n)\big(\psi(x_1, \dots, x_n) \to \varphi(x_1, \dots, x_n)\big)$$

.

Such formula $\psi(x_1, \dots, x_n)$ is called complete formula for theory of this model.

Theorem (R. Vaught)

A model of complete theory is prime iff this model is countable and atomic.

By Vaught criterion countable model is prime if and only if it is atomic.
Let us call a model *almost prime*, if it is prime in some finite enrichment with constants.

Autostability relative to strong constructivizations

Theorem (A. T. Nurtazin criterion)

Let $\mathfrak{M}$ – be strongly constructive model of complete theory T. Then following conditions are equivalent:
1) $\mathfrak{M}$ *is autostable relative to strong constructivizations;*
2) *there exists finite sequence* $\bar{a}$ *of elements from* M *such that enrichment* $(\mathfrak{M}, \bar{a})$ *of model* $\mathfrak{M}$ *with this constants is prime model and collection of sets of atoms of computable boolean Lindenbaum algebras* $F_n\big(Th(\mathfrak{M}, \bar{a})\big)$ *of theory* $Th(\mathfrak{M}, \bar{a})$ *over the set of formulas with* n *variables is uniformly computable.*

E. A. Palyutin has proposed an example of countably categorical theory with following algorithmic properties.

Theorem (E. A. Palyutin)

There exists decidable countably categorical theory T with elimination of quantifiers for which function $\alpha_n(T)$, giving cardinality of Lindenbaum algebra of theory $F_n(T)$ over the set of formulas with n free variables is not general recursive.

Function $\alpha_n(T)$ is called the function of Ryll-Nardzewsky for countably categorical theory T. From nonrecursiveness of Ryll-Nardzewsky function in the theory of E. A. Palyutin it follows that collection of complete formulas of this theory is not computable.

Corollary

There exists countable categorical theory with strongly constructivizable countable model, which is not autostable relative to strong constructivizations and is even not autostable.

R. L. Vaught has shown that the Ehrenfeucht theory with two countable models is impossible. But for every $n \geq 3$ there exists Ehrenfeucht theory T with n countable models.
M. G. Peretyatkin has shown that there is some decidable Ehrenfeucht theory has computable family of recursive principal types and only its prime model is decidable.
He proved that for any decidable Ehrenfeucht theory the prime model of this theory is decidable by N. Khisamiev results.

Theorem (M. G. Peretyatkin)

The prime model of decidable Ehrenfeucht theory is strongly constructivizable.

In general case was produced the following criterion for decidability of prime models.

Theorem (S. S. Goncharov and L. Harrington)

A prime model of decidable theory is strongly constructivizable if and only if the family of all principal types of this theory is computable.

Goncharov-Nurtazin constructed the theories ω-stable but without decidable prime models.
Millar-Goncharov problem. Can we prove that prime model of decidable complete theory with countable many countable models is decidable?
Morley problem. If saturated model of Ehrenfeucht theory is

Theorem

There is a decidable theory with a prime model in finite enrichment with constants such that it is decidable and is not autostable relative to strong constructivizations but prime model is decidable and is autostable relative to strong constructivizations.

Index set of autostable structures

Theorem (Downey, Kach, Lempp, Lewis-Pye, Montalbán, Turetsky, 2015)

The index set of autostable structures of a signature σ is m-complete Π^1_1.

Suppose that S is an m-complete Π^1_1 set. Downey et al. built a computable sequence $\{\mathfrak{A}_n\}_{n\in\omega}$ of directed graphs such that

- if $n \in S$, then $\mathfrak{A}_n$ is autostable;
- if $n \notin S$, then the hyperarithmetical dimension of $\mathfrak{A}_n$ is infinite, i. e. there are computable structures $\mathfrak{B}_0, \mathfrak{B}_1, \mathfrak{B}_2, \dots$ such that $\mathfrak{B}_i \cong \mathfrak{A}_n$ and $\mathfrak{B}_i \not\cong_{\mathrm{HYP}} \mathfrak{B}_j$ for $i \neq j$.

Finite computable dimension

The **computable dimension** of a computable structure $\mathfrak{M}$ is the number of computable copies of $\mathfrak{M}$ up to computable isomorphism.

Theorem (Goncharov, 1980)

Let n be a natural number such that $n \geq 2$. There exists a computable structure $\mathfrak{M}$ with computable dimension n.

Theorem (Bazhenov, Goncharov, Marchuk and ..)

Suppose that $2 \leq n < \omega$. The index set of computable structures of a signature σ with computable dimension n is m-complete Π^1_1.

Autostability spectra

The **autostability spectrum** of a computable structure $\mathfrak{M}$ is the set

$$\mathrm{AutSpec}(\mathfrak{M}) = \{\mathbf{d} : \mathfrak{M} \text{ is } \mathbf{d}\text{-autostable}\}\,.$$

A Turing degree $\mathbf{d}_0$ is the **degree of autostability** of $\mathfrak{M}$ if $\mathbf{d}_0$ is the least degree in the set $\mathrm{AutSpec}(\mathfrak{M})$.

Autostability spectra and degrees of autostability are also called **categoricity spectra** and **degrees of categoricity** respectively.

SC-autostability spectra

Let $\mathbf{d}$ be a Turing degree. A strongly constructivizable structure $\mathfrak{M}$ is **d-autostable relative to strong constructivizations** (**d-SC-autostable**) if for any decidable copies $\mathfrak{N}_0$ and $\mathfrak{N}_1$ of $\mathfrak{M}$, there exists a $\mathbf{d}$-computable isomorphism $f : \mathfrak{N}_0 \to \mathfrak{N}_1$.

Definition (Goncharov, 2009)

The **autostability spectrum relative to strong constructivizations** (**SC-autostability spectrum**) of a strongly constructivizable structure $\mathfrak{M}$ is the set

$$\mathrm{AutSpec}_{SC}(\mathfrak{M}) = \{\mathbf{d} : \mathfrak{M} \text{ is } \mathbf{d}\text{-}SC\text{-autostable}\}.$$

There exists a computable functors the category of computable models of finite non-trivial signature $\Rightarrow$ the category of computable models with one n-placed predicate $\Rightarrow$ the category of computable poset $\Rightarrow$ the category of computable graphs which preserves computability, decidability, autostability, autostability relative to decidability, algorithmic dimension and some model-theoretical properties.
Open problem. Is there the computable models with degree of stability from Π^0_{n+3} but not from Π^0_{n+2}? not arithmetical?

Index set of strongly computable almost prime structures

Let σ be a signature. We define the index set
$\mathrm{SC}-\mathrm{almostPrime}_\sigma$
$= \{e \in \omega : \mathfrak{M}_e$ is strongly constructivizable structure$\}$.

Theorem

(Goncharov, 2014) Suppose that σ is a nontrivial computable signature. Then the index set $\mathrm{SC}-\mathrm{almostPrime}_\sigma$ of structures of σ is m-complete $\Sigma^0_{\omega+2}$.

Autostable relative to strongly constructivizable structures

Theorem (Bazhenov, Goncharov, Marchuk)

The index set DecAut_σ of strongly constructivizable autostable structures of a signature σ is m-complete Σ^0_3.

Characterization of complexity in case of bounded decidability.

Theorem (Goncharov)

The index set of 2-decidable, computable categorical structures is Π^0_3 complete.

Theorem (Fokina, Goncharov, Harizanov, Kudinov, Turetsky, 2015)

The index set of 1-decidable, computable categorical structures is Π^0_4 complete.

Theorem (Fokina, Goncharov, Harizanov, Kudinov, Turetsky, 2015)

For any $n \geq 2$ and $m \leq n - 2$, the index set of n-decidable, categorical relative to m-decidable presentations structures is Σ^0_3 complete.

E. Fokina, I. Kallimulin and R. Miller proved the next result.

Theorem

For any Σ_2^{-1} Turing degree a there exists a model $\mathfrak{M}$ with degree of autostable a.

Σ_n^{-1} Ng, Shima, Shore - hyperarismetical

Degrees of SC-autostabilty

A Turing degree $\mathbf{d}_0$ is the **degree of autostability relative to strong constructivizations** (**degree of SC-autostability**) of $\mathfrak{M}$ if $\mathbf{d}_0$ is the least degree in the spectrum $\mathrm{AutSpec}_{SC}(\mathfrak{M})$.

Observation

Every degree of SC-autostability is a degree of autostability.

Open problem

Is the converse true?

Degrees of *SC*-autostabilty

A Turing degree $\mathbf{d}_0$ is the **degree of autostability relative to strong constructivizations** (**degree of *SC*-autostability**) of $\mathfrak{M}$ if $\mathbf{d}_0$ is the least degree in the spectrum $\mathrm{AutSpec}_{SC}(\mathfrak{M})$.

Observation

Every degree of *SC*-autostability is a degree of autostability.

Open problem

Is the converse true?

Turing spectrum of autostability

Theorem

For any constructivizable model, the Turing spectrum of autostability is nonempty.

Corollary

For any strongly constructivizable model, the Turing spectrum of autostability with respect to strong constructivizations has a c.e. degree

SC-autostability and degrees of autostability

Theorem

(Bazhenov) Suppose that $\beta \leq \omega$. There exists a SC-autostable linear ordering with the degree of autostability $\mathbf{0}^{(\beta)}$.

theorem

(Bazhenov, Marchuk) Let $\mathfrak{B}$ be a SC-autostable Boolean algebra. Then $\mathfrak{B}$ has the degree of autostability.

Corollary

Suppose that $\beta \leq \omega$. There exists a SC-autostable Boolean algebra with the degree of autostability $\mathbf{0}^{(\beta)}$.

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Suppose that $\beta \leq \omega$. There exists a SC-autostable Boolean algebra with the degree of autostability $\mathbf{0}^{(\beta)}$.

Theorem

If $\mathfrak{M}$ is a strongly constructivizable almost prime model of a complete theory T and, for a finite sequence $\overline{a}$ of elements M, an enrichment $(\mathfrak{M}, \overline{a})$ of the model $\mathfrak{M}$ with these constants is a prime model and the family of sets of atoms of the Lindenbaum computable Boolean algebras $F_n(Th(\mathfrak{M}, \overline{a}))$ of the theory $Th(\mathfrak{M}, \overline{a})$ over the set of formulas in n free variables is uniformly a-computable, then the model $\mathfrak{M}$ is a-autostable with respect to strong constructivizations.

Theorem

(2009) For any computable enumerable Turing degree a, there exists a prime model $\mathfrak{M}$ that is strongly constructivizable and has autostability degree a.

N.Bazhenov constructed example of model without degree of decidable categoricity and proved for hyperarithmetical degrees: If α is a computable successor ordinal and **d** is a degree such that $\mathbf{d} \geq \mathbf{0}^{(\alpha)}$ and **d** is c.e. in $\mathbf{0}^{(\alpha)}$, then **d** is a degree of decidable categoricity.

Theorem

(Goncharov-Harizanov-Miller, 2019) For any c.e. set D there exist a decidable theory T_D *with the prime decidable autostable model* $\mathfrak{M}_D$ *and decidable types* $p(x) \subset q(x,y)$ *such that the prime model* $\mathfrak{M}_{(D,p)}(c)$ *of* $p(c)$ *is decidable and the set of complete formulas of the theory* $p(c)$ *and the degree of decidable autostability and the degree of accostable of this model are the Turing degree of the set D, but the prime model of the theory* $q(c,d)$ *is decidable, autostable and the set of complete formulas of the theory* $q(c,d)$ *is computable.*

Proof. Let s be a computable function N onto D such that $s(i) \neq s(j)$ for $i \neq j$.
Let $\{P^1_{(n)} \mid n \in N\} \cup \{Q^1_{(n)} \mid n \in N\} \cup \{A^1_n \mid n \in N\} \mid n, i \in N\} \cup \{F^2_{(0,n)} \mid n \in N\} \cup \{F^2_{(1,n)} \mid n \in N\}$ be a signature of the theory T_D and c, d two different constants.

Axioms $(0, 1, n)$:
$(\forall x)(\neg P_0(x) \vee \neg A_n(x))$ for $n \in N$
$(\forall x)(\neg Q_0 \vee \neg A_n)$ for $n \in N$

$\neg(\exists x)(P_0(x)\&Q_0(x))$
$(\forall x)(P_{n+1}(x) \to P_n(x))\&(\forall x)(Q_{n+1}(x) \to Q_n(x))$ for $n \in N$.
Axioms $(0, 2, n, m)$: $\neg(\exists x)(A_n(x)\&A_m(x))$ for $n \neq m$.
Axioms $(0, 3, n)$: For any n and $k = s(n)$ the predicate $F_(0, n)$ is a function from P_n onto $A_{s(n)}$
and $(\forall x)(A_{s(n)}(x) \to (\exists y)^{\geq m}F_{(1,n)}(y, x)$ for $m \geq 2$. . The predicate $F_(1, n)$ is a function from $Q_(n)$ onto A_n
and $(\forall x)(A_n \to (\exists y)^{\geq m}F_{(1,n)}(y, x)$ for $m \geq 2$. .

Axioms $(0,4,n)$: $(\exists x)^{\geq m}((A_n(x))$ for $n \in N$ and $m \geq 2$. (A_n is infinite)
Axioms $(0,5,n)$: $(\exists x)^{\geq m} P_n(x)$ and $(\exists x)^{\geq m} Q_n(x)$ for $n, m \in N$. (P_n and Q_n are infinite) .
Axioms $(0,6,n)$: The sets $(P_n(x) \& \neg P_{n+1}(x))$ and $(Q_n(x) \& \neg Q_{n+1}(x))$ are infinite.

Axioms $(t+1,1)$:
$(\forall x_0)\dots(\forall x_t)(\forall y_0)\dots(\forall y_t)(\&_{0\leq i\leq t}A_i(x_i)\&\&_{0\leq i\leq t}A_{s(i)}(y_i))\rightarrow$
$(\exists^{\geq k}z)(\exists^{\geq k})w)(\&_{0\leq i\leq t+1}(F_{(0,i)}(z,x_i)\,\&P_t(z)\&\neg P_{t+1}(z)\&F_{(1,i}(w,y_i)\&$
for any $k\geq 2$.
Axioms :$(t+1,2)$:
$(\forall x_0)\dots(\forall x_t)(\forall y_0)\dots(\forall y_t)(\&_{0\leq i\leq t}A_i(x_i)\&\&_{0\leq i\leq t}A_{s(i)}(y_i))\rightarrow$
$(\exists^{\geq k}z)(\exists^{\geq k})w)(\&_{0\leq i\leq t+1}(F_{(0,i)}(z,x_i)\,\&P_{t+1}(z)\&F_{(1,i}(w,y_i)\&\ Q_t(w))$
for any $k\geq 2$.
Axioms for type $p(c)$: $P_n(c)$ for $n\in N$.
Axioms for type $(q(c,d),n)$: For $n\in N$ axiom $P_(c)\&Q_n(d)$ and
for any $n\in N$ axiom $(\exists y)(F_{(0,n)}(c,y)\&F_{(1,s(n))}(d,y))$.

Bounded theory T_n of the finite signature σ_n for any $n \in N$, where $\sigma_n \rightleftharpoons$
$\{A_0, ..., A_r(n), F_{(0,0)}, ..., F_{(0,n)}, F_{(1,0)}, ..., F_{(1,n)}, P_0, ..., P_n, Q_0, ..., Q_n\}$
but
$r(n) \rightleftharpoons \max(\{n\} \cup \{s(i) | i \leq n\})$.
Let axioms Ax_n of T_n be all axioms of T_D of signature σ_n and $(\exists z)^{\geq k}(\&_{i \leq s(n)} \neg A_i(z) \& \neg P_0(z) \& \neg Q_0(z))$ for $k \geq 2$.

Lemma

. *The theory T_n is consistent for $n \in N$.*

Let $\{D_n | m \in N\}$ be a strong computable family of infinite sets $D_n = \{d_{(n,0)} < ... < d_{(n,k)} < ....$
Let $\{Y_{(\text{o},(j_0,...,j_k))} \mid k \in N \& j_0, ..., j_k \in N\}$ and $\{Z_{(\text{o},(j_0,...,j_k))} \mid kN \& j_0, ..., j_k \in N\}$ be strong computable families of computable infinite sets without intersections for $k \leq n$.

Let $M_n \rightleftharpoons \cup_{i \in N}(D_i \cup \cup_{k \leq n} \cup_{j_0,\ldots,j_k \in N} (Y_{(k,(j_0,\ldots,j_k)} \cup Z_{(k,(j_0,\ldots,j_k)})$.
Let $A_i \rightleftharpoons D_i$ for $i \leq r(n)$.
Let $P_n \rightleftharpoons \cup_{j_0,\ldots,j_n \in N} Y_{(n,(j_0,\ldots,j_n))}$ and $Q_n \rightleftharpoons \cup_{j_0,\ldots,j_n \in N} Z_{(n,(j_0,\ldots,j_n))}$.
Let $P_m \rightleftharpoons P_{m+1} \cup \cup_{j_0,\ldots,j_m \in N} Y_{(m,(j_0,\ldots,j_m))}$ and
$Q_m \rightleftharpoons Q_{m+1} \cup \cup_{j_0,\ldots,j_m \in N} Z_{(m,(j_0,\ldots,j_m))}$.

For any $k \leq n$ and any z from $Y_{(k,(j_0,\ldots,j_k))}$ let
$F_(0,i)(z) \rightleftharpoons d_{(s(i),j_i)}$ for $i \leq k$.
For any $k \leq n$ and any z from $Z_{(k,(j_0,\ldots,j_k))}$ let $F_(1,i)(z) \rightleftharpoons d_{(i,j_i)}$
for $i \leq k$.
The constructed model $\mathfrak{M}_n$ is a model of T_n.

Lemma

The theories T_n for $n \in N$ have uniformly computable systems axioms, countably categorical and decidable and complete.

Corollary 1. Any complete formula of the theory T_n is equivalent to a $\exists$-formula

Lemma

. The theory T_D has prime decidable autostable model.

Lemma

The theory $p(c)$ complete, decidable and has a decidable prime model.

Lemma

The degree of decidable autostability of the prime model and Turing degree of complete formulas of the theory $p(c)$ is Turing degree of the set D.

Let $q(c,d) \rightleftharpoons p_n(c) \cup \{P_n(c) | n \in N\} \cup \{Q_n(d) | n \in N\} \cup \{(\exists y)(F_{(0,n)}(c,y) \& F_{(1,k)}(d,y)) | n,k \in N \& s(n) = k\}$ of signature $\sigma \cup \{c,d\}$.

Lemma

The theory $q(c,d)$ complete and decidable and the prime model of the theory $q(c,d)$ decidable and autostable.

Corollary

For any c.e. set D there exist a complete theory with the set of complete formulas and degree of decidable autostability of prime model $deg(D)$ and with decidable almost prime decidable autostable model.

Corollary

For any list c.e. sets $(D_0, ..., D_n$ *there exist a complete theory* T *with the set of complete formulas and degree of decidable autostability of prime model* $deg(D_0)$ *and with decidable almost prime decidable autostable models* $\mathfrak{M}_0 \preceq \mathfrak{M}_1 \preceq ... \preceq \mathfrak{M}_n$ *such that for any* $i \leq n$ *the theory model* $\mathfrak{M}_1, \overline{a}_0 ... \overline{a}_i$ *with the set of complete formulas and degree of decidable autostability of this model* $deg(D_i)$