Algebras and coalgebras

A categorical approach

Robert Wisbauer

University of Düsseldorf, Germany

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Preface

These are notes of lectures given at the Department of Mechanics and Mathematics of the Kazan State University in Tatarstan, Russian Federation, in September 2013. The author wants to thank the colleagues their, in particular Adel Abyzov, for their kind invitation and warm hospitality. He is also grateful to Bachuki Mesablishvili for proof reading this text.

The purpose of the talks is to show how algebraic notions can be introduced at an early stage in general categories, thus providing a framework which turns out to be most useful to decribe more advanced theories and research.

Together with the elementary notions for abelian groups, the corresponding terms are introduced in a categorical language. This leads naturally to a formalism which allows to handle algebraic and coalgebraic terminology in a general setting. At the end of the course the reader will be ready to deal with bimonads and Hopf monads in arbitrary categories.

Before beginning we will recall the notion of a Hopf algebra in vector spaces.

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Hopf algebras

A k-vector space H is called a *k-bialgebra* if it is an

algebra $\quad \mu : H \otimes_k H \to H, \quad \eta : k \to H,$ and a

coalgebra $\quad \Delta : H \to H \otimes_k H, \quad \varepsilon : H \to k,$

such that Δ and ε are algebra morphisms, where multiplication on $H \otimes_k H$ is derived from the canonical twist map

$$\mathsf{tw} : H \otimes_k H \to H \otimes_k H,\ a \otimes b \mapsto b \otimes a,$$

by defining $(a_1 \otimes a_2) \cdot (b_1 \otimes b_2) = a_1 b_1 \otimes a_2 b_2$.

Besides composition, $\mathrm{End}_k(H)$ allows for a *convolution product* for $f, g \in \mathrm{End}(H)$,

$$f * g\,(h) = (f \otimes g)(\Delta h),$$

making $(\mathrm{End}_k(H), *, +)$ a ring.

If the identity map $I : H \to H$ has an inverse S with respect to $*$, this is called an *antipode*, that is,

$$I * S = \eta \circ \varepsilon = S * I.$$

A bialgebra which has an antipode is a *Hopf algebra*.

As an example, consider the polynomial ring $k[X]$ with the usual multiplication of polynomials, a coproduct

$$\Delta : k[X] \to, \quad k[X] \otimes k[X], X \mapsto X \otimes 1 + 1 \otimes X,$$

and the antipode

$$S : k[X] \to k[X], \quad x \mapsto -x.$$

The purpose of this lecture is to analyse the structures involved and to formulate the notions for arbitrary categories.

Algebras and coalgebras

1 Abelian groups

In this section we recall fundamentals of abelian groups which we will need later on.

1.1. Abelian groups. An abelian group is defined as a set G with a map

$$+_G : G \times G \to G,\ (a, b) \mapsto a +_G b,$$

with the properties, for all $a, b, c \in G$,

$$\begin{array}{rl} \text{associativity} & (a +_G b) +_G c = a +_G (b +_G c), \\ \text{commutativity} & a +_G b = b +_G a, \\ \text{identity element} & \text{there exists } 0 \in G, \text{with } a +_G 0 = a = 0 +_G a, \\ \text{inverse element} & \text{there exists } -a \in G, \text{with } a +_G (-a) = 0 = (-a) +_G a. \end{array}$$

We will mostly write $+$ instead of $+_G$ if no confusion arises.

The integers $\mathbb{Z}$ form an abelian group; they act on any (abelian) group G by

$$\mathbb{Z} \times G \to G,\ (n, g) \mapsto n \cdot g = \begin{cases} g + \ldots + g,\ n\text{-times} & \text{if } n \geq 0 \\ (-g) + \ldots + (-g),\ (-n)\text{-times} & \text{if } n < 0 \\ 0 & \text{if } n = 0. \end{cases}$$

This means that every abelian group is a $\mathbb{Z}$-module (and vice versa).

1.2. Homomorphisms. Given two abelian groups $(G, +_G)$ and $(H, +_H)$, a map $f : G \to H$ is called a *(group) homomorphism* provided

$$f(a +_G b) = f(a) +_H f(b),\ \text{for all } a, b \in G.$$

The image of f, a subgroup of H, is defined as

$$\operatorname{Im}(f) = f(G) = \{f(g) \mid g \in G\} \subseteq H.$$

The set of homomorphisms from G to H is denoted by $\operatorname{Hom}(G, H)$. Note that these homomorphisms are nothing but $\mathbb{Z}$-linear maps.

For $f, g \in \operatorname{Hom}(G, H)$, the sum is defined by

$$(f +_{\operatorname{Hom}(G,H)} g)(a) = f(a) +_H g(a),\ \ \text{for all } a \in G,$$

making $\operatorname{Hom}(G, H)$ an abelian group.

Clearly, the identity map $I_G : G \to G$ is a group homomorphism and for any two homomorphism $f : G \to H$ and $g : H \to K$, the composition $g \circ f : G \to K$ is again a homomorphism.

Thus on $\mathrm{End}(G) := \mathrm{Hom}(G,G)$ we have a product $\circ$ (composition) and an addition $+_{\mathrm{End}(G)}$ induced by $+_G$. These operations are distributive making $(\mathrm{End}(G), \circ, +_{\mathrm{End}(G)})$ a *ring* (see 3.1).

1.3. Subgroups and factor groups. Let $(G,+)$ be an abelian group. A subset $U \subset G$ is a *subgroup* if it is closed under the group operation and inverses, that is,

$$u, v \in U \text{ implies } u + v \in U,\ -u \in U.$$

The subset $\{0\} \subset G$ is the smallest subgroup of G, we usually denote it just by 0. It is characterised as the smallest group generated by a single element.

There exists precisely one homomorphism $0 \to G$ and one $G \to 0$.

Every subgroup U induces an equivalence relation on G, by defining for $a, b \in G$,

$$a \sim_U b \Leftrightarrow a - b \in U.$$

The set of equivalence classes, denoted by G/U, has an abelian group structure given, for $a, g \in G$, by

$$G/U \times G/U \to G/U, \quad (\overline{a}, \overline{b}) \mapsto \overline{a+b},$$

where $\overline{x}$ denotes the equivalence class of $x \in G$.

By definition, the canonical projection $p : G \to G/U$, $a \mapsto \overline{a}$, is a surjective group homomorphism.

1.4. Products of abelian groups. Let $\{G_\lambda\}_\Lambda$ be a family of abelian groups. Then the cartesian product

$$\prod\nolimits_\Lambda G_\lambda = \{(g_\lambda)_\Lambda,\ g_\lambda \in G_\lambda\},$$

is an abelian group by componentwise addition and there are *projections*

$$\pi_\mu : \prod\nolimits_\Lambda G_\lambda,\ (g_\lambda)_\Lambda \mapsto g_\mu.$$

Denoting $P = \prod_\Lambda G_\lambda$ we observe the following property:

For every family of homomorphisms $\{f_\lambda : X \to G_\lambda\}_\Lambda$, there is a unique homomorphism $f : X \to P$ with $\pi_\lambda \circ f = f_\lambda$ for all $\lambda \in \Lambda$.

This corresponds just to the bijectivity of the map

$$\Phi : \mathrm{Hom}(X, \prod\nolimits_\Lambda G_\lambda) \to \prod\nolimits_\Lambda \mathrm{Hom}(X, G_\lambda),\ \ f \mapsto (\pi_\lambda \circ f)_\Lambda.$$

1.5. Coproducts of abelian groups. Let $\{G_\lambda\}_\Lambda$ be a family of abelian groups. The subset of the cartesian product

$$\coprod\nolimits_\Lambda G_\lambda = \{a \in \prod\nolimits_\Lambda G_\lambda \mid \pi_\lambda(a) \neq 0 \text{ only for finitely many } \lambda \in \Lambda\},$$

is a subgroup with injections

$$\epsilon_\mu : G_\mu \to \coprod\nolimits_\Lambda G_\lambda, \quad a_\mu \mapsto (a_\mu \delta_{\mu\lambda})_{\lambda\in\Lambda}.$$

Denoting $Q = \coprod_\Lambda G_\lambda$ we observe the following property:

For every family of homomorphisms $\{g_\lambda : G_\lambda \to Y\}_\Lambda$, there is a unique homomorphism $f : Q \to Y$ with $g \circ \epsilon_\lambda = g_\lambda$ for all $\lambda \in \Lambda$.

This corresponds just to the bijectivity of the map

$$\Psi : \mathrm{Hom}(\coprod\nolimits_\Lambda G_\lambda, Y) \to \prod\nolimits_\Lambda \mathrm{Hom}(G_\lambda, Y), \quad g \mapsto (g \circ \epsilon_\lambda)_\Lambda.$$

$\coprod_\Lambda G_\lambda$ is also called the (external) direct sum and written as $\bigoplus_\Lambda G_\lambda$.

1.6. Kernel. For a homomorphism $f : G \to H$ of abelian groups, the *kernel* is defined as

$$\mathrm{Ke}\ f = \{a \in G \mid f(a) = 0_H\}.$$

$\mathrm{Ke}\ f$ is a subgroup of G and characterised by the property:

For any homomorphism $g : L \to G$ with $f \circ g = 0$ there is a unique homomorphism $q : H \to \mathrm{Ke}\ f$ with commutative diagram (with inclusion i)

$$\begin{tikzcd} & L \arrow[dl, "q"'] \arrow[d, "g"] & \\ \mathrm{Ke}\ f \arrow[r, "i"] & G \arrow[r, "f"] & H. \end{tikzcd}$$

Furthermore, f factors as

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "p"'] & H \\ G/\mathrm{Ke}\ f \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

where $p : G \to G/Ke\, f$ is the canonical projection and $\bar{f}$ is injective.

1.7. Equaliser. Consider two homomorphisms $G \overset{f}{\underset{f'}{\rightrightarrows}} H$ of abelian groups. The *equaliser* of (f, f') is defined as the subgroup

$$\mathrm{Eq}(f, f') = \{a \in G \mid f(a) = f'(a)\}$$

and for the inclusion $k : \mathrm{Eq}(f, f') \to G$ we have the property:

for every homomorphism $g : L \to G$ with $f \circ g = f' \circ g$, there exists a unique homomorphism $u : L \to \mathrm{Eq}(f, f')$ such that $g = k \circ u$.

This is visualized by the commutative diagram

$$\begin{tikzcd} & L \arrow[dl, "u"'] \arrow[d, "g"] & \\ \mathrm{Eq}(f, f') \arrow[r, "k"] & G \arrow[r, shift left, "f"] \arrow[r, shift right, "f'"'] & H. \end{tikzcd}$$

By definition, $\mathrm{Eq}(f, f')$ is just the kernel of $f - f'$.

1.8. Pullback of homomorphisms. For any pair $f_1 : H_1 \to H$, $f_2 : G_2 \to H$ of homomorphisms of abelian groups, consider the homomorphism

$$p^* = f_1 \circ \pi_1 - f_2 \circ \pi_2 : \ G_1 \times G_2 \to H,$$

where $\pi_i : G_1 \times G_2 \to G_i$, $i = 1, 2$, are the canonical projections. With $P = \mathrm{Ke}\, p^*$ and the restrictions π_i' of π_i to $P \subset G_1 \times G_2$, the square

$$\begin{tikzcd} P \arrow[r, "\pi_2'"] \arrow[d, "\pi_1'"'] & G_2 \arrow[d, "f_2"] \\ G_1 \arrow[r, "f_1"] & H \end{tikzcd}$$

is called the *pullback* for (f_1, f_2) and has the property:

for every pair of homomorphisms $g_1 : X \to G_1$, $g_2 : X \to G_2$ with $f_1 \circ g_1 = f_2 \circ g_2$, there is a unique homomorphism $g : X \to P$ with $\pi_1' \circ g = g_1$ and $\pi_2' \circ g = g_2$.

1.9. Cokernel. For a homomorphism $f : G \to H$ of abelian groups, $f(G)$ is a subgroup of H, and the *cokernel of* f is defined as $\mathrm{Coke}\, f = H/f(G)$ with the canonical projection $q : H \to \mathrm{Coke}\, f$ and this has the property:

for any group homomorphism $g : H \to L$ with $g \circ f = 0$, there exists a unique homomorphism $\bar{g} : H/f(G) \to L$ with $g = v \circ p$, that is, we have the commutative diagram

$$\begin{tikzcd} G \arrow[r, "f"] & H \arrow[r, "p"] \arrow[d, "g"'] & H/f(G) \arrow[dl, "v"] \\ & L. & \end{tikzcd}$$

1.10. Coequaliser. Consider two homomorphisms $G \overset{f}{\underset{f'}{\rightrightarrows}} H$ of abelian groups. The *coequaliser* of (f, f') is defined as $\mathrm{Coeq}\,(f, f') = H/\mathrm{Im}\ (f{-}f)$ with the canonical projection $c : H \to \mathrm{Coeq}\,(f, f')$ and has the property:

for every homomorphism $h : H \to Y$ with $h \circ f = h \circ f'$, there exists a unique homomorphism $v : \mathrm{Coeq}\,(f, f') \to Y$ such that $h = v \circ c$.

This is visualized in the commutative diagram

$$\begin{tikzcd} G \arrow[r, shift left, "f"] \arrow[r, shift right, "f'"'] & H \arrow[r, "c"] \arrow[d, "h"'] & \mathrm{Coeq}\,(f, f') \arrow[dl, "v"] \\ & Y. & \end{tikzcd}$$

By definition, the coequaliser of (f, f') is just the cokernel of $f - f'$.

1.11. Pushout of homomorphisms. Let $g_1 : G \to H_1$, $g_2 : G \to H_2$ be two homomorphisms of abelian groups. With the injections $\varepsilon_i : H_i \to H_1 \oplus H_2$, $i = 1, 2$, we form the morphism

$$q^* = \varepsilon_1 \circ g_1 - \varepsilon_2 \circ g_2 : \ G \to H_1 \oplus H_2.$$

The $\mathrm{Coke}\, q^*$ together with the canonical homomorphisms $\bar\varepsilon_i : H_i \to H_1 \oplus H_2 \to \mathrm{Coke}\, q^*$ is called the *pushout* of (g_1, g_2). It has the property:

for any pair of homomorphisms $h_1 : H_1 \to Y$, $h_2 : H_2 \to Y$ with $h_1 \circ g_1 = h_2 \circ g_2$, there is a unique homomorphism $h : \mathrm{Coke}\, q^* \to Y$ with $h \circ \bar\varepsilon_1 = h_1$, $h \circ \bar\varepsilon_2 = h_2$, that is, we have a commutative diagram

$$\begin{tikzcd}
G \arrow[r, "g_2"] \arrow[d, "g_1"'] & H_2 \arrow[d, "\bar\varepsilon_2"'] \arrow[ddr, "h_2"] & \\
H_1 \arrow[r, "\bar\varepsilon_1"] \arrow[drr, "h_1"'] & Coke\, q^* \arrow[dr, "h"] & \\
 & & Y\,.
\end{tikzcd}$$

1.12. Special homomorphisms. A homomorphism $f : G \to H$ of abelian groups is called

monomorphism	if f is injective;
epimorphism	if f is surjective;
isomorphism	if f is bijective;
null morphism	if $f(g) = 0$ for all $g \in G$.

1.13. Characterisations of monomorphisms. *The following are equivalent for a homomorphism $f : G \to H$ of abelian groups:*

(a) *f is a monomorphisms;*

(b) *for any homomorphism $g, h : L \to G$, $f \circ g = f \circ h$ implies $g = h$;*

(c) *f is the kernel of the projection $p : H \to H/f(G)$;*

(d) *f is the coequaliser of* $H \overset{p}{\underset{0}{\rightrightarrows}} H/f(G)$.

1.14. Characterisations of epimorphisms. *The following are equivalent for a homomorphism $f : G \to H$ of abelian groups:*

(a) *f is an epimorphisms;*

(b) *for any homomorphism $g, h : H \to L$, $g \circ f = h \circ f$ implies $g = h$;*

(c) *f is the cokernel of the inclusion $i : \mathrm{Ke}\ f \to G$.*

(d) *f is the equaliser of* $\mathrm{Ke}\ f \overset{i}{\underset{0}{\rightrightarrows}} G$.

1.15. Characterisations of isomorphisms. *The following are equivalent for a homomorphism $f : G \to H$ of abelian groups:*

(a) *f is an isomorphisms (bijective);*

(b) *f is a monomorphism and an epimorphism;*

(c) *there exists a homomorphism $g : H \to G$ with $g \circ f = I_G$ and $f \circ g = I_H$.*

1.16. Exact sequences. A sequence of morphisms $G \xrightarrow{f} H \xrightarrow{g} L$ of abelian groups is called *exact* if $\mathrm{Im}\ f = \mathrm{Ke}\ g$. This means that $g \circ f = 0$ and in the resulting diagram

$$\begin{tikzcd}
G \arrow[rr, "f"] \arrow[dr, "\bar f"'] & & H \arrow[rr, "g"] \arrow[dr, "p"'] & & L \\
 & \mathrm{Ke}\ g \arrow[ur, "i"'] & & \mathrm{Coke}\, f \arrow[ur, "\bar g"'] &
\end{tikzcd} ,$$

$\bar{f}$ is epimorph and - equivalently - $\bar{g}$ is monomorph.

A sequence of group homomorphisms $\{f_i : A_i \to A_{i+1} \,|\, i \in \mathbb{N}\}$ is called *exact at* A_i if f_{i-1} and f_i form an exact sequence. It is called *exact* if it is everywhere exact.

For a homomorphism $f : G \to H$ *of abelian groups we have:*

(i) $0 \to G \xrightarrow{f} H$ *is exact if and only if* f *is monomorph;*

(ii) $G \xrightarrow{f} H \to 0$ *is exact if and only if* f *is epimorph;*

(iii) $0 \to G \xrightarrow{f} H \to 0$ *is exact if and only if* f *is an isomorphism;*

(iv) $0 \to K \xrightarrow{i} G \xrightarrow{f} H \xrightarrow{p} L \to 0$ *is exact if and only if* i *is the kernel of* f *and* p *is the cokernel of* f.

Exact sequences of the form $0 \to K \xrightarrow{i} G \xrightarrow{f} H \to 0$ are called *short exact sequences* or *extensions of* H *by* K.

1.17. Homotopy Lemma. Consider the commutative diagram of abelian groups with exact rows,

$$\begin{tikzcd} & G_1 \arrow[r, "f_1"] \arrow[d, "\varphi_1"'] & G_2 \arrow[r, "f_2"] \arrow[d, "\varphi_2"'] & G_3 \arrow[r] \arrow[d, "\varphi_3"'] & 0 \\ 0 \arrow[r] & H_1 \arrow[r, "g_1"] & H_2 \arrow[r, "g_2"] & H_3 & \end{tikzcd}$$

The following assertions are equivalent:

(a) there exists $\alpha : G_3 \to H_2$ with $g_2 \circ \alpha = \varphi_3$;

(b) there exists $\beta : G_2 \to H_1$ with $\beta \circ f_1 = \varphi_1$.

1.18. Bilinear maps. Let M, N and G be abelian groups. A map $\beta : M \times N \to G$ is called *bilinear* if

$$\begin{aligned} \beta(m_1 + m_2, n) &= \beta(m_1, n) + \beta(m_2, n), \\ \beta(m, n_1 + n_2) &= \beta(m, n_1) + \beta(m, n_2), \end{aligned}$$

for all $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$.

The set of all these maps we denote by $\mathrm{Bil}(M \times N, G)$. The sum of two $\beta, \beta' \in \mathrm{Bil}(M \times N, G)$ is defined by

$$(\beta + \beta')(m, n) = \beta(m, n) +_G \beta'(m, n), \quad \text{for } m \in M,\ n \in N,$$

making $\mathrm{Bil}(M \times N, G)$ an abelian group.

1.19. Tensor product. For abelian groups M, N, form the free $\mathbb{Z}$-module $\mathbb{Z}^{(M \times N)}$ over the set $M \times N$ and denote by $[m, n]$ the elements of the canonical basis. Let K be the submodule of $\mathbb{Z}^{(M \times N)}$ generated by elements of the form

$$[m_1 + m_2, n] - [m_1, n] - [m_2, n], \quad [m, n_1 + n_2] - [m, n_1] - [m, n_2],$$

with $m, m_i \in M$, $n, n_i \in N$. Put $M \otimes N := \mathbb{Z}^{(M \times N)}/K$ and define the map

$$\tau : M \times N \to M \otimes N, \quad (m, n) \mapsto m \otimes n := [m, n] + K\ .$$

By definition of K, the map τ is bilinear. Note that τ is not surjective but the image of τ, $\operatorname{Im}\tau = \{m \otimes n \mid m \in M,\ n \in N\}$, is a generating set (not a basis) of $M \otimes N$ as a $\mathbb{Z}$-module.

For a bilinear map $\beta : M \times N \to G$, G any abelian group, define a $\mathbb{Z}$-homomorphism

$$\tilde{\gamma} : \mathbb{Z}^{(M\times N)} \to G,\ [m,n] \mapsto \beta(m,n).$$

Obviously $K \subset \operatorname{Ke}\tilde{\gamma}$ and hence $\tilde{\gamma}$ factorises over τ. $(M \otimes N, \tau)$ is called the *tensor product* of M and N and we have observed the following property:

for any bilinear map $\beta : M \times N \to G$, there is a unique group homomorphisms $\gamma : M \otimes N \to G$ with commutative diagram

$$\begin{tikzcd} M \times N \arrow[r, "\beta"] \arrow[d, "\tau"'] & G \\ M \otimes N \arrow[ur, "\gamma"'] & \end{tikzcd}.$$

1.20. Tensor product and direct sums. Let M and $N = \bigoplus_\Lambda N_\lambda$ be abelian groups, with canonical injections $\epsilon_\lambda : N_\lambda \to N$ and projections $\pi_\lambda : N \to N_\lambda$. Then $(M \otimes N, I_M \otimes \epsilon_\lambda)$ is a direct sum of $\{M \otimes N_\lambda\}_\Lambda$, i.e.,

$$M \otimes (\bigoplus_\Lambda N_\lambda) \simeq \bigoplus_\Lambda (M \otimes N_\lambda),$$

that is, the tensor product *commutes with direct sums.*

Summarising the facts observed so far we have:

1.21. Properties of the tensor product. *Let M be any abelian group.*

(1) *$G \mapsto M \otimes G$ maps abelian groups to abelian groups.*

(2) *For any homomorphism $f : G \to H$, $I_M \otimes f : M \otimes G \to M \otimes H$ is a group homomorphism.*

(3) *For any homomorphism $f : G \to H$, $g : H \to L$,*

$$(I_M \otimes g) \circ (I_M \otimes f) = I_M \otimes g \circ f.$$

(4) *For any abelian group G, $\mathbb{Z} \otimes G \to G$, $n \otimes g \mapsto ng$, is an isomorphism.*

1.22. Hom-tensor relation. Let L, M and N be abelian groups and denote by $\operatorname{Bil}(L \times M, N)$ the set of the bilinear maps $L \times M \to N$. By the definition of $L \otimes M$, the canonical map $\tau : L \times M \to L \otimes M$ yields a bijection

$$\psi_1 : \operatorname{Hom}(L \otimes M, N) \to \operatorname{Bil}(L \times M, N),\quad \alpha \mapsto \alpha \circ \tau.$$

There is also map

$$\psi_2 : \operatorname{Bil}(L \times M, N) \to \operatorname{Hom}(M, \operatorname{Hom}(L,N)),\quad \beta \mapsto [m \mapsto \beta(-,m)],$$

with inverse $\psi_2^{-1} : \varphi \mapsto [(u,m) \mapsto \varphi(m)(u)]$, and $\psi_2 \circ \psi_1$ yields the isomorphism

$$\psi_{M,N} : \operatorname{Hom}(L \otimes M, N) \to \operatorname{Hom}(M, \operatorname{Hom}(L,N)),\quad \delta \mapsto [m \mapsto \delta(- \otimes m)],$$

with inverse map $\psi_M^{-1} : \varphi \mapsto [u \otimes m \mapsto \varphi(m)(u)]$.

Every homomorphism $f : N \to N'$ leads to a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(L\otimes M, N) \arrow[r, "\mathrm{Hom}(L\otimes M, f)"] \arrow[d, "\psi_{M,N}"'] & \mathrm{Hom}(L\otimes M, N') \arrow[d, "\psi_{M,N'}"] \\
\mathrm{Hom}(M, \mathrm{Hom}(L,N)) \arrow[r, "\mathrm{Hom}(M,\mathrm{Hom}(L,f))"] & \mathrm{Hom}(M, \mathrm{Hom}(L,N'))\,,
\end{tikzcd}$$

and any homomorphism $g : M \to M'$ yields a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(L\otimes M', N) \arrow[r, "\mathrm{Hom}(I\otimes g, N)"] \arrow[d, "\psi_{M',N}"'] & \mathrm{Hom}(L\otimes M, N) \arrow[d, "\psi_{M,N}"] \\
\mathrm{Hom}(M', \mathrm{Hom}(L,N)) \arrow[r, "\mathrm{Hom}(g,\mathrm{Hom}(L,N))"] & \mathrm{Hom}(M, \mathrm{Hom}(L,N))\,.
\end{tikzcd}$$

Related to any abelian groups L, G, we have the homomorphisms

$$\begin{array}{ll}
\varepsilon_G : & L \otimes \mathrm{Hom}(L,G) \to G, \quad u \otimes f \mapsto f(u), \\
\eta_G : & G \to \mathrm{Hom}(L, L\otimes G), \quad g \mapsto [u \mapsto u \otimes g],
\end{array}$$

satisfying the (triangular) identities

$$\epsilon_{L\otimes G} \circ (I_L \otimes \eta_G) = I_{L\otimes G}, \quad \mathrm{Hom}(L,\epsilon) \circ \eta_{\mathrm{Hom}(L,G)} = I_{\mathrm{Hom}(L,G)},$$

described by the commutative diagrams

$$\begin{tikzcd}
L\otimes G \arrow[r, "I\otimes\eta_G"] \arrow[d, "="'] & L\otimes \mathrm{Hom}(L, L\otimes G) \arrow[dl, "\varepsilon_{L\otimes G}"] \\
L\otimes G &
\end{tikzcd}, \qquad
\begin{tikzcd}
\mathrm{Hom}(L,G) \arrow[r, "\eta_{\mathrm{Hom}(L,G)}"] \arrow[d, "="'] & \mathrm{Hom}(L, L\otimes \mathrm{Hom}(L,G)) \arrow[dl, "\mathrm{Hom}(L,\varepsilon)"] \\
\mathrm{Hom}(L,G) &
\end{tikzcd}.$$

2 Categories

The data above give an example of the notion of a *category* which is basic for what will follow.

2.1. Categories. A *category* $\mathbb{A}$ is given by

(1) a class of *objects*, $\mathrm{Obj}(\mathbb{A})$;

(2) for any objects A, B in $\mathbb{A}$, there exists a set of *morphisms* $\mathrm{Mor}_{\mathbb{A}}(A,B)$, with

$$\mathrm{Mor}_{\mathbb{A}}(A,B) \cap \mathrm{Mor}_{\mathbb{A}}(A',B') = \emptyset \ \text{ for } \ (A,B) \neq (A',B');$$

(3) a *composition* of morphisms, that is a map

$$\diamond : \mathrm{Mor}_{\mathbb{A}}(A,B) \times \mathrm{Mor}_{\mathbb{A}}(B,C) \to \mathrm{Mor}_{\mathbb{A}}(A,C), (f,g) \mapsto g \diamond f,$$

for every triple (A,B,C) of objects, which is associative (in an obvious way);

(4) for every $A \in \mathrm{Obj}(\mathbb{A})$ there is an *identity morphisms* $I_A \in \mathrm{Mor}_{\mathbb{A}}(A,A)$, with $I_A \diamond f = f$ for any $f \in \mathrm{Mor}_{\mathbb{A}}(A,B)$ and $g \diamond I_A = g$ for any $g \in \mathrm{Mor}(B,A)$

We often write $\mathrm{Mor}_{\mathbb{A}}(A,B) = \mathrm{Mor}(A,B)$ and, for short, $A \in \mathbb{A}$ instead of $A \in \mathrm{Obj}(\mathbb{A})$. The composition $g \diamond f$ is usually denoted by gf. For $f \in \mathrm{Mor}(A,B)$ we also write $f : A \to B$ or $A \xrightarrow{f} B$; A is called the *source* and B the *target* of f.

The following notions can be defined in any category without saying anything about their existence. Throughout $\mathbb{A}$ will always denote a category.

2.2. Product of objects. Let $\{A_\lambda\}_\Lambda$ be a family of objects in $\mathbb{A}$. An object P in $\mathbb{A}$ with morphisms (projections) $\{\pi_\lambda : P \to A_\lambda\}_\Lambda$ is called the *product* of $\{A_\lambda\}_\Lambda$, if for every family $\{f_\lambda : X \to A_\lambda\}_\Lambda$, there is a unique morphism $f : X \to P$ with $\pi_\lambda \diamond f = f_\lambda$ for all $\lambda \in \Lambda$.

As for abelian groups, the object P is often denoted by $\prod_\Lambda A_\lambda$. Note that this is not meant as a hint how to construct such an object in general.

The definition is equivalent to bijectivity of the map

$$\Phi : \mathrm{Mor}_{\mathbb{A}}(X, \prod\nolimits_\Lambda A_\lambda) \to \prod\nolimits_\Lambda \mathrm{Mor}_{\mathbb{A}}(X, A_\lambda), \quad f \mapsto (\pi_\lambda \diamond f)_{\lambda \in \Lambda}.$$

2.3. Coproducts of objects. Let $\{A_\lambda\}_\Lambda$ be a family of objects in $\mathbb{A}$. An object Q with morphisms (injections) $\{\epsilon_\lambda : A_\lambda \to Q\}_\Lambda$ is called the *coproduct* of $\{A_\lambda\}_\Lambda$, if for every family $\{g_\lambda : A_\lambda \to Y\}_\Lambda$, there is a unique morphism $g : Q \to Y$ with $g \diamond \epsilon_\lambda = g_\lambda$ for all $\lambda \in \Lambda$.

Writing $Q =: \coprod_\Lambda G_\lambda$ this corresponds to the bijectivity of the map

$$\Psi : \mathrm{Mor}_{\mathbb{A}}(\coprod\nolimits_\Lambda A_\lambda, Y) \to \prod\nolimits_\Lambda \mathrm{Mor}_{\mathbb{A}}(G_\lambda, Y), \quad g \mapsto (g \diamond \epsilon_\lambda)_\Lambda.$$

2.4. Equaliser. The *equaliser* (*difference kernel*) of two morphisms $G \overset{f}{\underset{f'}{\rightrightarrows}} H$ in $\mathbb{A}$ is defined as a morphism $k : K \to G$ with $f \diamond k = f' \diamond k$ and the property that for every morphism $g : L \to G$ with $f \diamond g = f' \diamond g$, there exists a unique morphism $u : L \to K$ such that $g = k \diamond u$.

2.5. Coequaliser. The *coequaliser* (*difference cokernel*) of two morphisms $G \overset{f}{\underset{f'}{\rightrightarrows}} H$ in $\mathbb{A}$ is defined as a morphism $c : H \to C$ with $c \diamond f = c \diamond f'$ and the property that for every morphism $h : H \to Y$ with $h \diamond f = h \diamond f'$, there exists a unique morphism $v : C \to Y$ such that $h = v \diamond c$.

2.6. Special morphisms. A morphism $f : G \to H$ in $\mathbb{A}$ is called

monomorphism	if for any $g, h : L \to G$, $f \diamond g = f \diamond h$ implies $g = h$;
epimorphism	if for any $g, h : H \to L$, $g \diamond f = h \diamond f$ implies $g = h$;
bimorphism	if f is monomorph and epimorph;
retraction	if there exists $g : H \to G$ with $f \diamond g = I_H$;
coretraction	if there exists $g : H \to G$ with $g \diamond f = I_G$;
isomorphism	if f is a retraction and a coretraction.

2.7. Special objects. In any category $\mathbb{A}$, an object A is called

initial object	if $\mathrm{Mor}_{\mathbb{A}}(A, B)$ has just one element, for any $B \in \mathbb{A}$;
terminal object	if $\mathrm{Mor}_{\mathbb{A}}(C, A)$, has just one element, for any $C \in \mathbb{A}$; .
zero object	if A is an initial and a terminal object.

2.8. Zero morphism. Let $\mathbb{A}$ be a category with zero object 0. Then for any objects A, B, there is exactly one morphism $A \to B$ which factors through 0, that is, it can be written as $A \to 0 \to B$. This is called the *zero morphism* and denoted by $0_{A,B}$ or just 0.

2.9. Kernel and cokernel. Let $f : G \to H$ be a morphism in a category $\mathbb{A}$ with zero object.

(1) The *kernel* of f is defined as a morphism $k : K \to G$ with $f \diamond k = 0$, such that for any morphism $g : L \to G$ with $f \diamond g = 0$, there is a unique morphism $u : H \to K$ with $g = k \diamond u$.

Clearly, the kernel of f is just the equaliser of $G \overset{f}{\underset{0}{\rightrightarrows}} H$

(2) The *cokernel of* f is defined as a morphism with $c : H \to C$ $c \diamond f = 0$, such that for any morphism $h : H \to L$ with $h \diamond f = 0$, there exists a unique homomorphism $v : C \to L$ with $h = v \diamond c$.

The cokernel of f is just the coequaliser of $G \overset{f}{\underset{0}{\rightrightarrows}} H$.

2.10. Pullback of morphisms. Let $f_1 : B_1 \to B$, $f_2 : A_2 \to B$ be morphisms in $\mathbb{A}$. A commutative diagram

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & A_2 \arrow[d, "f_2"] \\ A_1 \arrow[r, "f_1"] & B \end{tikzcd}$$

is called the *pullback* for (f_1, f_2) if, for every pair of morphisms $g_1 : X \to A_1$, $g_2 : X \to A_2$ with $f_1 \diamond g_1 = f_2 \diamond g_2$, there is a unique morphism $g : X \to P$ with $p_1 \diamond g = g_1$ and $p_2 \diamond g = g_2$.

2.11. Pushout of morphisms. Let $g_1 : A \to B_1$, $g_2 : A \to B_2$ be two morphisms in $\mathbb{A}$. A commutative diagram in $\mathbb{A}$

$$\begin{tikzcd} G \arrow[r, "g_2"] \arrow[d, "g_1"'] & B_2 \arrow[d, "q_2"] \\ B_1 \arrow[r, "q_1"] & Q \end{tikzcd}$$

is called the *pushout* for (g_1, g_2) if, for every pair of morphisms $h_1 : B_1 \to Y$, $h_2 : B_2 \to Y$ with $h_1 \diamond g_1 = h_2 \diamond g_2$, there is a unique morphism $h : Q \to Y$ with $h \diamond q_1 = h_1$ and $h \diamond q_2 = h_2$.

2.12. Additive categories. A category $\mathbb{A}$ is called *additive* if for any objects $A, B \in \mathbb{A}$, the set $\mathrm{Mor}_{\mathbb{A}}(A, B)$ has an additive group structure $+$ satisfying the distributive laws for $f, g \in \mathrm{Mor}_{\mathbb{A}}(A, B)$, $h \in \mathrm{Mor}_{\mathbb{A}}(C, A)$, $k \in \mathrm{Mor}_{\mathbb{A}}(B, D)$,

$$(f + g) \diamond h = f \diamond h + g \diamond h, \quad k \diamond (f + g) = k \diamond f + k \diamond g.$$

2.13. Abelian categories. The category $\mathbb{A}$ is called *abelian* if

(i) it has a zero-object,

(ii) it has finite products and coproducts,

(iii) every morphism has a kernel and a cokernel,

(iv) every monomorphism is a kernel and every epimorphism is a cokernel.

It can be shown that abelian categories are also additive.

The abelian groups form a category Ab with the *objects* all abelian groups and *morphisms* between abelian groups G, H are the homomorphisms, i.e. $\mathrm{Mor}_{\mathsf{Ab}}(G, H) = \mathrm{Hom}(A, B)$. This is (the prototype of) an abelian category.

The non-commutative groups form a category Grp (Objects: groups, morphisms: group homomorphisms) in which monomorphisms need not be kernels and which is not additive.

Another basic example is the category Set where the objects are sets and the morphisms between sets X, Y are just the maps, i.e. $\mathrm{Mor}_{\mathsf{Set}}(X, Y) = \mathrm{Map}(X, Y)$. In Set the initial object is $\{\emptyset\}$ and the terminal object is presented by a singleton; thus there is no zero-object in Set.

The connection between two categories is given by

2.14. Functors. A *covariant functor* $F : \mathbb{A} \to \mathbb{B}$ between two categories consists of assignments

$$\begin{array}{lrcl} \mathrm{Obj}(\mathbb{A}) \to \mathrm{Obj}(\mathbb{B}), & A & \mapsto & F(A), \\ \mathrm{Mor}(\mathbb{A}) \to \mathrm{Mor}(\mathbb{B}), & f : A \to B & \mapsto & F(f) : F(A) \to F(B), \end{array}$$

such that $F(I_A) = I_{F(A)}$ and $F(fg) = F(f)F(g)$.

Contravariant functors reverse the composition of morphisms.

The composition of two covariant functors again yields a covariant functor.

A functor $F : \mathbb{A} \to \mathbb{B}$ is said to *preserve* properties of an object $A \in \mathrm{Obj}(\mathbb{A})$ or a morphism $f \in \mathrm{Mor}(\mathbb{A})$, if $T(A)$, resp. $T(f)$, again have the same properties.

The functor F *reflects* a property of A, resp. of f, if whenever $F(A)$, resp. $F(f)$, has this property, then this is also true for A, resp. f.

By definition, all functors preserve identities and commutativity of diagrams. Any covariant functor $F : \mathbb{A} \to \mathbb{B}$ assigns to a morphism $A \to B$ in $\mathbb{A}$ a morphism $F(A) \to F(B)$, i.e. for every pair A, B in $\mathrm{Obj}(\mathbb{A})$ we have a (set) map

$$F_{A,B} : \mathrm{Mor}_{\mathbb{A}}(A, B) \to \mathrm{Mor}_{\mathbb{B}}(F(A), F(B)).$$

Any contravariant functor $F : \mathbb{A} \to \mathbb{B}$ induces the map

$$F_{A,B} : \mathrm{Mor}_{\mathbb{A}}(A, B) \to \mathrm{Mor}_{\mathbb{B}}(F(B), F(A))\,.$$

Properties of these maps lead to the definition of

2.15. Special functors. A functor $F : \mathbb{A} \to \mathbb{B}$ is called

faithful	if $F_{A,B}$ is injective for all $A, B \in \mathrm{Obj}(\mathbb{A})$;
full	if $F_{A,B}$ is surjective for all $A, B \in \mathrm{Obj}(\mathbb{A})$;
fully faithful	if F is full and faithful;
an *embedding*	if the assignment $F : \mathrm{Mor}(\mathbb{A}) \to \mathrm{Mor}(\mathbb{B})$ is injective;
representative	if for every $B \in \mathrm{Obj}(\mathbb{B})$ there is an $A \in \mathrm{Obj}(\mathbb{A})$ with $B \simeq F(A)$.

Instead of *representative* one also says *surjective on objects*.

The relation between two functors is described by

2.16. Natural transformations. A *natural transformation* $\alpha : F \to F'$ between two covariant functors $F, F' : \mathbb{A} \to \mathbb{B}$ is given by a family of morphisms

$$\alpha_A : F(A) \to F'(A) \text{ in } \mathbb{B},\ A \in \mathrm{Obj}(\mathbb{A}),$$

such that any $f : A \to B$ in $\mathbb{A}$ induces the commutative diagram in $\mathbb{B}$

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(B) \arrow[d, "\alpha_B"] \\ F'(A) \arrow[r, "F'(f)"] & F'(B)\,. \end{tikzcd}$$

Given another pair of functors $G, G' : \mathbb{B} \to \mathbb{C}$ with any natural transformation $\beta : G \to G'$, the diagram

$$\begin{tikzcd} GF \arrow[r, "G\alpha"] \arrow[d, "\beta F"'] & GF' \arrow[d, "\beta F'"] \\ G'F \arrow[r, "G'\alpha"] & G'F' \end{tikzcd}$$

is commutative and thus there is a natural transformation (*Godement product*)

$$\beta\alpha := \beta F' \diamond G\alpha = G'\alpha \diamond \beta F : GF \to G'F'.$$

In what follows we will use functors and natural transformations as basic tools for general constructions.

2.17. Mor-functors. Let A, B, C be objects in $\mathbb{A}$. Any morphisms $f : B \to C$ yields the following maps between morphism sets,

$$\begin{aligned} \mathrm{Mor}(A, f) &: \mathrm{Mor}_{\mathbb{A}}(A, B) \to \mathrm{Mor}_{\mathbb{A}}(A, C), \quad u \mapsto f \diamond u, \\ \mathrm{Mor}(f, A) &: \mathrm{Mor}_{\mathbb{A}}(C, A) \to \mathrm{Mor}_{\mathbb{A}}(B, A), \quad v \mapsto v \diamond f. \end{aligned}$$

These induce a covariant functor $\mathrm{Mor}_{\mathbb{A}}(A, -) : \mathbb{A} \to \mathsf{Set}$,

$$\begin{array}{rll} \mathrm{Mor}(A, -) : & \mathrm{Obj}(\mathbb{A}) \to \mathrm{Obj}(\mathsf{Set}), & B \mapsto \mathrm{Mor}_{\mathbb{A}}(A, B), \\ & \mathrm{Mor}_{\mathbb{A}}(\mathbb{A}) \to \mathrm{Map}, & f \mapsto \mathrm{Mor}_{\mathbb{A}}(A, f), \end{array}$$

and a contravariant functor $\mathrm{Mor}_{\mathbb{A}}(-, A) : \mathbb{A} \to \mathsf{Set}$,

$$\begin{array}{rll} \mathrm{Mor}_{\mathbb{A}}(-, A) : & \mathrm{Obj}(\mathbb{A}) \to \mathrm{Obj}(\mathsf{Set}), & B \mapsto \mathrm{Mor}_{\mathbb{A}}(B, A), \\ & \mathrm{Mor}_{\mathbb{A}}(\mathbb{A}) \to \mathrm{Map}, & f \mapsto \mathrm{Mor}_{\mathbb{A}}(f, A). \end{array}$$

Note that $\mathrm{Mor}_{\mathbb{A}}(A, -)$ always preserves monomorphisms while $\mathrm{Mor}_{\mathbb{A}}(-, A)$ converts epimorphisms into monomorphisms; $\mathrm{Mor}_{\mathbb{A}}(A, f)$ is injective if and only if f is monomorph, $\mathrm{Mor}_{\mathbb{A}}(f, A)$ is injective if and only if f is epimorph.

Properties of the Mor-functors may be used to specify special objects.

2.18. Definitions. An object A in $\mathbb{A}$ is called

generator if $\mathrm{Mor}_{\mathbb{A}}(A, -)$ is faithful;
projective if $\mathrm{Mor}_{\mathbb{A}}(A, -)$ preserves epimorphisms;
cogenerator if $\mathrm{Mor}_{\mathbb{A}}(-, A)$ is faithful;
injective if $\mathrm{Mor}_{\mathbb{A}}(-, A)$ converts monomorphisms to epimorphisms.

Depending on the properties of the category under consideration these objects can be characterised in different ways.

In the category Ab, the integers $\mathbb{Z}$ form a projective generator since for any abelian group G, $\mathrm{Hom}(\mathbb{Z}, G) \simeq G$ and hence $\mathrm{Hom}(\mathbb{Z}, -) : \mathsf{Ab} \to \mathsf{Set}$ is faithful and preserves epimorphisms.

2.19. Adjoint functors. Let $L : \mathbb{A} \to \mathbb{B}$ and $R : \mathbb{B} \to \mathbb{A}$ be (covariant) functors between any categories $\mathbb{A}$, $\mathbb{B}$. The pair (L, R) is called *adjoint* (or an *adjunction*) if any of the two equivalent conditions holds:
(a) there is an isomorphism, natural in $A \in \mathbb{A}$ and $B \in \mathbb{B}$,

$$\varphi_{A,B} : \mathrm{Mor}_{\mathbb{B}}(L(A), B) \to \mathrm{Mor}_{\mathbb{A}}(A, R(B)),$$

that is, any morphisms $f : A \to A'$, $g : B \to B'$ induce commutative diagrams

$$\begin{tikzcd} \mathrm{Mor}_{\mathbb{B}}(L(A'), B) \arrow[r, "\varphi_{A',B}"] \arrow[d, "\mathrm{Mor}(L(f),B)"'] & \mathrm{Mor}_{\mathbb{A}}(A', R(B)) \arrow[d, "\mathrm{Mor}(f,R(B))"'] \\ \mathrm{Mor}_{\mathbb{B}}(L(A), B) \arrow[r, "\varphi_{A,B}"] & \mathrm{Mor}_{\mathbb{A}}(A, R(B)), \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Mor}_{\mathbb{B}}(L(A), B) \arrow[r, "\varphi_{A,B}"] \arrow[d, "\mathrm{Mor}(L(A),g)"] & \mathrm{Mor}_{\mathbb{A}}(A, R(B)) \arrow[d, "\mathrm{Mor}(A,R(g))"] \\ \mathrm{Mor}_{\mathbb{B}}(L(A), B') \arrow[r, "\varphi_{A,B'}"] & \mathrm{Mor}_{\mathbb{A}}(A, R(B')); \end{tikzcd}$$

(b) there are natural transformations $\eta : I_{\mathbb{A}} \to RL$ and $\varepsilon : LR \to I_{\mathbb{B}}$ (*unit* and *counit*) with commutative diagrams (*triangular identities*)

$$\begin{tikzcd} L \arrow[r, "L\eta"] \arrow[dr, "="'] & LRL \arrow[d, "\varepsilon L"] \\ & L, \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "\eta R"] \arrow[dr, "="'] & RLR \arrow[d, "R\varepsilon"] \\ & R. \end{tikzcd}$$

Unit and counit are obtained by

$$\eta_A = \varphi_{L(A),L(A)}(I_{L(A)}), \quad \varepsilon_B = \varphi^{-1}_{R(B),R(B)}(I_{R(B)}),$$

and we have the properties

$$\begin{aligned} \varphi : \quad & L(A) \xrightarrow{f} B \quad \longmapsto \quad A \xrightarrow{\eta_A} RL(A) \xrightarrow{R(f)} R(B), \\ \varphi^{-1} : \quad & A \xrightarrow{g} R(B) \quad \longmapsto \quad L(A) \xrightarrow{L(g)} LR(B) \xrightarrow{\varepsilon_B} B. \end{aligned}$$

If (L, R) form an adjoint pair, then L is called *left adjoint* to R and R is said to be *right adjoint* to L. Adjoints are unique up to natural isomorphisms.

2.20. Properties of adjoint functors. *Let (L, R) be an adjoint pair of functors (as in* 2.19*).*

(1) *L preserves epimorphisms and coproducts.*

(2) *R preserves monomorphisms and coproducts.*

2.21. Properties of unit and counit. *Let (L, R) be an adjoint pair of functors.*

(1) (i) *R is faithful if and only if ε_B is an epimorphism for each $B \in \mathbb{B}$.*

(ii) *R is full if and only if ε_B is a coretraction for each $B \in \mathbb{B}$.*

(iii) *R is full and faithful if and only if ε is an isomorphism.*

(2) (i) *L is faithful if and only if η_A is a monomorphism for each $A \in \mathbb{A}$.*

(ii) *L is full if and only if η_A is a retraction for each $A \in \mathbb{A}$.*

(iii) *L is full and faithful if and only if η is an isomorphism.*

2.22. Natural transformations for adjoints. For two adjunctions (L, R) and $(\widetilde{L}, \widetilde{R})$ between $\mathbb{A}$ and $\mathbb{B}$, with respective units η, $\widetilde{\eta}$ and counits ε, $\widetilde{\varepsilon}$, there is an isomorphism between the natural transformations

$$\begin{aligned} h : \mathrm{Nat}(L, \widetilde{L}) \to \mathrm{Nat}(\widetilde{R}, R), &\quad \alpha \mapsto \bar{\alpha} := R\widetilde{\varepsilon} \circ R\alpha\widetilde{R} \circ \eta\widetilde{R}, \\ h^{-1} : \mathrm{Nat}(\widetilde{R}, R) \to \mathrm{Nat}(L, \widetilde{L}), &\quad \bar{\alpha} \mapsto \alpha := \varepsilon\widetilde{L} \circ L\bar{\alpha}\widetilde{L} \circ L\widetilde{\eta}. \end{aligned}$$

We say that α and $\bar{\alpha}$ are *mates under the adjunctions* (L, R) and $(\widetilde{L}, \widetilde{R})$.

These maps are obtained from the commutative diagram

$$\begin{tikzcd} \mathrm{Mor}_{\mathbb{B}}(\widetilde{L}\widetilde{R}(B), B) \arrow[r, "\simeq"] \arrow[d, "\mathrm{Mor}_{\mathbb{B}}(\alpha_{\widetilde{R}(B)}, B)"'] & \mathrm{Mor}_{\mathbb{A}}(\widetilde{R}(B), \widetilde{R}(B)) \arrow[d, "\mathrm{Mor}_{\mathbb{A}}(B, \bar{\alpha}_B)"] \\ \mathrm{Mor}_{\mathbb{B}}(L\widetilde{R}(B), B) \arrow[r, "\simeq"] & \mathrm{Mor}_{\mathbb{A}}(\widetilde{R}(B), R(B)), \end{tikzcd}$$

by considering the image of $\widetilde{\varepsilon} \in \mathrm{Mor}_{\mathbb{B}}(\widetilde{L}\widetilde{R}(B), B)$.

2.23. The pair ($U \otimes -$, $\mathrm{Hom}(U,-)$). For any abelian group U, the endofunctors

$$U \otimes - : \mathsf{Ab} \to \mathsf{Ab}, \qquad \mathrm{Hom}(U,-) : \mathsf{Ab} \to \mathsf{Ab},$$

form an adjoint pair by the natural isomorphism (see 1.22)

$$\mathrm{Hom}(U \otimes M, N) \to \mathrm{Hom}(M, \mathrm{Hom}(U,N)), \quad \delta \mapsto [m \mapsto \delta(- \otimes m)].$$

We may consider the notion of *tensor product* also in an arbitrary category. For this we need the

2.24. Product of categories. The *product* $\mathbb{A} \times \mathbb{B}$ of two categories $\mathbb{A}$, $\mathbb{B}$ has as objects the ordered pairs (A,B) of objects $A \in \mathrm{Obj}(\mathbb{A})$, $B \in \mathrm{Obj}(\mathbb{B})$, the morphisms sets are

$$\mathrm{Mor}_{\mathbb{A}\times\mathbb{B}}((A,B),(A',B')) = \mathrm{Mor}_{\mathbb{A}}(A,A') \times \mathrm{Mor}_{\mathbb{B}}(B,B'),$$

and componentwise composition

$$(f,g) \diamond (f,g) = (f' \diamond f, g \diamond g).$$

Hereby $I_{(A,B)} = (I_A, I_B)$.

2.25. Monoidal category. A category $\mathbb{A}$ is said to be *monoidal* if there is a functor

$$\oslash : \mathbb{A} \times \mathbb{A} \to \mathbb{A}, \quad (A,B) \mapsto A \oslash B,$$

called *tensor product*, a *unit object* $\mathbb{I} \in \mathbb{A}$, and natural families of isomorphisms in $\mathbb{A}$,

$$\alpha_{A,B,C} : (A \oslash B) \oslash C \to A \oslash (B \oslash C),$$

$$r_A : A \oslash \mathbb{I} \to A, \qquad \ell_A : \mathbb{I} \oslash A \to A,$$

called the *associativity*, *right unit*, and *left unit constraints*, respectively, inducing commutativity of the diagram ....

$$\begin{tikzcd}
 & (A \oslash B) \oslash (C \oslash D) \arrow[dr, "\alpha_{A,B,C\oslash D}"] & \\
((A \oslash B) \oslash C) \oslash D \arrow[ur, "\alpha_{A\oslash B,C,D}"] \arrow[d, "\alpha_{A,B,C}\oslash I_D"'] & & A \oslash (B \oslash (C \oslash D)) \\
(A \oslash (B \oslash C)) \oslash D \arrow[rr, "\alpha_{A,B\oslash C,D}"] & & A \oslash ((B \oslash C) \oslash D) \arrow[u, "I_A\oslash\alpha_{B,C,D}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(A \oslash \mathbb{I}) \oslash B \arrow[rr, "\alpha_{A,\mathbb{I},B}"] \arrow[dr, "r_A\oslash I_B"'] & & A \oslash (\mathbb{I} \oslash B) \arrow[dl, "I_A\oslash\ell_B"] \\
 & A \oslash B &
\end{tikzcd}$$

By MacLane's coherence theorem we may assume that α, r and ℓ are the identity maps.

Clearly, the category Ab is a monoidal category with $\oslash = \otimes$. However, many properties known for $\otimes$ need not hold for $\oslash$ in monoidal categories in general.

3 Rings and modules

Based on the knowledge about abelian groups we introduce associative rings and the category of their modules.

3.1. Rings. A *ring* is an abelian group R with a bilinear map

$$\tilde{m} : R \times R \to R,\ (r,s) \mapsto rs,$$

called *multiplication*, satisfying the associativity condition $(rs)t = r(st)$, for $r,s,t \in R$, and a *unit element* $1_R \in R$, that is $r1_R = r = 1_R r$, for all $r \in R$. Note that the unit can be characterised by the group homomorphism $\eta : \mathbb{Z} \to R$, $1_{\mathbb{Z}} \mapsto 1_R$.

By the property of the tensor product $(R \otimes R, \tau)$, we have the commutative diagram

$$\begin{tikzcd} R \times R \arrow[r, "\tilde{m}"] \arrow[d, "\tau"'] & R \\ R \otimes R \arrow[ur, "m"'] & \end{tikzcd}$$

and this shows that - using the tensor product - rings can be defined by referring to a group homomorphisms m. Then the associativity and unitality conditions can be expressed by commutativity of the diagrams

$$\begin{tikzcd} R \otimes R \otimes R \arrow[r, "I_R \otimes m"] \arrow[d, "m \otimes I_R"'] & R \otimes R \arrow[d, "m"] \\ R \otimes R \arrow[r, "m"] & M \end{tikzcd}, \qquad \begin{tikzcd} \mathbb{Z} \otimes R \arrow[r, "\eta \otimes I_R"] \arrow[dr, "\simeq"'] & R \otimes R \arrow[d, "m"] & R \otimes \mathbb{Z} \arrow[l, "I_R \otimes \eta"'] \arrow[dl, "\simeq"] \\ & R\,. & \end{tikzcd}$$

3.2. Ring morphisms. Given rings R and R', a linear map $f : R \to R'$ is said to be a *ring (homo)morphism* provided the diagrams

$$\begin{tikzcd} R \otimes R \arrow[r, "f \otimes f"] \arrow[d, "m"'] & R' \otimes R' \arrow[d, "m'"'] \\ R \arrow[r, "f"] & R', \end{tikzcd} \qquad \begin{tikzcd} \mathbb{Z} \arrow[d, "\eta"'] \arrow[dr, "\eta'"] & \\ R \arrow[r, "f"'] & R' \end{tikzcd}$$

are commutative, that is, for $a, b \in R$,

$$f(ab) = f(a)f(b), \quad f(1_R) = 1_{R'}.$$

3.3. R-modules. Let (R, m, η) be a ring. A *left R-module* is an abelian group M with a bilinear map $\tilde{\varrho}_M : R \times M \to M$, called the *action*, subject to the associativity and unitality conditions,

$$r(sm) = (rs)m \text{ and } 1_R m = m, \text{ for any } r, s \in R,\ m \in M.$$

Similar to the ring case, the tensor product $(R \otimes M, \tau)$ allows to replace the bilinear map in the definition by the homomorphism ϱ_M in the diagram

$$\begin{tikzcd} R \times M \arrow[r, "\tilde{\varrho}_M"] \arrow[d, "\tau"'] & M \\ R \otimes M \arrow[ur, "\varrho_M"'] & , \end{tikzcd}$$

and the conditions on ϱ_M are expressed by commutativity of the diagrams

$$
\begin{tikzcd}
R \otimes R \otimes M \arrow[r, "I_R \otimes \varrho_M"] \arrow[d, "\mu \otimes I"'] & R \otimes M \arrow[d, "\varrho_M"] \\
R \otimes M \arrow[r, "\varrho_M"] & M ,
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{Z} \otimes M \arrow[r, "\eta \otimes I_M"] \arrow[dr, "\simeq"'] & R \otimes M \arrow[d, "\varrho_M"] \\
 & M .
\end{tikzcd}
$$

Right R-modules are defined symmetrically by interchanging $R \times M$ with $M \times R$ and making the appropriate adaptions.

3.4. R-morphisms. A group homomorphism $g : M \to N$ between left R-modules is called an *R-homomorphism* or *R-linear* provided $g(rm) = rg(m)$, for any $r \in R$, $m \in M$, this means commutativity of the diagram

$$
\begin{tikzcd}
R \otimes M \arrow[r, "I_R \otimes g"] \arrow[d, "\varrho_M"'] & R \otimes N \arrow[d, "\varrho_N"] \\
M \arrow[r, "g"] & N .
\end{tikzcd}
$$

The set of all these maps is denoted by $\mathrm{Hom}_R(M, N)$. With the induced addition this is a subgroup of the abelian group $\mathrm{Hom}(M, N)$ ($=\mathrm{Hom}_{\mathbb{Z}}(M, N)$). It can be characterized as an equaliser

$$
\begin{tikzcd}
\mathrm{Hom}_R(M, N) \arrow[r] & \mathrm{Hom}(M, N) \arrow[r, shift left, "\varrho_N \circ (R \otimes -)"] \arrow[r, shift right, "\mathrm{Hom}(\varrho_M, N)"'] & \mathrm{Hom}(R \otimes M, N).
\end{tikzcd}
$$

The composition of two R-morphisms is again an R-morphism and $\mathrm{End}_R(M) := \mathrm{Hom}_R(M, M)$ is a subring of $\mathrm{End}(M)$.

For R-morphisms of right R-modules the formulas are to be adapted in an obvious way. In the expression $\mathrm{Hom}_R(M, N)$, the subscript R indicates the module structure of the objects M, N which we have in mind. In case of ambiguity we will write $\mathrm{Hom}_{R-}(M, N)$ for left R-module morphisms and $\mathrm{Hom}_{-R}(M, N)$ for right R-module morphisms.

3.5. Category of R-modules. By ${}_R\mathbb{M}$ we denote the category of left R-modules, that is, the objects are left R-modules and the morphisms are the R-module homomorphisms (R-linear maps).

For any abelian group X, $R \otimes X$ is a left R-module by $m \otimes I_X : R \otimes R \otimes X \to R \otimes X$, and this induces the functor

$$R \otimes - : \mathsf{Ab} \to {}_R\mathbb{M}, \quad X \mapsto (R \otimes X, m \otimes I_X),$$

which is left adjoint to the forgetful functor $U_R : {}_R\mathbb{M} \to \mathsf{Ab}$, $(M, \rho_M) \mapsto M$, by the isomorphism

$$\mathrm{Hom}_R(R \otimes X, M) \to \mathrm{Hom}(X, M), \quad f \mapsto f \circ (\eta \otimes I_X),$$

with inverse map

$$X \xrightarrow{h} M \quad \mapsto \quad R \otimes X \xrightarrow{I_R \otimes h} R \otimes M \xrightarrow{\rho_M} M.$$

${}_R\mathbb{M}$ is an abelian category: The zero-object is the 0-module. Products (coproducts) of R-modules are obtained from the product (coproduct) of abelian groups endowed with an R-module structure. Kernels and cokernels of R-linear maps are defined in the same way as for abelian groups. Monomorphisms are the same as injective linear maps and can be considered as equalisers, epimorphisms are just surjective R-linear maps and are coequalisers.

3.6. The Kleisli category of a ring. For the ring R, the *Kleisli category* ${}_R\widetilde{\mathbb{M}}$ is defined as the category whose objects are those of Ab and whose morphisms between X and Y are

$$\mathrm{Mor}_{{}_R\widetilde{\mathbb{M}}}(X,Y) = \mathrm{Hom}(X, R\otimes Y),$$

with composition of $g \in \mathrm{Mor}_{{}_R\widetilde{\mathbb{M}}}(X,Y)$ and $h \in \mathrm{Mor}_{{}_R\widetilde{\mathbb{M}}}(Y,Z)$ given by

$$X \xrightarrow{g} R\otimes Y \xrightarrow{I_R\otimes h} R\otimes R\otimes Z \xrightarrow{m\otimes I_Z} R\otimes Z.$$

There are functors

$$\begin{aligned}
\Phi : \mathsf{Ab} \to {}_R\widetilde{\mathbb{M}}, \quad X \xrightarrow{f} Y \quad &\mapsto \quad X \xrightarrow{\eta\otimes X} R\otimes X \xrightarrow{I_R\otimes f} R\otimes Y,\\
\Psi : {}_R\widetilde{\mathbb{M}} \to {}_R\mathbb{M}, \quad X \xrightarrow{g} R\otimes Y \quad &\mapsto \quad R\otimes X \xrightarrow{I_R\otimes g} R\otimes R\otimes Y \xrightarrow{m\otimes I_Y} R\otimes Y,
\end{aligned}$$

yielding the commutative diagram

$$\begin{tikzcd}
\mathsf{Ab} \arrow[rr, "R\otimes -"] \arrow[dr, "\Phi"'] & & {}_R\mathbb{M} \\
& {}_R\widetilde{\mathbb{M}}. \arrow[ur, "\Psi"'] &
\end{tikzcd}$$

Ψ is a full embedding and hence corestriction yields an equivalence between ${}_R\widetilde{\mathbb{M}}$ and the image of Ψ, a subcategory of ${}_R\mathbb{M}$.

3.7. Category of bimodules. Let R and S be rings. An abelian group M which is a left R-module $\rho_M : R\otimes M \to M$ and a right S-module ${}_M\rho : M\otimes S \to M$, is called an (R,S)-bimodule, if

$$(rm)s = r(ms), \text{ for any } m\in M,\ r\in R, \text{ and } s\in S,$$

that means commutativity of the diagram

$$\begin{tikzcd}
R\otimes M\otimes S \arrow[r, "\rho_M\otimes I"] \arrow[d, "I\otimes {}_M\rho"'] & M\otimes S \arrow[d, "{}_M\rho"] \\
R\otimes M \arrow[r, "\rho_M"] & M.
\end{tikzcd}$$

Morphisms between (R,S)-bimodules M,N are group morphisms which are R-linear as well as S-linear, we denote them by $\mathrm{Hom}_{R,S}(M,N)$.

These data define the category of (R,S)-bimodules which is denoted by ${}_R\mathbb{M}_S$; it is also an abelian category.

3.8. Tensor product of modules. Given a ring R, let ${}_M\rho : M \otimes R \to M$ be a right module, $\rho_N : R \otimes N \to N$ a left module, and G an abelian group. A bilinear map $\beta : M \times N \to G$ is called *R-balanced* if

$$\beta(mr, n) = \beta(m, rn), \text{ for all } m \in M,\ n \in N \text{ and } r \in R.$$

An abelian group T with an R-balanced map $\tau : M \times N \to T$ is called the *tensor product of M and N* if every R-balanced map

$$\beta : M \times N \to G,\ G \text{ an abelian group },$$

can be uniquely factorised over τ, that is, there is a unique homomorphism $\gamma : T \to G$ with commutative diagram

$$\begin{tikzcd} M \times N \arrow[r, "\beta"] \arrow[d, "\tau"'] & G \\ T \arrow[ur, "\gamma"'] & \end{tikzcd} .$$

Such a T is unique up to isomorphism and is usually written $T = M \otimes_R N$. It is determined by the coequaliser diagram

$$M \otimes R \otimes N \underset{{}_M\rho \otimes I_N}{\overset{I_M \otimes \rho_N}{\rightrightarrows}} M \otimes N \longrightarrow M \otimes_R N.$$

3.9. Module structure of tensor products. By construction, the tensor product $M \otimes_R N$ of M_R and ${}_RN$ is only an abelian group. However, if ${}_TM_R$ or ${}_RN_S$ are bimodules, then $M \otimes_R N$ becomes a (T, S)-bimodule by the actions of $t \in T$, $s \in S$,

$$t(\sum m_i \otimes n_i)s = \sum (tm_i) \otimes (n_i s).$$

3.10. Tensor product with *R*. Regarding R as an (R, R)-bimodule, for every R-module ${}_RN$, there is an R-isomorphism

$$\mu_N : R \otimes_R N \to N,\ \sum r_i \otimes n_i \mapsto \sum r_i n_i.$$

The map exists since the map $R \times N \to RN$, $(r, n) \mapsto rn$, is balanced; it obviously has the given properties.

3.11. Associativity of the tensor product. Given rings R, S and three modules M_R, ${}_RN_S$ and ${}_SL$, the tensor products $(M \otimes_R N) \otimes_S L$ and $M \otimes_R (N \otimes_S L)$ can be formed and there is an isomorphism

$$\alpha_{M,N,L} : (M \otimes_R N) \otimes_S L \to M \otimes_R (N \otimes_S L),\ \ (m \otimes n) \otimes l \mapsto m \otimes (n \otimes l)\,.$$

This can be derived from the corresponding property of abelian groups.

3.12. Hom-tensor relation. For rings R, S, let ${}_RP_S$ be an (R, S)-bimodule.

(1) $\mathrm{Hom}_R(P, -) : {}_R\mathbb{M} \to {}_S\mathbb{M}$, $\quad M \mapsto \mathrm{Hom}_R(P, M)$,
$\qquad f \mapsto \mathrm{Hom}_R(P, f)$

is a left exact covariant functor preserving direct products.

(2) $P \otimes_S - : {}_S\mathbb{M} \to {}_R\mathbb{M}, \quad X \mapsto P \otimes_S X,$
$\qquad g \mapsto I_P \otimes g.$
is a right exact covariant functor preserving direct sums.

(3) The two functors are adjoint by the natural isomorphism for $M \in {}_R\mathbb{M}$, $N \in {}_S\mathbb{M}$,

$$\psi'_{M,N} : \mathrm{Hom}_R(P \otimes_S N, M) \to \mathrm{Hom}_S(N, \mathrm{Hom}_R(P, M)), \ \delta \mapsto [n \mapsto \delta(- \otimes n)].$$

It is straightforward to verify that the objects involved have the module structure required for these assertions. The isomorphism $\psi'_{M,N}$ in (3) is the restriction of the corresponding isomorphism $\psi_{M,N}$ for abelian groups (see 1.22). It is just to verify that the specified subsets correspond to each other.

Again properties of related functors can be used to define special objects. So U_R is called *flat* provided $U \otimes_R -$ preserves monomorphisms.

3.13. Tensor product and linear maps. Let M, N be left modules over a ring R. The *dual space* $M = \mathrm{Hom}_R(M, R)$ is a right R-module by the action of $s \in R$ on $f \in M$, $f \cdot s\,(m) := f(m)s$ for all $m \in M$.

(1) The map $M^* \times N \to \mathrm{Hom}_R(M, N)$, $(f, n) \mapsto [m \mapsto f(m)n]$, is (obviously) R-balanced and hence it induces a group homomorphism

$$\vartheta : M^* \otimes_R N \to \mathrm{Hom}_R(M, N).$$

(2) ϑ is an isomorphisms provided M has a finite dual basis (i.e., M_R is finitely generated and projective).

(3) The evaluation $\mathrm{ev} : M^* \times M \to R$, $(f, m) \mapsto f(m)$, is R-balanced and hence factorises over a group homomorphism

$$\overline{\mathrm{ev}} : M^* \otimes_R M \to R.$$

(4) If M has a finite dual basis, we get the linear map

$$\mathrm{End}_R(M) \xrightarrow{\vartheta^{-1}} M^* \otimes_R M \xrightarrow{\overline{\mathrm{ev}}} R.$$

This is the trace map on matrix rings.

3.14. Category of (R, R)-bimodules. For any ring R, the category of (R, R)-bimodules, denoted by ${}_R\mathbb{M}_R$ (see 3.7), is an abelian category. For any two bimodules M, N, $M \otimes_R N$ is again an (R, R)-bimodule and $R \otimes_R M \simeq M$.

Thus $({}_R\mathbb{M}_R, \otimes_R, R)$ is a monoidal category (with unit object R).

3.15. Commutative rings. If R is a commutative ring, left R-modules may be considered as (R, R)-bimodules canonically. Then for R-modules M, N, L, the R-balanced maps $\beta : M \times N \to L$ are just as R-bilinear maps. $M \otimes_R N$ is an R-module and the factoring map $\gamma : M \otimes_R N \to L$ is R-linear.

The category $\mathbb{M}_R$ is monoidal with unit object R and the twist map is defined,

$$\mathsf{tw} : M \otimes_R N \to N \otimes_R M, \quad m \otimes n \mapsto n \otimes m.$$

These observation apply in particular for vector spaces over fields. Over the real numbers $\mathbb{R}$, the *scalar products* are familiar examples of $\mathbb{R}$-bilinear maps.

Recall that a ring R was defined in 3.1 by a bilinear map and modules were considered in the monoidal category Ab. Since over a commutative ring R, ${}_R\mathbb{M}$ is also a monoidal category, these definitions can be generalised to the following situation.

3.16. Algebras and their modules. Let R be a commutative ring. An algebra over R is an R-module A with R-linear maps

$$\mu : A \otimes_R A \to A, \quad \eta : R \to A,$$

the *multiplication* and *unit*, subject to associativity and unitality conditions (see 3.1).

An R-module M is said to be a *left A-module* provided there is an *R-linear map*

$$\rho_M : A \otimes_R M \to M, \quad a \otimes m \mapsto am,$$

subject to associativity and unitality conditions (see 3.3). Morphisms between A-modules (M, ρ_M) and (N, ρ_N) are defined as R-linear maps which are also A-linear and they can be characterised as an equaliser

$$\mathrm{Hom}_A(M,N) \longrightarrow \mathrm{Hom}_R(M,N) \underset{\mathrm{Hom}(\varrho_M,N)}{\overset{\varrho_N\circ(A\otimes_R -)}{\rightrightarrows}} \mathrm{Hom}_R(A \otimes_R M, N).$$

Of course, every algebra A is also a ring and rings may be seen as $\mathbb{Z}$-algebras.

The category of all left A-modules is denoted by ${}_A\mathbb{M}$. It is an abelian category but not monoidal. Similar to 3.5, there is a functor

$$A \otimes_R - : {}_R\mathbb{M} \to {}_A\mathbb{M}, \quad X \mapsto (A \otimes_R X, m \otimes I_X),$$

which is left adjoint to the forgetful functor $U_A : {}_A\mathbb{M} \to {}_R\mathbb{M}$, $(M, \rho_M) \mapsto M$, by the isomorphism

$$\mathrm{Hom}_A(A \otimes_R X, M) \to \mathrm{Hom}_R(X, M), \quad f \mapsto f \circ \eta \otimes_R X.$$

As in 3.6, the Kleisli category ${}_A\widetilde{\mathbb{M}}$ of an R-algebra A is defined as the image of the free functor $A \otimes_R -$ in ${}_A\mathbb{M}$.

4 Coalgebras and comodules

As pointed out in the preceding section, the definition of algebras over a commutative ring is essentially the same as a ring over $\mathbb{Z}$. Nevertheless properties of the base ring can have an influence on the behavior of the modules over algebra. So we gain some generality if we introduce coalgebras over an arbitrary commutative ring. In this section R will be a commutative ring.

4.1. Coalgebras. A *coalgebra* over R is an R-module C with linear maps

$$\Delta : C \to C \otimes_R C, \quad \varepsilon : C \to R,$$

the *comultiplication* and the *counit*, inducing commutative diagrams

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes_R C \arrow[d, "I_C \otimes \Delta"] \\ C \otimes_R C \arrow[r, "\Delta \otimes I_C"] & C \otimes_R C \otimes_R C, \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "I_C"] & C \otimes_R C \arrow[d, "\varepsilon \otimes I_C"] \\ C \otimes_R C \arrow[r, "I_C \otimes \varepsilon"'] & C\,. \end{tikzcd}$$

The coalgebra C is called *cocommutative* if $\Delta = \mathsf{tw} \circ \Delta$

4.2. Sweedler's Σ-notation. For an elementwise description of the maps we use the Σ-*notation*, writing for $c \in C$

$$\Delta(c) = \sum c_{\underline{1}} \otimes c_{\underline{2}},$$

where $c_{\underline{1}}$ and $c_{\underline{2}}$ do not denote single elements but families of elements of C representing the element $\Delta(c)$; they are by no means uniquely determined. With this notation, the coassociativity of Δ is expressed by

$$\sum \Delta(c_{\underline{1}}) \otimes c_{\underline{2}} = \sum c_{\underline{1}\,\underline{1}} \otimes c_{\underline{1}\,\underline{2}} \otimes c_{\underline{2}} = \sum c_{\underline{1}} \otimes c_{\underline{2}\,\underline{1}} \otimes c_{\underline{2}\,\underline{2}} = \sum c_{\underline{1}} \otimes \Delta(c_{\underline{2}}),$$

and hence the notation is often shortened to

$$(\Delta \otimes I_C)\Delta(c) = (I_C \otimes \Delta)\Delta(c) \quad = \quad \sum c_{\underline{1}} \otimes c_{\underline{2}} \otimes c_{\underline{3}}.$$

The conditions for the counit are described by

$$\sum \varepsilon(c_{\underline{1}})c_{\underline{2}} = c = \sum c_{\underline{1}}\varepsilon(c_{\underline{2}}).$$

Cocommutativity is equivalent to the equality $\sum c_{\underline{1}} \otimes c_{\underline{2}} = \sum c_{\underline{2}} \otimes c_{\underline{1}}$.

Coalgebraic structures are closely related to algebraic ones. For example, the module of R-linear maps from a coalgebra C to any R-algebra is an R-algebra (e.g. [15, 1.3]).

4.3. The algebra $\mathrm{Hom}_R(C, A)$. *For any R-linear map $\Delta : C \to C \otimes_R C$ and an R-algebra A, $\mathrm{Hom}_R(C, A)$ is an R-algebra by the* convolution product

$$f * g = \mu \circ (f \otimes g) \circ \Delta, \quad i.e., \quad f * g(c) = \sum f(c_{\underline{1}})g(c_{\underline{2}}),$$

for $f, g \in \mathrm{Hom}_R(C, A)$ and $c \in C$. Furthermore,

(1) Δ *is coassociative if and only if* $\mathrm{Hom}_R(C,A)$ *is an associative* R*-algebra, for any* R*-algebra* A.

(2) C *is cocommutative if and only if* $\mathrm{Hom}_R(C,A)$ *is a commutative* R*-algebra, for any commutative* R*-algebra* A.

(3) C *has a counit if and only if* $\mathrm{Hom}_R(C,A)$ *has a unit, for all* R*-algebras* A *with a unit.*

4.4. Coalgebra morphisms. Given R-coalgebras (C,Δ,ε) and $(C',\Delta',\varepsilon')$, an R-linear map $f : C \to C'$ is said to be a *coalgebra morphism* provided the diagrams

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta"'] & C' \arrow[d, "\Delta'"] \\ C \otimes_R C \arrow[r, "f\otimes f"] & C' \otimes_R C' , \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "f"] \arrow[dr, "\varepsilon"'] & C' \arrow[d, "\varepsilon'"] \\ & R \end{tikzcd}$$

are commutative, that is $\Delta' \circ f = (f \otimes f) \circ \Delta$ and $\varepsilon' \circ f = \varepsilon$, i.e., for any $c \in C$,

$$\sum f(c_{\underline{1}}) \otimes f(c_{\underline{2}}) = \sum f(c)_{\underline{1}} \otimes f(c)_{\underline{2}}, \quad \varepsilon'(f(c)) = \varepsilon(c).$$

4.5. Coproduct of coalgebras. For a family $\{C_\lambda\}_\Lambda$ of R-coalgebras, put $C = \bigoplus_\Lambda C_\lambda$, the coproduct in $\mathbf{M}_R$, $i_\lambda : C_\lambda \to C$ the canonical inclusions, and consider the R-linear maps

$$C_\lambda \xrightarrow{\Delta_\lambda} C_\lambda \otimes C_\lambda \subset C \otimes C, \quad \varepsilon : C_\lambda \to R.$$

By the properties of coproducts of R-modules there exist unique maps

$$\Delta : C \to C \otimes_R C \text{ with } \Delta \circ i_\lambda = \Delta_\lambda, \quad \varepsilon : C \to R \text{ with } \varepsilon \circ i_\lambda = \varepsilon_\lambda.$$

(C,Δ,ε) is called the *coproduct* (or *direct sum*) of the coalgebras C_λ. It is obvious that the $i_\lambda : C_\lambda \to C$ are coalgebra morphisms.

The coproduct constructed above is the *coproduct in the category of coalgebras*, that is, objects are coalgebras and morphisms are coalgebra morphisms.

The definition of comodules over coalgebras is derived from modules over algebras by reversing the arrows.

4.6. C-comodules. A *right C-comodule* is an R-module M with an R-linear map

$$\rho^M : M \longrightarrow M \otimes_R C,$$

inducing commutative diagrams

$$\begin{tikzcd} M \arrow[r, "\varrho^M"] \arrow[d, "\varrho^M"'] & M \otimes_R C \arrow[d, "I_M\otimes\Delta"] \\ M \otimes_R C \arrow[r, "\varrho^M\otimes I_C"] & M \otimes_R C \otimes_R C, \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\varrho^M"] \arrow[dr, "I_M"'] & M \otimes_R C \arrow[d, "I_M\otimes\varepsilon"] \\ & M. \end{tikzcd}$$

For the value of ϱ^M on elements $m \in M$ we write

$$\varrho^M(m) = \sum m_{\underline{0}} \otimes m_{\underline{1}}.$$

Then coassociativity and counitality are expressed by the equalities

$$\sum \varrho^M(m_{\underline{0}}) \otimes m_{\underline{1}} = \sum m_{\underline{0}} \otimes \Delta(m_{\underline{1}}), \quad m = \sum m_{\underline{0}}\varepsilon(m_{\underline{1}}).$$

In view of the first of these equations the notation is often shortened to

$$(I_M \otimes \Delta) \circ \varrho^M(m) = \sum m_{\underline{0}} \otimes m_{\underline{1}} \otimes m_{\underline{2}}.$$

Note that the elements with subscript 0 are in M while all the elements with positive subscripts are in C.

An R-module with a coassociative and counital right coaction is called a *C-comodule*.

4.7. C-comodule morphisms. Given left C-comodules M and N, a *C-comodule morphism* is an R-linear map $f : M \to N$ with a commutative diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\rho^M"'] & N \arrow[d, "\rho^M"] \\ M \otimes_R C \arrow[r, "f \otimes I"] & N \otimes_R C, \end{tikzcd}$$

that is $\varrho^N \circ f = (f \otimes I_C) \circ \varrho^M$, and for any $m \in M$,

$$\sum f(m)_{\underline{0}} \otimes f(m)_{\underline{1}} = \sum f(m_{\underline{0}}) \otimes m_{\underline{1}}.$$

The set $\mathrm{Hom}^C(M,N)$ of C-morphisms from M to N is an R-module and by definition that it is characterized as an equaliser

$$\mathrm{Hom}^C(M,N) \longrightarrow \mathrm{Hom}_R(M,N) \underset{\mathrm{Hom}_R(M,\varrho^N)}{\overset{(-\otimes I_C)\circ\varrho^M}{\rightrightarrows}} \mathrm{Hom}(M, N \otimes_R C).$$

The composition of two C-morphisms is again a C-morphism and the endomorphisms $\mathrm{End}^C(M) := \mathrm{Hom}^C(M,M)$ form a subring of $\mathrm{End}_R(M)$.

The right C-comodules together with the comodule homomorphisms form the *category of right C-comodules* which we denote by $\mathbb{M}^C$. As mentioned before, $\mathbb{M}^C$ is an additive category.

4.8. Coproducts in $\mathbf{M}^C$. Let $\{M_\lambda, \varrho^M_\lambda\}_\Lambda$ be a family of C-comodules. Put $M = \bigoplus_\Lambda M_\lambda$, the coproduct in $\mathbf{M}_R$, $i_\lambda : M_\lambda \to M$ the canonical inclusions, and consider the linear maps

$$M_\lambda \xrightarrow{\varrho^M_\lambda} M_\lambda \otimes_R C \subset M \otimes_R C.$$

By the properties of coproducts of R-modules, there exists a unique R-linear map

$$\varrho^M : M \to M \otimes_R C, \text{ such that } \varrho^M \circ i_\lambda = \varrho^M_\lambda,$$

and this map is coassociative and counital (since all the ϱ^M_λ are).

(M, i_λ) is the coproduct of the $\{M_\lambda\}_\Lambda$ in $\mathbb{M}^C$.

4.9. Kernels and cokernels in $\mathbf{M}^C$. Let $f : M \to N$ be a morphism in $\mathbf{M}^C$. The cokernel g of f in $\mathbf{M}_R$ yields the exact commutative diagram

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "\varrho^M"'] & N \arrow[r, "g"] \arrow[d, "\varrho^N"] & L \arrow[r] & 0 \\
M \otimes_R C \arrow[r, "f \otimes I_C"] & N \otimes_R C \arrow[r, "g \otimes I_C"] & L \otimes_R C \arrow[r] & 0,
\end{tikzcd}$$

which can be completed commutatively in $\mathbf{M}_R$ by some $\varrho^L : L \to L \otimes_R C$. It is a minor exercise to show that this is a coassociative and counital coaction ([15, 3.5]).

This shows that cokernels exist in the category $\mathbf{M}^C$.

Dually, for the kernel h of f in $\mathbf{M}_R$ there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "h"] & M \arrow[r, "f"] \arrow[d, "\varrho^M"'] & N \arrow[d, "\varrho^N"] \\
0 \arrow[r] & K \otimes_R C \arrow[r, "h \otimes I_C"] & M \otimes_R C \arrow[r, "f \otimes I_C"] & N \otimes_R C \, ,
\end{tikzcd}$$

where the top sequence is always exact while the bottom sequence is exact provided f is C-pure as an R-morphism. If this is the case, the diagram can be extended commutatively by an R-linear map $\varrho^K : K \to K \otimes_R C$ which can be shown to be coassociative and counital.

Thus, for example, kernels exist in $\mathbb{M}^C$ provided C is flat as an R-module.

For modules N, M, N is said to be *(sub-)generated* by M if N is a (submodule of a) homomorphic image of a direct sum of copies of M. If every module in a category $\mathbb{A}$ is subgenerated by M, then M is called a *subgenerator* of $\mathbb{A}$. A similar terminology is applied for comodules.

4.10. The category $\mathbb{M}^C$. For any R-module X, $X \otimes_R C$ is a right C-comodule with coaction induced by Δ. This yields a functor

$$- \otimes_R C : {}_R\mathbb{M} \to \mathbb{M}^C, \quad X \mapsto (X \otimes_R C, I_X \otimes \Delta),$$

which is right adjoint to the forgetful functor $U^C : \mathbb{M}^C \to {}_R\mathbb{M}$, $(M, \rho^M) \mapsto M$, by the isomorphism

$$\mathrm{Hom}^C(M, X \otimes_R C) \to \mathrm{Hom}_R(M, X), \quad f \mapsto (I_X \otimes \varepsilon) \circ f.$$

(1) *$\mathbb{M}^C$ is an additive category with coproducts and cokernels.*

(2) *For any generator $P \in {}_R\mathbb{M}$, $C \otimes_R P$ is a subgenerator in $\mathbb{M}^C$, in particular, C is a subgenerator in ${}^C\mathbb{M}$.*

(3) *For any monomorphism $f : X \to Y$ in ${}_R\mathbb{M}$, $f \otimes_R I_C : X \otimes_R C \to Y \otimes_R C$ is a monomorphism in $\mathbb{M}^C$.*

(4) *For any family X_λ of R-modules, $(\prod_\Lambda X_\lambda) \otimes_R C$ is the product of the C-comodules $X_\lambda \otimes_R C$.*

Note that in $\mathbb{M}^C$ monomorphisms need not be injective maps. This is a consequence of the fact that the functor $- \otimes_R C : {}_R\mathbb{M} \to {}_R\mathbb{M}$ need not preserve monomorphisms. However, functors which have a left adjoint always preserve monomorphisms and products and hence so does $- \otimes_R C : {}_R\mathbb{M} \to \mathbb{M}^C$. This explains properties (3) and (4).

The R-module structure of C has a strong influence on the properties of $\mathbb{M}^C$, [15, 3.14].

4.11. Proposition. *For an R-coalgebra C, the following are equivalent:*

(a) *C is flat as an R-module;*

(b) *every monomorphism in $\mathbb{M}^C$ is injective;*

(c) *the forgetful functor $\mathbb{M}^C \to {}_R\mathbb{M}$ respects monomorphisms.*

In this case, $\mathbb{M}^C$ is an abelian category and

(i) *for any cogenerator $Q \in {}_R\mathbb{M}$, $Q \otimes_R C$ is a cogenerator in $\mathbb{M}^C$;*

(ii) *for any injective $X \in {}_R\mathbb{M}$, $X \otimes_R C$ is injective in ${}^C\mathbb{M}$.*

4.12. The Kleisli category of a coalgebra. For an R-coalgebra C, the *Kleisli category* $\widetilde{\mathbb{M}}^C$ is defined as the category whose objects are those of ${}_R\mathbb{M}$ and whose morphisms between X and Y are

$$\mathrm{Mor}_{\widetilde{\mathbb{M}}^C}(X,Y) = \mathrm{Hom}_R(X \otimes_R C, Y),$$

with composition of $g \in \mathrm{Mor}_{\widetilde{\mathbb{M}}^C}(X,Y)$ and $h \in \mathrm{Mor}_{\widetilde{\mathbb{M}}^C}(Y,Z)$ given by

$$X \otimes_R C \xrightarrow{I_X \otimes \Delta} X \otimes_R C \otimes_R C \xrightarrow{g \otimes I_C} Y \otimes C \xrightarrow{g} Z.$$

The identity in $\mathrm{Mor}_{\widetilde{\mathbb{M}}^C}(X,X)$ is given by $I_X \otimes \varepsilon : X \otimes_R C \to X$. There are functors

$$\Phi : {}_R\mathbb{M} \to \widetilde{\mathbb{M}}^C, \quad X \xrightarrow{f} Y \quad \mapsto \quad X \otimes_R C \xrightarrow{f \otimes I_C} Y \otimes_R C \xrightarrow{I_Y \otimes \varepsilon} Y,$$

$$\Psi : \widetilde{\mathbb{M}}^C \to \mathbb{M}^C, \quad X \otimes_R C \xrightarrow{g} Y \quad \mapsto \quad X \otimes_R C \xrightarrow{I_X \otimes \Delta} X \otimes_R C \otimes_R C \xrightarrow{g \otimes I_C} Y \otimes_R C,$$

yielding the commutative diagram

$$\begin{tikzcd}
{}_R\mathbb{M} \arrow[rr, "- \otimes_R C"] \arrow[dr, "\Phi"'] & & \mathbb{M}^C \\
& \widetilde{\mathbb{M}}^C . \arrow[ur, "\Psi"'] &
\end{tikzcd}$$

Ψ is a full embedding and hence corestriction yields an equivalence between $\widetilde{\mathbb{M}}^C$ and the image of Ψ, a subcategory of $\mathbb{M}^C$.

Left comodules and related notions over a coalgebra C are defined symmetrically to the right handed case. The category of left C-comodules is denoted by $\mathbb{M}$.

4.13. Cotensor product of comodules. For $M \in \mathbf{M}^C$ and $N \in {}^C\mathbf{M}$, the *cotensor product* $M \otimes^C N$ is defined as an equaliser in $\mathbf{M}_R$,

$$M \otimes^C N \longrightarrow M \otimes_R N \underset{I_M \otimes {}^N\varrho}{\overset{\varrho^M \otimes I_N}{\rightrightarrows}} M \otimes_R C \otimes_R N.$$

4.14. Cotensor product of comodule morphisms. Let $f : M \to M'$, $g : N \to N'$ be morphisms of right, resp. left, C-comodules. Then there is a unique R-linear map,

$$f \otimes^C g : \; M \otimes^C N \longrightarrow M' \otimes^C N',$$

yielding a commutative diagram

$$\begin{tikzcd}
M \otimes^C N \arrow[r] \arrow[d, "f\otimes^C g"'] & M \otimes_R N \arrow[r, shift left, "\varrho^M \otimes I_N"] \arrow[r, shift right, "I_M \otimes {}^N\varrho"'] \arrow[d, "f\otimes g"'] & M \otimes_R C \otimes_R N \arrow[d, "f\otimes I_C \otimes g"] \\
M' \otimes^C N' \arrow[r] & M' \otimes_R N' \arrow[r, shift left, "\varrho^{M'} \otimes I_{N'}"] \arrow[r, shift right, "I_{M'} \otimes {}^{N'}\varrho"'] & M' \otimes_R C \otimes_R N' .
\end{tikzcd}$$

4.15. The cotensor functor. *For any $M \in \mathbf{M}^C$ there is a covariant functor*

$$\begin{array}{llcl} M \otimes^C - : {}^C\mathbf{M} \to \mathbf{M}_R, & N & \mapsto & M \otimes^C N, \\ & f : N \to N' & \mapsto & I_M \otimes^C f : M \otimes^C N \to M \otimes^C N'. \end{array}$$

Similarly, every left C-comodule N yields a functor $- \otimes^C N : \mathbf{M}^C \to \mathbf{M}_R$.

5 Monads and comonads

Let R be a commutative ring. Recall that, for any R-algebra (A, μ, η), we have the tensor functor $A \otimes_R - : {}_R\mathbb{M} \to {}_R\mathbb{M}$. By the isomorphism

$$(A \otimes_R A) \otimes_R M \simeq A \otimes_R (A \otimes_R M),$$

the functor $(A \otimes_R A) \otimes_R -$ can be seen as composition of $A \otimes_R -$ with $A \otimes_R -$ and the multiplication induces a natural transformation

$$\mu \otimes - : A \otimes_R A \otimes_R - \to A \otimes_R -.$$

A-modules are defined by R-linear maps $A \otimes_R M \to M$ (A-actions). This leads to the following definition for endofunctors on arbitrary categories.

Let $\mathbb{A}$ denote any category.

5.1. Actions of endofunctors and their morphisms. Given an endofunctor $F : \mathbb{A} \to \mathbb{A}$, an *$F$-action* on an object $A \in \mathrm{Obj}(\mathbb{A})$ is a morphism

$$\varrho_A : F(A) \to A \text{ in } \mathbb{A}.$$

We say a morphism $f : A \to A'$ in $\mathbb{A}$ respects F-actions on objects if it induces commutativity of the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\varrho_A"'] & F(A') \arrow[d, "\varrho_{A'}"] \\ A \arrow[r, "f"] & A'. \end{tikzcd}$$

Obviously, the composition of two morphisms respecting F-actions is again of this type and thus we have the *category of objects with F-actions*.

Note that F-actions are defined for *any* functors $F : \mathbb{A} \to \mathbb{A}$. So, for example, any R-module N gives rise to a functor $N \otimes_R - : {}_R\mathbb{M} \to {}_R\mathbb{M}$. For algebras A we usually require associativity and put more conditions on the A-actions. Similarly, we define a product on endofunctors satisfying associativity conditions and consider their actions compatible with those.

5.2. Monads. A *monad* on $\mathbb{A}$ is a triple $\mathbf{F} = (F, \mu, \eta)$, where $F : \mathbb{A} \to \mathbb{A}$ is a functor and

$$\mu : FF \to F, \quad \eta : I_{\mathbb{A}} \to F,$$

are natural transformations with commutative diagrams

$$\begin{tikzcd} FFF \arrow[r, "\mu_F"] \arrow[d, "F\mu"'] & FF \arrow[d, "\mu"] \\ FF \arrow[r, "\mu"'] & F, \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "\eta F"] \arrow[d, "F\eta"'] \arrow[dr, "="] & FF \arrow[d, "\mu"] \\ FF \arrow[r, "\mu"'] & F. \end{tikzcd}$$

Thus $A \otimes_R - : {}_R\mathbb{M} \to {}_R\mathbb{M}$ is a monad if and only if A is an associative R-algebra with unit.

5.3. Morphisms of monads. Given two monads $\mathbb{F} = (F, \mu, \eta)$ and $\mathbb{F}' = (F', \mu', \eta')$ on $\mathbb{A}$, a natural transformation $\alpha : F \to F'$ is called a *morphism of monads* if the following induced diagrams commute:

$$\begin{tikzcd} FF \arrow[r, "\alpha\alpha"] \arrow[d, "\mu"'] & F'F' \arrow[d, "\mu'"] \\ F \arrow[r, "\alpha"] & F', \end{tikzcd} \qquad \begin{tikzcd} I_{\mathbb{A}} \arrow[r, "\eta"] \arrow[dr, "\eta'"'] & F \arrow[d, "\alpha"] \\ & F'. \end{tikzcd}$$

The definitions of A-modules and their morphisms are generalised to

5.4. Modules for monads. Given a monad $\mathbf{F} = (F, \mu, \eta)$ on a category $\mathbb{A}$, an $\mathbf{F}$-*module* is an object $A \in \mathrm{Obj}(\mathbb{A})$ with an F-action $\varrho_A : F(A) \to A$ inducing commutative diagrams

$$\begin{tikzcd} FF(A) \arrow[r, "\mu_A"] \arrow[d, "F\varrho_A"'] & F(A) \arrow[d, "\varrho_A"] \\ F(A) \arrow[r, "\varrho_A"'] & A\,, \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "I_A"'] & F(A) \arrow[d, "\varrho_A"] \\ & A\,. \end{tikzcd}$$

The $\mathbf{F}$-modules together with the morphisms preserving the $\mathbf{F}$-action form a category which we denote by $\mathbb{A}_F$.

In particular, for any $A \in \mathrm{Obj}(\mathbb{A})$, $F(A)$ is an $\mathbf{F}$-module by

$$\mu_A : FF(A) \to F(A).$$

This yields the *free functor*

$$\phi_F : \mathbb{A} \to \mathbb{A}_F, \quad A \mapsto (F(A), \mu_A),$$

which is *left adjoint* to the forgetful functor $U_F : \mathbb{A}_F \to \mathbb{A}$ by the bijection

$$\mathrm{Mor}_{\mathbb{A}_F}(F(A), B) \to \mathrm{Mor}_{\mathbb{A}}(A, U_F(B)), \quad f \mapsto f \circ \eta_A.$$

The construction of the Kleisli category for a ring can readily be transferred to

5.5. The Kleisli category of a monad. For a monad F on a category $\mathbb{A}$, the *Kleisli category* $\widetilde{\mathbb{A}}_F$ is defined as the category whose objects are those of $\mathbb{A}$ and whose morphisms between X and Y are

$$\mathrm{Mor}_{\widetilde{\mathbb{A}}_F}(X, Y) = \mathrm{Mor}_{\mathbb{A}}(X, F(Y)),$$

with composition of $g \in \mathrm{Mor}_{\widetilde{\mathbb{A}}_F}(X, Y)$ and $h \in \mathrm{Mor}_{\widetilde{\mathbb{A}}_F}(Y, Z)$ given by

$$X \xrightarrow{g} F(Y) \xrightarrow{F(h)} FF(Z) \xrightarrow{\mu_Z} F(Z).$$

There are functors

$$\Phi : \mathbb{A} \to \widetilde{\mathbb{A}}_F, \quad X \xrightarrow{f} Y \quad \mapsto \quad X \xrightarrow{\eta_X} F(X) \xrightarrow{F(f)} F(Y),$$
$$\Psi : \widetilde{\mathbb{A}}_F \to \mathbb{A}_F, \quad X \xrightarrow{g} F(Y) \quad \mapsto \quad F(X) \xrightarrow{F(g)} FF(Y) \xrightarrow{\mu_Y} F(Y),$$

yielding the commutative diagram

$$\begin{tikzcd} \mathbb{A} \arrow[rr, "\phi_F"] \arrow[dr, "\Phi"'] & & \mathbb{A}_F \\ & \widetilde{\mathbb{A}}_F . \arrow[ur, "\Psi"'] & \end{tikzcd}$$

Ψ is a full embedding and hence corestriction yields an equivalence between $\widetilde{\mathbb{A}}_F$ and the image of Ψ, the subcategory of $\mathbb{A}_F$ generated by all free F-modules.

Although the modules for a monad $\mathbf{F}$ are fairly close to the modules over an associative unital algebra, there are many properties of the category of A-modules which are not shared by all $\mathbf{F}$-modules. This depends on the special properties of $A \otimes_R -$: it is a right exact functor which preserves direct sums and cokernels. This implies, for example, that $A \otimes_R R$, the image of R, is a (projective) generator in ${}_A\mathbb{M}$.

The notions of coalgebras and comodules as considered in 4.1 and 4.6 are the blueprint for the introduction of comonads and their comodules.

5.6. G-coactions and their morphisms. For an endofunctor $G : \mathbb{A} \to \mathbb{A}$, a *$G$-coaction* on an $A \in \mathrm{Obj}(\mathbb{A})$ is a morphism

$$\varrho^A : A \to G(A) \ \text{ in } \mathbb{A}.$$

We say a morphism $f : A \to A'$ between objects with *G-coactions* respects the coaction if it induces commutativity of the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\varrho^A"'] & A' \arrow[d, "\varrho^{A'}"] \\ G(A) \arrow[r, "G(f)"] & G(A') . \end{tikzcd}$$

With this morphisms, the objects with G-coactions form a category.

5.7. Comonads. A *comonad* on a category $\mathbb{A}$ is a triple $\mathbf{G} = (G, \delta, \varepsilon)$, where $G : \mathbb{A} \to \mathbb{A}$ is a functor and

$$\delta : G \to GG, \quad \varepsilon : G \to I_{\mathbb{A}},$$

are natural transformations with commuting diagrams

$$\begin{tikzcd} G \arrow[r, "\delta"] \arrow[d, "\delta"'] & GG \arrow[d, "G\delta"] \\ GG \arrow[r, "\delta_G"] & GGG , \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\delta"] \arrow[d, "\delta"'] \arrow[dr, "="] & GG \arrow[d, "\varepsilon_G"] \\ GG \arrow[r, "G\varepsilon"'] & G . \end{tikzcd}$$

5.8. Morphisms of comonads. Given two comonads $\mathbb{G} = (G, \delta, \varepsilon)$ and $\mathbb{G}' = (G', \delta', \varepsilon')$, a natural transformation $\beta : G \to G'$ is called a *morphism of comonads* if it induces commutativity of the diagrams

$$\begin{tikzcd} G \arrow[r, "\beta"] \arrow[d, "\delta"'] & G' \arrow[d, "\delta'"] \\ GG \arrow[r, "\beta\beta"] & G'G' , \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\beta"] \arrow[dr, "\varepsilon"'] & G' \arrow[d, "\varepsilon'"] \\ & I_{\mathbb{A}}. \end{tikzcd}$$

5.9. Comodules for comonads. A **G**-comodule is an object $A \in \mathrm{Obj}(\mathbb{A})$ with a G-coaction $\varrho^A : A \to G(A)$ in $\mathbb{A}$ and commutative diagrams

$$\begin{tikzcd} A \arrow[r, "\varrho^A"] \arrow[d, "\varrho^A"'] & G(A) \arrow[d, "\delta_A"] \\ G(A) \arrow[r, "G\varrho^A"] & GG(A) , \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\varrho^A"] \arrow[dr, "I_A"'] & G(A) \arrow[d, "\varepsilon_A"] \\ & A. \end{tikzcd}$$

The **G**-comodules together with the morphisms preserving the G-coaction form a category which we denote by $\mathbb{A}^G$.

For any object $A \in \mathrm{Obj}(\mathbb{A})$, $G(A)$ is a comodule canonically and thus we have the *free functor*

$$\phi^G : \mathbb{A} \to \mathbb{A}^G, \quad A \mapsto (G(A), \delta_A),$$

which is right adjoint to the forgetful functor $U^G : \mathbb{A}^G \to \mathbb{A}$ by the bijection

$$\mathrm{Mor}_{\mathbb{A}^G}(B, G(A)) \to \mathrm{Mor}_{\mathbb{A}}(U^G(B), A), \quad f \mapsto \varepsilon_A \circ f.$$

The construction of the Kleisli category for coalgebras is the blueprint for

5.10. The Kleisli category of a comonad. For a comonad G on a category $\mathbb{A}$, the *Kleisli category* $\widetilde{\mathbb{A}}^G$ is defined as the category whose objects are those of $\mathbb{A}$ and whose morphisms between X and Y are

$$\mathrm{Mor}_{\widetilde{\mathbb{A}}^G}(X, Y) = \mathrm{Hom}_R(G(X), Y),$$

with composition of $g \in \mathrm{Mor}_{\widetilde{\mathbb{A}}^G}(X, Y)$ and $h \in \mathrm{Mor}_{\widetilde{\mathbb{A}}^G}(Y, Z)$ given by

$$G(X) \xrightarrow{\delta_X} GG(X) \xrightarrow{G(g)} G(Y) \xrightarrow{h} Z.$$

The identity in $\mathrm{Mor}_{\widetilde{\mathbb{A}}^G}(X, X)$ is given by $\varepsilon_X : G(X) \to X$. There are functors

$$\Phi : \mathbb{A} \to \widetilde{\mathbb{A}}^G, \quad X \xrightarrow{f} Y \quad \mapsto \quad G(X) \xrightarrow{G(f)} G(Y) \xrightarrow{\varepsilon_Y} Y,$$

$$\Psi : \widetilde{\mathbb{A}}^G \to \mathbb{A}^G, \quad G(X) \xrightarrow{g} Y \quad \mapsto \quad G(X) \xrightarrow{\delta_X} GG(X) \xrightarrow{G(g)} G(Y),$$

yielding the commutative diagram

$$\begin{tikzcd} \mathbb{A} \arrow[rr, "\phi^G"] \arrow[dr, "\Phi"'] & & \mathbb{A}^G \\ & \widetilde{\mathbb{A}}^G. \arrow[ur, "\Psi"'] & \end{tikzcd}$$

Ψ is a full embedding and hence corestriction yields an equivalence between $\widetilde{\mathbb{A}}^G$ and the image of Ψ, the subcategory of $\mathbb{A}^G$ generated by the cofree G-comodules.

Properties of these categories depend heavily on the properties of the comonad $\mathbf{G}$.

Recall that an endofunctor $F : \mathbb{A} \to \mathbb{A}$ is left adjoint to an endofunctor $G : \mathbb{A} \to \mathbb{A}$ provided there is a functorial isomorphism

$$\mathrm{Mor}_{\mathbb{A}}(F(X), Y) \xrightarrow{\simeq} \mathrm{Mor}_{\mathbb{A}}(X, G(Y)).$$

5.11. Right adjoints of monads. Let (F, G) be an adjoint pair of endofunctors on $\mathbb{A}$ and assume (F, m, e) to be a monad. The FF is again an endofunctor with right adjoint GG by the isomorphism

$$\mathrm{Mor}_{\mathbb{A}}(FF(X), Y) \simeq \mathrm{Mor}_{\mathbb{A}}(F(X), G(Y)) \simeq \mathrm{Mor}_{\mathbb{A}}(X, GG(Y)),$$

with unit and counit

$$\bar{\varepsilon} :\ FFGG \xrightarrow{F\varepsilon G} FG \xrightarrow{\varepsilon} I_{\mathbb{B}}, , \quad \bar{e} :\ I_{\mathbb{A}} \xrightarrow{\eta} GF \xrightarrow{G\eta F} GGFF\ .$$

By 2.22, the natural transformation $m : FF \to F$ has a mate $\delta := \bar{m} : G \to GG$, and $e : I_{\mathbb{A}} \to F$ has a mate $\varepsilon := \bar{e} : G \to I_{\mathbb{A}}$ under the given adjunctions and we obtain the commutative diagrams

$$\begin{tikzcd}
\mathrm{Mor}_{\mathbb{A}}(F(X), Y) \arrow[r, "\simeq"] \arrow[d, "\mathrm{Mor}_{\mathbb{A}}(m,Y)"'] & \mathrm{Mor}_{\mathbb{A}}(X, G(Y)) \arrow[d, "\mathrm{Mor}_{\mathbb{A}}(X,\delta)"] \\
\mathrm{Mor}_{\mathbb{A}}(FF(X), Y) \arrow[r, "\simeq"] & \mathrm{Mor}_{\mathbb{A}}(X, GG(Y))
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Mor}_{\mathbb{A}}(F(X), Y) \arrow[r, "\simeq"] \arrow[d, "\mathrm{Mor}_{\mathbb{A}}(e,Y)"'] & \mathrm{Mor}_{\mathbb{A}}(X, G(Y)) \arrow[d, "\mathrm{Mor}_{\mathbb{A}}(X,\varepsilon)"] \\
\mathrm{Mor}_{\mathbb{A}}(X, Y) \arrow[r, "="] & \mathrm{Mor}_{\mathbb{A}}(X, Y).
\end{tikzcd}$$

Thus the right adjoint functor G has a comonad structure (G, δ, ε).

It may come as a surprise that, in the situation considered in 5.11, the category of F-modules is isomorphic to the category of G-comodules.

5.12. Modules and comodules for an adjoint pair. *Let (F, G) be an adjoint pair of functors with unit $\eta : I_{\mathbb{A}} \to GF$ and counit $\varepsilon : FG \to I_{\mathbb{A}}$. Let F be a monad and consider G with the induced comonad structure. Then the (Eilenberg-Moore) categories $\mathbb{A}_F$ and $\mathbb{A}^G$ are isomorphic to each other.*

Proof. The isomorphism is given by a functor leaving objects and morphisms unchanged and turning F-module structure maps to G-comodule structure maps and vice versa. An F-module $\varrho_A : F(A) \to A$ induces a morphism

$$A \xrightarrow{\eta A} GF(A) \xrightarrow{G\varrho_A} G(A),$$

making A an G-comodule. Similarly, a comodule $\varrho^A : A \to G(A)$ induces

$$F(A) \xrightarrow{F\varrho^A} FG(A) \xrightarrow{\varepsilon_A} A,$$

defining an F-module structure on A.

It remains to show that module morphisms are also comodule morphisms. An F-module morphism $f : A' \to A$ yields a commutative diagram

$$\begin{tikzcd} F(A') \arrow[r, "Ff"] \arrow[d, "\varrho_{A'}"'] & F(A) \arrow[d, "\varrho_A"] \\ A' \arrow[r, "f"] & A \end{tikzcd} ,$$

from which we obtain the commutative diagram

$$\begin{tikzcd} A' \arrow[r, "f"] \arrow[d, "\eta A'"'] & A \arrow[d, "\eta A"] \\ GF(A') \arrow[r, "GFf"] \arrow[d, "G\varrho_{A'}"'] & GF(A) \arrow[d, "G\varrho_A"] \\ G(A') \arrow[r, "Gf"] & G(A). \end{tikzcd}$$

Commutativity of the outer rectangle shows that f is also a G-comodule morphism. Similarly one proves that G-comodule morphisms are also F-module morphisms. $\square$

5.13. Right adjoints of comonads. Let (F, G) be an adjoint pair of endofunctors on $\mathbb{A}$ and assume (F, δ, ε) to be a comonad. With similar arguments as in 5.11 we deduce from 2.22 that the mates $m = \bar{\delta} : GG \to G$ and $e = \bar{\varepsilon} : I_{\mathbb{A}} \to G$ lead to a monad (G, m, e).

Thus for an adjoint pair (F, G), F is a comonad if and only if G is a monad.

Related to such a pair we have the categories $\mathbb{A}^F$ and $\mathbb{A}_G$. Unlike to the monad-comonad case considered in 5.12, here we do not get an equivalence between $\mathbb{A}^F$ and $\mathbb{A}_G$. In this case the Kleisli categories $\widetilde{\mathbb{A}}^F$ and $\widetilde{\mathbb{A}}_G$ are isomorphic to each other. This follows by the canonical isomorphisms for $A, A' \in \mathbb{A}$,

$$\begin{aligned} \mathrm{Mor}_{\widetilde{\mathbb{A}}^F}(A, A') &\simeq \mathrm{Mor}_{\mathbb{A}^F}(\phi^F A, \phi^F A') \simeq \mathrm{Mor}_{\mathbb{A}}(F(A), A') \\ &\simeq \mathrm{Mor}_{\mathbb{A}}(A, G(A')) \simeq \mathrm{Mor}_{\mathbb{A}_G}(\phi_G A, \phi_G A') \simeq \mathrm{Mor}_{\widetilde{\mathbb{A}}_G}(A, A'). \end{aligned}$$

The interest in monads and comonads arose originally from the fact that any adjunction produces a monad and a comonad.

5.14. Comonads and monads induced by adjunctions. *Consider an adjoint pair of functors $F : \mathbb{A} \to \mathbb{B}$, $G : \mathbb{B} \to \mathbb{A}$ with unit $\eta : I_{\mathbb{A}} \to GF$ and counit $\varepsilon : FG \to I_{\mathbb{B}}$.*

(1) (i) $\mathbf{T} = (T = GF, G\varepsilon F, \eta)$ *is a monad on* $\mathbb{A}$;

(ii) *there is a (comparison) functor* $\widehat{G} : \mathbb{B} \to \mathbb{A}_T,\ B \mapsto (G(B), G\varepsilon_B)$;

(iii) *there is a functor* $\widetilde{F} : \widetilde{\mathbb{A}}_T \to \mathbb{B}$,

$$A \xrightarrow{g} T(A') \;\mapsto\; F(A) \xrightarrow{F(g)} FGF(A') \xrightarrow{\varepsilon F(A')} F(A').$$

(2) (i) $\mathbf{S} = (S = FG, F\eta G, \varepsilon)$ *is a comonad on* $\mathbb{B}$;

(ii) *there is a (comparison) functor* $\widehat{F} : \mathbb{A} \to \mathbb{B}^S,\; A \mapsto (F(A), F\eta_A)$;

(iii) *there is a functor* $\widetilde{G} : \widetilde{\mathbb{A}}^S \to \mathbb{B}$,

$$S(A) \xrightarrow{g} A' \;\mapsto\; G(A) \xrightarrow{\eta F(A)} GFG(A) \xrightarrow{G(g)} G(A').$$

The functors defined above, called comparison functors, *yield commutative diagrams*

$$\begin{tikzcd}
 & \widetilde{\mathbb{A}}_T \arrow[d, "\widetilde{F}"] \arrow[dr, "U_T"] & \\
\mathbb{A} \arrow[ur, "\phi_T"] \arrow[r, "F"] \arrow[dr, "\phi_T"'] & \mathbb{B} \arrow[r, "G"] \arrow[d, "\widehat{G}"] & \mathbb{A} \\
 & \mathbb{A}_T \arrow[ur, "U_T"'] &
\end{tikzcd} , \qquad
\begin{tikzcd}
 & \widetilde{\mathbb{B}}^S \arrow[d, "\widetilde{G}"] \arrow[dr, "U^S"] & \\
\mathbb{B} \arrow[ur, "\phi^S"] \arrow[r, "G"] \arrow[dr, "\phi^S"'] & \mathbb{A} \arrow[r, "F"] \arrow[d, "\widehat{F}"] & \mathbb{B} \\
 & \mathbb{B}^S \arrow[ur, "U^S"'] &
\end{tikzcd} .$$

Proof. (1.i) The unitality conditions are expressed by the diagram

$$\begin{tikzcd}
GF \arrow[r, "\eta GF"] \arrow[dr, "="'] & GFGF \arrow[d, "G\varepsilon F"] & GF \arrow[l, "GF\eta"'] \arrow[dl, "="] \\
 & GF &
\end{tikzcd} ,$$

whose commutativity follows from the triangular identities (see 2.19).

Naturality of ε implies commutativity of

$$\begin{tikzcd}
FGFG \arrow[r, "\varepsilon FG"] \arrow[d, "FG\varepsilon"'] & FG \arrow[d, "\varepsilon"] \\
FG \arrow[r, "\varepsilon"] & I_{\mathbb{B}},
\end{tikzcd}$$

and this can be extended to the commutative diagram

$$\begin{tikzcd}
GFGFGF \arrow[r, "G\varepsilon FGF"] \arrow[d, "GFG\varepsilon F"'] & GFGF \arrow[d, "G\varepsilon F"] \\
GFGF \arrow[r, "G\varepsilon F"] & GF,
\end{tikzcd}$$

showing associativity of the product $G\varepsilon F : GFGF \to GF$.

(1.ii) We show that for any $B \in \mathbb{B}$, $G\varepsilon_B : GFG(B) \to G(B)$ defines a GF-module. Consider again the first square in the proof (1.i). Action of G from the left and application to B yields the commutative diagram

$$\begin{tikzcd} GFGFG(B) \arrow[r, "G\varepsilon FG_B"] \arrow[d, "GFG\varepsilon_B"'] & GFG(B) \arrow[d, "G\varepsilon_B"] \\ GFG(B) \arrow[r, "G\varepsilon_B"] & G(B). \end{tikzcd}$$

This proves the associativity condition for the GF-module $G(B)$. Unitality follows from the triangular identities (2.19). Again by naturality of ε, for any $f \in \mathbb{B}$, $G(f)$ is a GF-module morphism.

(1.iii) Clearly morphisms in $\widetilde{\mathbb{A}}_T$ are taken to morphisms in $\mathbb{B}$. It remains to show that the assignment respects composition of morphisms $A \xrightarrow{g} T(A')$ and $A' \xrightarrow{h} T(A'')$. For this consider the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(g)"] & FGF(A') \arrow[r, "FT(h)"] \arrow[d, "\varepsilon F(A')"'] & FTT(A'') \arrow[r, "FG\varepsilon F(A'')"] \arrow[d, "\varepsilon FGF(A'')"] & FGF(A'') \arrow[d, "\varepsilon F(A'')"] \\ & F(A') \arrow[r, "F(h)"] & FGF(A'') \arrow[r, "\varepsilon F(A'')"] & F(A'') \end{tikzcd}$$

in which all partial diagrams are commutative. This shows that composition of morphisms is preserved by our assignment.

The proofs for (2) follow by similar arguments and commutativity of the diagrams is seen by direct verification. □

6 Monads and comonads in module categories

In this section we study the notions considered in the preceding section for arbitrary categories in the category of R-modules, R a commutative ring.

6.1. Adjoint endofunctors on ${}_R\mathbb{M}$. For any R-module M, the endofunctors $-\otimes_R M$ and $\mathrm{Hom}_R(M,-)$ form an adjoint pair by the isomorphism

$$\psi^M_{X,Y} : \mathrm{Hom}_R(X \otimes_R M, Y) \to \mathrm{Hom}_R(X, \mathrm{Hom}_R(M,Y)), \quad \gamma \mapsto [x \mapsto \gamma(x \otimes -)],$$

with unit $\nu : I \to \mathrm{Hom}_R(M, - \otimes_R M)$ and counit $\varepsilon : \mathrm{Hom}(M,-) \otimes_R M \to I$.

First we apply 5.11 and 5.12 to algebras.

6.2. Proposition. *For an R-module A, the following are equivalent:*

(a) *A is an R-algebra;*

(b) *$- \otimes_R A : {}_R\mathbb{M} \to {}_R\mathbb{M}$ is a monad;*

(c) *$\mathrm{Hom}_R(A,-) : {}_R\mathbb{M} \to {}_R\mathbb{M}$ is a comonad.*

In this case the categories $\mathbb{M}_A = \mathbb{M}_{-\otimes_R A}$ and $\mathbb{M}^{\mathrm{Hom}(A,-)}$ are isomorphic.

Proof. The assertions are special cases of 5.11 and 5.12. The algebra structure on A, $m : A \otimes_R A \to A$ and $e : R \to A$, correspond to the comonad structure on $\mathrm{Hom}_R(A,-)$,

$$\mathrm{Hom}_R(A,-) \xrightarrow{m^*} \mathrm{Hom}_R(A \otimes_R A, -) \xrightarrow{\psi^A_{A,-}} \mathrm{Hom}_R(A, \mathrm{Hom}_R(A,-)),$$
$$\mathrm{Hom}_R(A,-) \xrightarrow{e^*} \mathrm{Hom}_R(R,-),$$

where $m^* = \mathrm{Hom}_R(m,-)$ and $e^* = \mathrm{Hom}_R(e,-)$.

The right A-module structure $\rho_N : N \otimes_R A \to N$ induces a $\mathrm{Hom}_R(A,-)$-comodule structure on N,

$$\widehat{\rho_N} : N \xrightarrow{\nu_N} \mathrm{Hom}_R(A, N \otimes_R A) \xrightarrow{\mathrm{Hom}(A,\rho_N)} \mathrm{Hom}_R(A,N).$$

On the other hand, a comodule structure $\rho^N : N \to \mathrm{Hom}_R(A,N)$ induces an A-module structure on N,

$$\widetilde{\rho^N} : N \otimes_R A \xrightarrow{\rho^N \otimes I} \mathrm{Hom}_R(A,N) \otimes_R A \xrightarrow{\varepsilon_N} N.$$

The equivalence between module and comodule category is given by

$$\mathbb{M}_{A\otimes_R -} \to \mathbb{M}^{\mathrm{Hom}(A,-)}, \quad (M, \rho_N) \mapsto (M, \widehat{\rho_N}),$$

keeping the morphisms unchanged. □

6.3. Value at R. Since R is a generator in ${}_R\mathbb{M}$ and $-\otimes_R A$ preserves direct sums and epimorphisms, the functor $- \otimes_R A : {}_R\mathbb{M} \to {}_R\mathbb{M}$ is fully determined by the value at R, that is by $R \otimes_R A \simeq A$. Similarly, a natural transformation $\varphi : - \otimes_R A \to - \otimes_R B$

between tensor functors is of the form $\varphi_R \otimes_R -$, where $\varphi_R : A \to B$ is an R-linear map.

In general, $\mathrm{Hom}_R(A,-)$ is not determined by $A^* = \mathrm{Hom}_R(A,R)$, unless it preserves direct sums and epimorphisms, that is, unless A is a finitely generated and projective R-module. However, $\mathrm{Hom}_R(A,-)$ is determined by $\mathrm{Hom}_R(A,Q)$ for any cogenerator $Q \in {}_R\mathbb{M}$ since it is left exact and preserves direct products. For a natural transformation $\psi : \mathrm{Hom}_R(A,-) \to \mathrm{Hom}_R(B,-)$ between Hom functors, it follows by the Yoneda Lemma that $\psi = \mathrm{Hom}_R(\psi_R,-)$, where $\psi_R : B \to A$ is an R-linear map.

Now let (C,Δ,ε) be an R-coalgebra, 4.1. For properties of the category $\mathbb{M}^C$ of right C-comodules we refer to 4.10 and 4.11.

In view of the adjointness of the endofunctors $C \otimes_R -$ and $\mathrm{Hom}_R(C,-)$, the latter has a monad structure by 5.13.

6.4. Proposition. *For an R-module C, the following are equivalent:*

(a) *C is an R-coalgebra;*

(b) *$- \otimes_R C : {}_R\mathbb{M} \to {}_R\mathbb{M}$ has a comonad structure;*

(c) *$\mathrm{Hom}_R(C,-) : {}_R\mathbb{M} \to {}_R\mathbb{M}$ has a monad structure.*

Hereby the coalgebra structure maps $\Delta : C \to C \otimes_R C$, $\varepsilon : C \to R$, correspond to the monad structure

$$\mathrm{Hom}_R(C,\mathrm{Hom}_R(C,-)) \xrightarrow{\simeq} \mathrm{Hom}_R(C \otimes_R C,-) \xrightarrow{\Delta^*} \mathrm{Hom}_R(C,-),$$
$$\mathrm{Hom}_R(R,-) \xrightarrow{\varepsilon^*} \mathrm{Hom}_R(C,-),$$

where $\Delta^* = \mathrm{Hom}_R(\Delta,-)$, $\varepsilon^* = \mathrm{Hom}_R(\varepsilon,-)$, and the isomorphism is from 6.1.

Henceforth we will write $[C,-] = \mathrm{Hom}_R(C,-)$ for short. Applying 5.4 we have the definition of $[C,-]$-modules and the category $\mathbb{M}_{[C,-]}$ with the (free) functor

$$\phi_{[C,-]} : {}_R\mathbb{M} \to \mathbb{M}_{[C,-]}$$

being left adjoint to the forgetful functor $U_{[C,-]} : \mathbb{M}_{[C,-]} \to {}_R\mathbb{M}$. Following [21], the $[C,-]$-modules are also called *C-contramodules*.

By left exactness of the functor $[C,-]$ we obtain (with similar arguments as in the comodule case) special properties of $[C,-]$-modules.

6.5. The category $\mathbb{M}_{[C,-]}$. *Let C be an R-coalgebra.*

(1) *$\mathbb{M}_{[C,-]}$ is an additive category with products and kernels.*

(2) *For any $M \in \mathbb{M}_{[C,-]}$, $\mathrm{Hom}_{[C,-]}([C,R],M) \simeq M$.*

(3) *For any epimorphism $h : X \to Y$ in ${}_R\mathbb{M}$, $[C,h] : [C,X] \to [C,Y]$ is an epimorphism (not necessarily surjective) in $\mathbb{M}_{[C,-]}$.*

(4) *For any family X_λ of R-modules, $[C,\bigoplus_\Lambda X_\lambda]$ is the coproduct of the $[C,-]$-modules $[C,X_\lambda]$.*

For C-comodules it is not always clear if the kernel of a C-comodule morphism is a subcomodule. Here we need some condition to make the cokernel of a $[C,-]$-module again a $[C,-]$-module.

6.6. Proposition. *Let C be an R-algebra. If C_R is projective, then for any $[C,-]$-submodule $K \subset M$, the R-module M/K is a $[C,-]$-module.*

Proof. By assumption we have the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & {[C,K]} \arrow[r] \arrow[d] & {[C,M]} \arrow[r] \arrow[d] & {[C,M/K]} \arrow[r] & 0 \\
0 \arrow[r] & K \arrow[r] & M \arrow[r] & M/K \arrow[r] & 0,
\end{tikzcd}$$

which can be completed commutatively by an R-morphism $[C, M/K] \to M/K$. It is routine to check that this provides M/K with a $[C,-]$-module structure. □

As observed above, in $\mathbb{M}_{[C,-]}$ epimorphisms need not be surjective maps. This is the case under the following conditions.

6.7. Proposition. *For an R-coalgebra C, the following are equivalent:*

(a) *C_R is projective;*

(b) *every epimorphism in $\mathbb{M}_{[C,-]}$ is surjective;*

(c) *the forgetful functor $\mathbb{M}_{[C,-]} \to \mathbb{M}_R$ respects epimorphisms.*

In this case, $\mathbb{M}_{[C,-]}$ is an abelian category and

(i) *for any generator $P \in {}_R\mathbb{M}$, $[C,P]$ is a generator in $\mathbb{M}_{[C,-]}$;*

(ii) *for any projective $Y \in {}_R\mathbb{M}$, $[C,Y]$ is projective in $\mathbb{M}_{[C,-]}$.*

Proof. (b)⇒(a) For any epimorphism $f : K \to L$ in ${}_R\mathbb{M}$, $\mathrm{Hom}(C,f) : \mathrm{Hom}_R(C,K) \to \mathrm{Hom}_R(C,K)$ is an epimorphism in $\mathbb{M}_{[C,-]}$ and hence surjective by (b). This means that C_R is projective.

(c)⇒(a) is shown with a similar argument.

(a)⇒(c) Assume C_R to be projective and consider an epimorphism $f : M \to N$ in $\mathbb{M}_{[C,-]}$. Then the cokernel $h : N \to N/f(M)$ is a morphism in $\mathbb{M}_{[C,-]}$ and $0f = hf = 0$. Since f is an epimorphism this implies $N = f(M)$. □

Recall that for any R-coalgebra C, the dual space $C^* = \mathrm{Hom}_R(C,R)$ has a ring structure by the convolution product for $f, g \in C^*$, $f * g = (g \otimes f) \circ \Delta$ (convention opposite to [15, 1.3]). The relation between C-comodules and modules over the dual ring of C is well studied (e.g. [15, Section 19]). Now it follows from the general observations in 5.11 and 5.13 that a coalgebra C gives rise to two comonads and two monads on ${}_R\mathbb{M}$,

$$- \otimes_R C \text{ and } \mathrm{Hom}_R(C^*,-), \qquad - \otimes_R C^* \text{ and } \mathrm{Hom}_R(C,-).$$

Relation between those is given by morphisms which are well-known in module theory - but usually not viewed under this aspect.

6.8. The comonads $- \otimes_R C$ and $[C^*,-]$. *The comonad morphism*

$$\alpha : - \otimes_R C \to \mathrm{Hom}_R(C^*,-), \quad c \otimes - \mapsto [f \mapsto f(c)-],$$

yields a faithful functor

$$\begin{array}{rcl} G_\alpha : \quad \mathbb{M}^C & \longrightarrow & \mathbb{M}^{[C^*,-]} \simeq \mathbb{M}_{C^*}\ , \\ N \xrightarrow{\varrho^N} N \otimes_R C & \longmapsto & N \xrightarrow{\varrho^N} N \otimes_R C \xrightarrow{\alpha_N} \mathrm{Hom}_R(C^*, N), \end{array}$$

and the following are equivalent:

(a) α_N *is injective for each* $N \in {}_R\mathbb{M}$*;*

(b) G_α *is a full functor;*

(c) C *is a locally projective* R*-module.*

If these conditions are satisfied, $\mathbb{M}^C$ can be identified with $\sigma[C_{C^*}]$, the full subcategory of $\mathbb{M}_{C^*}$ subgenerated by C. This follows from the fact that C is a subgenerator in $\mathbb{M}^C$.

6.9. C-comodules and C^*-modules. The relation between C-comodules and C^*-modules can be given directly by observing that (e.g. [15, 4.1])

(i) for any $M \in \mathbb{M}^C$ is a (unital) right C^*-module by

$$\rightharpoonup : M \otimes_R C^* \to M, \quad m \otimes f \mapsto (I_M \otimes f) \circ \varrho^M(m) = \sum m_{\underline{0}} f(m_{\underline{1}}).$$

(ii) any morphism $h : M \to N$ in $\mathbb{M}^C$ is a C^*-module morphism, that is,

$$\mathrm{Hom}^C(M, N) \subset \mathrm{Hom}_{C^*}(M, N);$$

(iii) this yields a faithful functor from $\mathbf{M}^C$ to $\sigma[C_{C^*}]$.

(iv) this is an equivalence $\mathbf{M}^C \to \sigma[C_{C^*}]$ if and only if α (in 6.8) is a monomorphisms (α-condition).

Similar to 6.8, C-contramodules can be related to C^*-modules.

6.10. The monads $[C, -]$ and $- \otimes_R C^*$. *The monad morphism*

$$\beta : - \otimes_R C^* \to \mathrm{Hom}_R(C, -), \quad - \otimes f \mapsto [c \mapsto f(c)-],$$

yields a faithful functor

$$\begin{array}{rcl} F_\beta : \quad \mathbb{M}_{[C,-]} & \longrightarrow & \mathbb{M}_{C^*}\ , \\ \mathrm{Hom}_R(C, M) \xrightarrow{\varrho_M} M & \longmapsto & M \otimes_R C^* \xrightarrow{\beta_M} \mathrm{Hom}_R(C, M) \xrightarrow{\varrho_M} M, \end{array}$$

and the following are equivalent:

(a) β *is surjective for all* $M \in {}_R\mathbb{M}$*;*

(b) F_β *is an isomorphism;*

(c) C *is a finitely generated and projective* R*-module.*

In general, C is not a $[C, -]$-module and $[C, R]$ is not a C-comodule. In fact, $[C, R] \in {}^C\mathbb{M}$ holds provided C is finitely generated and projective as an R-module.

We now consider the relationship between the categories of C-comodules and C-contramodules. As indicated in 5.13, the two categories need not be equivalent. This becomes evident from their properties listed in 6.5 and 4.10, respectively. However they are related by an adjoint pair of functors.

6.11. Correspondence of categories. Let C be an R-coalgebra. Then

$$\mathrm{Hom}^C(C,-) : \mathbb{M}^C \to \mathbb{M}_{[C,-]}, \quad M \mapsto \mathrm{Hom}^C(C,M),$$

is a functor which has a left adjoint given by the *contratensor product* defined for $(N,\rho) \in \mathbb{M}^C$ and $(M,\alpha) \in \mathbb{M}_{[C,-]}$ as the coequaliser

$$\begin{tikzcd} \mathrm{Hom}_R(C,M) \otimes_R N \arrow[r, shift left, "h"] \arrow[r, shift right, "\alpha \otimes_R I_N"'] & M \otimes_R N \arrow[r] & M \otimes^c N, \end{tikzcd}$$

for the map $h : f \otimes_R n \mapsto (I_N \otimes_R f) \circ \rho(n)$.

This follows by Dubuc's Adjoint Triangle Theorem applied to the diagram

$$\begin{tikzcd} \mathbb{M}^C \arrow[rr, "\mathrm{Hom}^C(C,-)"] \arrow[dr, shift left, "\mathrm{Hom}^C(C,-)"] & & \mathbb{M}_{[C,-]} \arrow[dl, shift right, "U_{[C,-]}"'] \\ & \mathbb{M}_A \arrow[ul, shift left, "C \otimes_R -"] \arrow[ur, shift right, "\phi_{[C,-]}"'] & \end{tikzcd}$$

by the existence of the functor in the upper line and the fact that $\mathbb{M}^C$ has coequalisers (see Dubuc [19, Appendix], Positselski [42], [8, Section 4]).

6.12. Equivalence of subcategories. Take any $X \in {}_R\mathbb{M}$. Since $- \otimes_R C$ is right adjoint to the forgetful functor, $\mathrm{Hom}^C(C,-)$ takes

$$X \otimes_R C \mapsto \mathrm{Hom}^C(C, X \otimes_R C) \simeq \mathrm{Hom}_R(C,X).$$

On the other hand, the functor $- \otimes_{[C,-]} C$ transfers

$$\mathrm{Hom}_R(C,X) \mapsto \mathrm{Hom}_R(C,X) \otimes_{[C,-]} C \simeq X \otimes_R C.$$

This shows that the full subcategory of $\mathbb{M}^C$, whose objects are of the form $X \otimes_R C$, is equivalent to the full subcategory of $\mathbb{M}_{[C,-]}$, with objects $\mathrm{Hom}_R(C,X)$, $X \in {}_R\mathbb{M}$ (*Kleisli subcategories*, e.g. [8]).

7 Tensor product of algebras

7.1. R-rings. An (R,R)-bimodule A is called an *R-ring* if there are (R,R)-homomorphisms $m_A : A \otimes_R A \to A$ (multiplication) and $\iota_A : R \to A$, $\iota_A(1_R) = 1_A$ (unit), subject to associativity and unitality conditions. Thus A is just an associative unital ring with ring homomorphism $\iota_A : R \to A$.

$A \otimes_R -$ is an endofunctor on ${}_R\mathbb{M}$ and is left adjoint to the endofunctor $\mathrm{Hom}_R(A,-)$. We will often write $\otimes$ instead of $\otimes_R$ for short (in diagrams).

7.2. Tensor product of R-rings. Given two R-rings (A, m_A, ι_A) and (B, m_B, ι_B), the tensor product $A \otimes_R B$ is again an R-bimodule. An (R,R)-bilinear map

$$\varphi : B \otimes_R A \to A \otimes_R B,$$

allows for the definition of a product on $A \otimes_R B$,

$$\mathrm{m}_\varphi : A \otimes B \otimes A \otimes B \xrightarrow{I_A \otimes \varphi \otimes I_B} A \otimes A \otimes B \otimes B \xrightarrow{m_A \otimes m_B} A \otimes B,$$

that is, $\mathrm{m}_\varphi = (m_A \otimes m_B) \circ (I_A \otimes \varphi \otimes I_B)$. For computational reasons we write

$$\varphi(b \otimes a) = \sum a_\varphi \otimes b_\varphi = \sum\nolimits_r a_r \otimes b_r,$$

and thus for $a, c \in A$, $b, d \in B$,

$$(a \otimes b) \cdot_\varphi (c \otimes d) =: \mathrm{m}_\varphi(a \otimes b, c \otimes d) = \sum a c_\varphi \otimes b_\varphi d.$$

If $(A \otimes_R B, m_{AB}, 1_A \otimes 1_B)$ is a unital associative R-ring, it is called a *smash product* of A and B and we denote it by $A \otimes_\varphi B$. For this certain properties of φ are required: The conditions

$$1_A \otimes b = (1_A \otimes b) \cdot_\varphi (1_A \otimes 1_B) = \varphi(b \otimes 1_A), \tag{7.1}$$

$$a \otimes 1_B = (a \otimes 1_B) \cdot_\varphi (1_A \otimes 1_B) = \varphi(1_B \otimes a), \tag{7.2}$$

are called *normality conditions*. Applying these the associativity conditions

$$(1_A \otimes b) \cdot_\varphi ((a \otimes 1_B) \cdot_\varphi (c \otimes 1_B)) = ((1_A \otimes b) \cdot_\varphi (a \otimes 1_B)) \cdot_\varphi (c \otimes 1_B), \tag{7.3}$$

$$((1_A \otimes b) \cdot_\varphi (1_A \otimes d)) \cdot_\varphi (a \otimes 1_B) = (1_A \otimes b) \cdot_\varphi ((1_A \otimes d) \cdot_\varphi (a \otimes 1_B)), \tag{7.4}$$

can be written as

$$\varphi(b \otimes ac) = (m_A \otimes I)(I \otimes \varphi)(\varphi(b \otimes a) \otimes c), \tag{7.5}$$

$$\varphi(bd \otimes a) = (I \otimes m_B)(\varphi \otimes I)(b \otimes \varphi(d, a)). \tag{7.6}$$

On the other hand, multiplying (7.3) by $1_A \otimes d$ from the right shows that $c \otimes d$ associates with the two other elements in (7.3). Continuing with similar arguments one can show that the normality conditions together with 7.3 and 7.4 imply associativity of m_φ. Expressing these conditions by commutativity of diagrams we have shown:

7.3. Algebra entwinings. *Consider R-rings A, B. For an (R,R)-bilinear map $\varphi : B \otimes_R A \to A \otimes_R B$, the following are equivalent:*

(a) $(A \otimes_R B, \mathrm{m}_\varphi, 1_A \otimes 1_B)$ *is an associative unital R-ring;*

(b) $A \otimes_R -$ *induces a monad* $\widehat{A\otimes_R}- : {}_B\mathbb{M} \to {}_B\mathbb{M}$, *by*

$$B \otimes M \xrightarrow{\varrho} M \quad \longmapsto \quad B \otimes A \otimes M \xrightarrow{\varphi\otimes I_M} A \otimes B \otimes M \xrightarrow{I_A\otimes\varrho} A \otimes M;$$

(c) $A\otimes_\varphi B \otimes_B - : {}_B\mathbb{M} \to {}_B\mathbb{M}$ *is a monad with commutative diagram*

$$\begin{tikzcd}
{}_B\mathbb{M} \arrow[r, "A\otimes_\varphi B\otimes_B-"] \arrow[d, "U_B"'] & {}_B\mathbb{M} \arrow[d, "U_B"] \\
{}_R\mathbb{M} \arrow[r, "A\otimes_R-"] & {}_R\mathbb{M},
\end{tikzcd}$$

where U_B *is the forgetful functor;*

(d) φ *induces functors* $\widehat{A\otimes_R}-$ *and* $\widetilde{B\otimes_R}-$ *with commutative diagrams*

$$\begin{tikzcd}
{}_B\mathbb{M} \arrow[r, "\widehat{A\otimes_R}-"] \arrow[d, "U_B"'] & {}_B\mathbb{M} \arrow[d, "U_B"] \\
{}_R\mathbb{M} \arrow[r, "A\otimes_R-"] & {}_R\mathbb{M},
\end{tikzcd}
\qquad
\begin{tikzcd}
{}_R\mathbb{M} \arrow[r, "B\otimes_R-"] \arrow[d, "\phi_A"'] & {}_R\mathbb{M} \arrow[d, "\phi_A"] \\
{}_A\widetilde{\mathbb{M}} \arrow[r, "\widetilde{B\otimes_R}-"] & {}_A\widetilde{\mathbb{M}}
\end{tikzcd}$$

where U_B *is the forgetful and* ϕ_A *the free functor and* ${}_A\widetilde{\mathbb{M}}$ *is the Kleisli category;*

(e) φ *induces commutativity of the diagrams*

$$\begin{tikzcd}
B\otimes B\otimes A \arrow[rr, "m_B\otimes I_A"] \arrow[d, "I_B\otimes\varphi"'] & & B\otimes A \arrow[d, "\varphi"] \\
B\otimes A\otimes B \arrow[r, "\varphi\otimes I_B"] & A\otimes B\otimes B \arrow[r, "I_A\otimes m_B"] & A\otimes B,
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\iota_B\otimes I_A"] \arrow[dr, "I_A\otimes\iota_B"'] & B\otimes A \arrow[d, "\varphi"] \\
& A\otimes B,
\end{tikzcd} \tag{7.7}$$

$$\begin{tikzcd}
B\otimes A\otimes A \arrow[rr, "I_B\otimes m_A"] \arrow[d, "\varphi\otimes I_A"'] & & B\otimes A \arrow[d, "\varphi"] \\
A\otimes B\otimes A \arrow[r, "I_A\otimes\varphi"] & A\otimes A\otimes B \arrow[r, "m_A\otimes I_B"] & A\otimes B,
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "I_B\otimes\iota_A"] \arrow[dr, "\iota_A\otimes I_B"'] & B\otimes A \arrow[d, "\varphi"] \\
& A\otimes B.
\end{tikzcd} \tag{7.8}$$

If these conditions ares satisfied, we call φ - more precisely $(A, B)_\varphi$ - an *algebra entwining.*

Proof. (a)$\Leftrightarrow$(e) follows from the considerations in 7.2.

(e)$\Rightarrow$(b) Consider a B-module $\varrho : B \otimes_R M \to M$. To show associativity of the B-action on $A \otimes_R M$, we need commutativity of the diagram

$$\begin{tikzcd}
B\otimes B\otimes A\otimes M \arrow[r, "I_B\otimes\varphi\otimes I_M"] \arrow[dd, "m_B\otimes I_A\otimes I_M"'] & B\otimes A\otimes B\otimes M \arrow[r, "I_B\otimes I_A\otimes\varrho"] \arrow[d, "\varphi\otimes I_B\otimes I_M"'] & B\otimes A\otimes M \arrow[d, "\varphi\otimes I_M"] \\
& A\otimes B\otimes B\otimes M \arrow[r, "I_A\otimes I_B\otimes\varrho"] \arrow[d, "I_A\otimes m_B\otimes I_M"'] & A\otimes B\otimes M \arrow[d, "I_A\otimes\varrho"] \\
B\otimes A\otimes M \arrow[r, "\varphi\otimes I_M"] & A\otimes B\otimes M \arrow[r, "I_A\otimes\varrho"] & A\otimes M.
\end{tikzcd}$$

Indeed, the left hand rectangle is commutative by commutativity of the rectangle in (7.7), the right upper rectangle is commutative by functoriality of the tensor product, and the right lower diagram is commutative since ϱ is a B-homomorphism.

The triangle in (7.7) implies commutativity of the upper triangle in the diagram

$$\begin{tikzcd}
A \otimes M \arrow[r, "\iota_B \otimes I_A \otimes I_M"] \arrow[d, "="'] \arrow[dr, "I_A \otimes \iota_B \otimes I_M"] & B \otimes A \otimes M \arrow[d, "\varphi \otimes I_M"] \\
A \otimes M & A \otimes B \otimes M, \arrow[l, "I_A \otimes \varrho"]
\end{tikzcd}$$

while the lower triangle is commutative by unitality of the B-module M. This shows unitality of the $A \otimes_R M$ as B-module.

Any B-module morphism $f : M \to N$ is transferred to $I_A \otimes f : A \otimes_R M \to A \otimes_R N$ and to show that this is a B-module morphisms we need commutativity of the diagram

$$\begin{tikzcd}
B \otimes A \otimes M \arrow[r, "I_B \otimes I_A \otimes f"] \arrow[d, "\varphi \otimes I_M"'] & B \otimes A \otimes N \arrow[d, "\varphi \otimes I_N"] \\
A \otimes B \otimes M \arrow[r, "I_A \otimes I_B \otimes f"] \arrow[d, "I_A \otimes \varrho_M"'] & A \otimes B \otimes N \arrow[d, "I_A \otimes \varrho_N"] \\
A \otimes M \arrow[r, "I_A \otimes f"] & A \otimes N,
\end{tikzcd}$$

which is obvious since the lower rectangle is commutative since f is a morphism in ${}_B\mathbb{M}$ and the upper rectangle is commutative by functoriality of the tensor product.

To show that $\widehat{A \otimes_r} -$ is a monad on ${}_B\mathbb{M}$, we need that $m_A \otimes_R M : A \otimes_R A \otimes M \to A \otimes M$ and $\iota_A : R \otimes_R M \to A \otimes_R M$ give morphisms in ${}_B\mathbb{M}$ for any $\varrho : B \otimes_R M \to M$ in ${}_B\mathbb{M}$. The first condition follows from the diagram

$$\begin{tikzcd}
B \otimes A \otimes A \otimes M \arrow[r, "\varphi \otimes I \otimes I"] \arrow[d, "I_B \otimes m_A \otimes I_M"'] & A \otimes B \otimes A \otimes M \arrow[r, "I \otimes \varphi \otimes I"] & A \otimes A \otimes B \otimes M \arrow[r, "I \otimes I \otimes \varrho"] \arrow[d, "m_A \otimes I_B \otimes I_M"] & A \otimes A \otimes M \arrow[d, "m_A \otimes I_M"] \\
B \otimes A \otimes M \arrow[rr, "\varphi \otimes I_M"] & & A \otimes B \otimes M \arrow[r, "I_A \otimes \varrho"] & A \otimes M,
\end{tikzcd}$$

in which the left rectangle is commutative by commutativity of the rectangle in (7.8) and the right square is commutative by naturality of $m_A \otimes -$.

For the condition on ι_A, consider the diagram

$$\begin{tikzcd}
B \otimes M \arrow[r, "I_B \otimes \iota_A \otimes I_M"] \arrow[dr, "\iota_A \otimes I_B \otimes I_M"'] \arrow[dd, "\varrho"'] & B \otimes A \otimes M \arrow[d, "\varphi \otimes I_M"] \\
 & A \otimes B \otimes M \arrow[d, "I_A \otimes \varrho"] \\
M \arrow[r, "\iota_A \otimes I_M"] & A \otimes M,
\end{tikzcd}$$

in which the triangle is commutative by the commutative triangle in (7.8) and the trapezium is commutative by naturality of $\iota_A \otimes_R -$. This shows that $\iota_A \otimes_R -$ is a natural transformation of endofunctors of ${}_B\mathbb{M}$.

(e)$\Rightarrow$(d) To define the functor $\widetilde{M \otimes_R} - : \widetilde{\mathbb{M}} \to \widetilde{\mathbb{M}}$, recall that the bijection $\mathrm{Hom}_A(A \otimes X, A \otimes Y) \to \mathrm{Hom}_R(X, A \otimes Y)$.

Thus to describe the functor it is enough to explain its action on $g \in \mathrm{Hom}_R(X, A \otimes Y)$. This is sent to

$$B \otimes_R X \xrightarrow{I_B \otimes g} B \otimes_R A \otimes_R X \xrightarrow{\varphi \otimes I_X} A \otimes_R B \otimes_R X.$$

It follows from the rectangle in (7.8) that this definition defines a functor. The triangle in (7.8) leads for $f : X \to Y$ in ${}_R\mathbb{M}$ to commutativity of the diagram

$$\begin{tikzcd}
B \otimes X \arrow[r, "I_B \otimes \eta_A \otimes X"] \arrow[dr, "\eta_A \otimes I_B"'] & B \otimes A \otimes X \arrow[r, "I_B \otimes I_A \otimes f"] \arrow[d, "\varphi \otimes I_X"] & B \otimes A \otimes Y \arrow[d, "\varphi \otimes I_Y"] \\
 & A \otimes B \otimes X \arrow[r, "I_A \otimes I_B \otimes f"] & A \otimes B \otimes Y
\end{tikzcd}$$

which shows commutativity of the right diagram in (d). The proof is now complete. □

So far we did not give any example for an $\varphi : B \otimes_R A \to A \otimes_R B$ with the properties required above. We mention two

7.4. Examples.

(1) For any ring R and R-ring A, the map

$$\varphi : A \otimes_R A \to A \otimes_R A, \quad a \otimes b \mapsto ab \otimes 1_A + 1_A \otimes ab - a \otimes b, \tag{7.9}$$

is an algebra entwining (thus making $A \otimes_R A$ an associative ring).

(2) For commutative rings R and R-algebras A, B, the canonical twist map is an algebra entwining. This gives the product usually considered on $A \otimes_R B$.

7.5. $(A, B)_\varphi$**-modules.** For an algebra entwining $(A, B)_\varphi$, φ-bimodules are defined as R-modules M which are both A-modules $\nu : A \otimes_R M \to M$ as well and B-modules $\varrho : B \otimes_R M \to M$ implying commutativity of the diagram

$$\begin{tikzcd}
B \otimes A \otimes M \arrow[rr, "\varphi \otimes I_M"] \arrow[d, "I_B \otimes \nu"'] & & A \otimes B \otimes M \arrow[d, "I_A \otimes \varrho"] \\
B \otimes M \arrow[r, "\varrho"] & M & A \otimes M \arrow[l, "\nu"']
\end{tikzcd} \tag{7.10}$$

For elements $a \in A$, $b \in B$ and $m \in M$ this conditions reads $b(am) = \sum a^\varphi(b^\varphi m)$.

Morphisms between φ-bimodules are R-linear maps which are A-module and B-module morphisms and we denote the resulting category by ${}_{A,B}\mathbb{M}(\varphi)$.

The following categories are isomorphic for an algebra entwining $(A, B)_\varphi$:

(a) ${}_{A,B}\mathbb{M}(\varphi)$ - the category of φ-bimodules;

(b) $({}_B\mathbb{M})_{\widehat{A\otimes_R -}}$ - the category of $\widehat{A\otimes}_R$-modules over ${}_B\mathbb{M}$;

(c) ${}_{A\otimes_\varphi B}\mathbb{M}$ - the category of left $A \otimes_\varphi B$-modules.

Proof. (c)⇒(b) By the normality conditions, the maps

$$I_A \otimes \iota_B : A \to A \otimes_\varphi B, \quad \iota_A \otimes I_B : B \to A \otimes_\varphi B$$

are algebra morphisms and thus every $A \otimes_\varphi B$-module has a canonical A-module and B-module structure. Any $A \otimes_\varphi B$-module $\gamma : A \otimes_\varphi B \otimes_R M \to M$ leads to a commutative diagram

$$\begin{tikzcd}
B \otimes M \arrow[r, "\iota_A \otimes I_B \otimes I_A"] \arrow[dr, "\varrho"'] & A \otimes_\varphi B \otimes M \arrow[d, "\gamma"] & A \otimes M \arrow[l, "I_A \otimes \iota_B I_M"'] \arrow[dl, "\nu"] \\
 & M &
\end{tikzcd},$$

where ϱ and ν are defined as the corresponding compositions. It is straightforward to verify that these maps satisfy the condition (7.10). □

7.6. Yang-Baxter equation. Let A, B, C be R-modules with linear maps

$$\varphi_{BA} : B \otimes_R A \to A \otimes_R B, \quad \varphi_{CB} : C \otimes_R B \to B \otimes_R C, \quad \varphi_{CA} : C \otimes_R A \to A \otimes_R C.$$

The triple $(\varphi_{BA}, \varphi_{CB}, \varphi_{CA})$ is said to satisfy the *Yang-Baxter equation* if it induces commutativity of the diagram

$$\begin{tikzcd}
C \otimes B \otimes A \arrow[r, "\varphi_{CB} \otimes I_A"] \arrow[d, "I_C \otimes \varphi_{BA}"'] & B \otimes C \otimes A \arrow[r, "I_B \otimes \varphi_{CA}"] & B \otimes A \otimes C \arrow[d, "\varphi_{BA} \otimes I_C"] \\
C \otimes A \otimes B \arrow[r, "\varphi_{CA} \otimes I_B"] & A \otimes C \otimes B \arrow[r, "I_A \otimes \varphi_{CB}"] & A \otimes B \otimes C.
\end{tikzcd}$$

7.7. Tensor product of three algebras. *Let (A, m_A, ι_A), (B, m_B, ι_B) and (C, m_C, ι_C) be R-algebras with algebra entwinings*

$$\varphi_{BA} : B \otimes_R A \to A \otimes_R B, \quad \varphi_{CB} : C \otimes_R B \to B \otimes_R C, \quad \varphi_{CA} : C \otimes_R A \to A \otimes_R C.$$

The following statements are equivalent:

(a) $(\varphi_{BA}, \varphi_{CB}, \varphi_{CA})$ *satisfies the Yang-Baxter equation;*

(b) $A \otimes_{\varphi_{BA}} B \otimes -$ *lifts to a monad* $A \widehat{\otimes_{\varphi_{BA}} B \otimes -} : \mathbb{M}_C \to \mathbb{M}_C$*;*

(c) $A \otimes_R B \otimes_R C$ *has a ring structure with product*

$$(m_A \otimes m_B \otimes m_C) \circ (I_A \otimes \varphi_{BA} \otimes \varphi_{CB} \otimes I_C) \circ (I_A \otimes I_B \otimes \varphi_{CA} \otimes I_B \otimes I_C)$$

and unit $\iota_A \otimes \iota_B \otimes \iota_C$*;*

(d) $(I_A \otimes \varphi_{CB}) \circ (\varphi_{CA} \otimes I_B) : C \otimes_R A \otimes_{\varphi_{BA}} B \to A \otimes_{\varphi_{BA}} B \otimes_R C$ *is an algebra entwining.*

Proof. (a)⇒(b) The algebra entwinings φ_{CA} and φ_{CB} give rise to monads $\widehat{A\otimes}-$ and $\widehat{B\otimes}-$ on $\mathbb{M}_C$. By the Yang-Baxter condition, every C-module $\varrho : C \otimes_R M \to M$ leads to the commutative diagram

$$\begin{tikzcd}
C\otimes B\otimes A\otimes M \arrow[r, "\varphi_{CB}\otimes I_A\otimes I_M"] \arrow[d, "I_C\otimes\varphi_{BA}\otimes I_M"'] & B\otimes C\otimes A\otimes M \arrow[r, "I_B\otimes\varphi_{CA}\otimes I_M"] & B\otimes A\otimes C\otimes M \arrow[r, "I_B\otimes I_A\otimes\varrho"] \arrow[d, "\varphi_{BA}\otimes I_C\otimes I_M"] & B\otimes A\otimes M \arrow[d, "\varphi_{BA}\otimes I_M"] \\
C\otimes A\otimes B\otimes M \arrow[r, "\varphi_{CA}\otimes I_B\otimes I_M"] & A\otimes C\otimes B\otimes M \arrow[r, "I_A\otimes\varphi_{CB}\otimes I_M"] & A\otimes B\otimes C\otimes M \arrow[r, "I_A\otimes I_B\otimes\varrho"] & A\otimes B\otimes M.
\end{tikzcd}$$

(a)⇒(d) The normality conditions on φ_{CA} and φ_{CB} yield the commutative diagrams

$$\begin{tikzcd}
A\otimes B \arrow[r, "\iota_C\otimes I_A\otimes I_B"] \arrow[dr, "I_A\otimes\iota_C\otimes I_B"] \arrow[ddr, "I_A\otimes I_B\otimes\iota_C"'] & C\otimes A\otimes B \arrow[d, "\varphi_{CA}\otimes I_B"] \\
& A\otimes C\otimes B \arrow[d, "I_A\otimes\varphi_{CB}"] \\
& A\otimes B\otimes C,
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "I_C\otimes\iota_A\otimes\iota_B"] \arrow[dr, "\iota_A\otimes I_C\otimes\iota_B"] \arrow[ddr, "I_A\otimes I_B\otimes\iota_C"'] & C\otimes A\otimes B \arrow[d, "\varphi_{CA}\otimes I_B"] \\
& A\otimes C\otimes B \arrow[d, "I_A\otimes\varphi_{CB}"] \\
& A\otimes B\otimes C,
\end{tikzcd}$$

which show normality for the entwining under consideration.

The commutativity of the rectangle in (7.7) follows from the diagram (where obvious identity transformations are deleted)

$$\begin{tikzcd}
C\otimes C\otimes A\otimes B \arrow[r, "m_C"] \arrow[d, "\varphi_{CA}"'] & C\otimes A\otimes B \arrow[r, "\varphi_{CA}"] & A\otimes C\otimes B \arrow[d, "\varphi_{CB}"] \\
C\otimes A\otimes C\otimes B \arrow[r, "\varphi_{CA}"] \arrow[d, "\varphi_{CB}"'] & A\otimes C\otimes C\otimes B \arrow[ur, "m_C"] \arrow[d, "\varphi_{CB}"'] & A\otimes B\otimes C \\
C\otimes A\otimes B\otimes C \arrow[r, "\varphi_{CA}"] & A\otimes C\otimes B\otimes C \arrow[r, "\varphi_{CB}"] & A\otimes B\otimes C\otimes C, \arrow[u, "m_C"']
\end{tikzcd}$$

where the pentagons are commutative by the properties of φ_{CA} and φ_{CB}, respectively, and the square is commutative by naturality.

To prove commutativity of the rectangle in (7.8), consider the diagram

$$\begin{tikzcd}
C\otimes A\otimes B\otimes A\otimes B \arrow[r, "\varphi_{BA}"] \arrow[d, "\varphi_{CA}"'] & C\otimes A\otimes A\otimes B\otimes B \arrow[r, "m_A"] \arrow[d, "\varphi_{CA}"'] \arrow[dr, phantom, "(1)"] & C\otimes A\otimes B\otimes B \arrow[r, "m_B"] \arrow[d, "\varphi_{CA}"'] & C\otimes A\otimes B \arrow[d, "\varphi_{CA}"'] \\
A\otimes C\otimes B\otimes A\otimes B \arrow[r, "\varphi_{BA}"] \arrow[d, "\varphi_{CB}"'] \arrow[dr, phantom, "(2)"] & A\otimes C\otimes A\otimes B\otimes B \arrow[d, "\varphi_{CA}"'] & A\otimes C\otimes B\otimes B \arrow[r, "m_B"] \arrow[d, "\varphi_{CB}"] \arrow[dr, phantom, "(3)"] & A\otimes C\otimes B \arrow[d, "\varphi_{CB}"] \\
A\otimes B\otimes C\otimes A\otimes B \arrow[d, "\varphi_{CA}"'] & A\otimes A\otimes C\otimes B\otimes B \arrow[ur, "m_A"] \arrow[d, "\varphi_{CB}"'] & A\otimes B\otimes C\otimes B \arrow[dr, "\varphi_{CB}"] & A\otimes B\otimes C \\
A\otimes B\otimes A\otimes C\otimes B \arrow[r, "\varphi_{BA}"] \arrow[dr, "\varphi_{CB}"'] & A\otimes A\otimes B\otimes C\otimes B \arrow[ur, "m_A"'] \arrow[drr, "\varphi_{CB}"] & & A\otimes B\otimes B\otimes C \arrow[u, "m_B"'] \\
& A\otimes B\otimes A\otimes B\otimes C \arrow[rr, "\varphi_{BA}"'] & & A\otimes A\otimes B\otimes B\otimes C, \arrow[u, "m_A"']
\end{tikzcd}$$

in which (1) and (3) are commutative since φ_{CA} and φ_{CB} are algebra entwinings. (2) is commutative because of the Yang-Baxter equation, and the other inner diagrams are commutative because of naturality of the transformations involved. The outer morphisms yield the rectangle in (7.8) for the entwining between $A \otimes_{\varphi_{BA}} B$ and C.

(d)⇒(a) Assume (d) holds. Then the diagram for (7.8) in the preceding proof has to be commutative. Entering the diagram with the map

$$I_C \otimes \iota_A \otimes I_B \otimes I_A \otimes \iota_B : C \otimes_R B \otimes_R A \to C \otimes_R A \otimes_R B \otimes_R A \otimes_R B,$$

a short argument shows that the Yang-Baxter equation is satisfied.

The remaining conclusions are left to the reader. □

8 Tensor product of coalgebras

Similar as for algebras we may also consider coalgebra structures on the tensor product of two coalgebras. We will assume R to be a commutative ring.

8.1. Tensor product of coalgebras. Given R-coalgebras $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$, the tensor product $C \otimes_R D$ is again an R-module and an R-linear map

$$\omega : C \otimes_R D \to D \otimes_R C, \quad c \otimes d \mapsto \omega(c \otimes d) =: \sum d^\omega \otimes c^\omega,$$

induces a coproduct on $C \otimes_R D$ by

$$\Delta_\omega : C \otimes D \xrightarrow{\Delta_C \otimes \Delta_D} C \otimes C \otimes C \otimes C \xrightarrow{I_C \otimes \omega \otimes I_D} C \otimes D \otimes C \otimes D\ ,$$

that is $\Delta_\omega = (I_C \otimes \omega \otimes I_D)(\Delta_C \otimes \Delta_D)$, and for $c \in C$, $d \in D$,

$$\Delta_\omega(c \otimes d) = \sum\sum c_{\underline{1}} \otimes {d_{\underline{1}}}^\omega \otimes {c_{\underline{2}}}^\omega \otimes d_{\underline{2}}.$$

To make $(C \otimes_R D, \Delta_\omega, \varepsilon_\omega)$ a coassociative coalgebra with counit

$$\varepsilon_\omega = \varepsilon_C \otimes \varepsilon_D : C \otimes_R D \to R,$$

we have the *conormality condition*

$$(I_D \otimes \varepsilon_C)\,\omega(c \otimes d) = \varepsilon_C(c)d, \quad (\varepsilon_D \otimes I_C)\,\omega(c \otimes d) = \varepsilon_D(d)c. \tag{8.1}$$

By definition, right counitality of ε_ω requires $(I_{C\otimes_R D} \otimes \varepsilon_\omega) \circ \Delta_\omega = I_{C\otimes_R D}$, that is,

$$c \otimes d = \sum c_{\underline{1}} \otimes (I_D \otimes \varepsilon_C)\,\omega(c_{\underline{2}} \otimes d_{\underline{1}})\varepsilon_D(d_{\underline{2}}) = \sum c_{\underline{1}} \otimes (I_D \otimes \varepsilon_C)\,\omega(c_{\underline{2}} \otimes d).$$

Applying $\varepsilon_C \otimes I_D$, we obtain the first equality for ω in (8.1). Similarly, from the identity $(\varepsilon_\omega \otimes I_{C\otimes_R D}) \circ \Delta_\omega = I_{C\otimes_R D}$, the second equality in (8.1) is derived.

Coassociativity of Δ_ω means commutativity of the diagram

$$\begin{tikzcd}
C \otimes C \otimes D \otimes D \arrow[r, "I\otimes\omega\otimes I"] & C \otimes D \otimes C \otimes D \arrow[r, "I\otimes I\otimes\Delta\otimes\Delta"] & C \otimes D \otimes C \otimes C \otimes D \otimes D \arrow[d, "I\otimes I\otimes I\otimes\omega\otimes I"] \\
C \otimes D \arrow[u, "\Delta_C\otimes\Delta_D"] \arrow[d, "\Delta_C\otimes\Delta_D"'] & (*) & C \otimes D \otimes C \otimes D \otimes C \otimes D \\
C \otimes C \otimes D \otimes D \arrow[r, "I\otimes\omega\otimes I"] & C \otimes D \otimes C \otimes D \arrow[r, "\Delta\otimes\Delta\otimes I\otimes I"] & C \otimes C \otimes D \otimes D \otimes C \otimes D\ , \arrow[u, "I\otimes\omega\otimes I\otimes I\otimes I"']
\end{tikzcd}$$

Applying the map $\varepsilon_C \otimes I_D \otimes I_C \otimes I_D \otimes I_C \otimes \varepsilon_D$ to the last module in the diagram $(*)$ this reduces to the commutative diagram

$$\begin{tikzcd}
C \otimes D \otimes D \arrow[r, "\omega\otimes I"] & D \otimes C \otimes D \arrow[r, "I\otimes\Delta_C\otimes I"] & D \otimes C \otimes C \otimes D \arrow[d, "I\otimes I\otimes\omega"] \\
C \otimes D \arrow[u, "I\otimes\Delta_D"] \arrow[d, "\Delta_C\otimes I"'] & (**) & D \otimes C \otimes D \otimes C \\
C \otimes C \otimes D \arrow[r, "I\otimes\omega"] & C \otimes D \otimes C \arrow[r, "I\otimes\Delta_D\otimes I"] & C \otimes D \otimes D \otimes C\ , \arrow[u, "\omega\otimes I\otimes I"']
\end{tikzcd}$$

Applying $I_D \otimes \varepsilon_C \otimes I_D \otimes I_C$ and $I_D \otimes I_C \otimes \varepsilon_D \otimes I_C$ to the last module in the diagram $(**)$ and using conormality, the conditions reduce to the rectangular diagrams in subsequent statement. The commutativity of the triangles corresponds to the conormality conditions in (8.1).

8.2. Coalgebra entwinings. *For R-coalgebras C, D and a linear map $\omega : C \otimes_R D \to D \otimes_R C$, the following are equivalent:*

(a) *$(C \otimes_R D, \Delta_\omega, \varepsilon_\omega)$ is a counital coalgebra;*

(b) *$C \otimes_R -$ induces a comonad $\widehat{C \otimes_R} - : {}^D\mathbb{M} \to {}^D\mathbb{M}$ by*

$$N \xrightarrow{\varrho^N} D \otimes N \quad \mapsto \quad C \otimes N \xrightarrow{I_C \otimes \varrho^N} C \otimes D \otimes N \xrightarrow{\omega \otimes I_N} D \otimes C \otimes N;$$

(c) *ω induces commutativity of the diagrams*

$$\begin{tikzcd}
C \otimes D \arrow[r, "I_C \otimes \Delta_D"] \arrow[d, "\omega"'] & C \otimes D \otimes D \arrow[r, "\omega \otimes I_D"] & D \otimes C \otimes D \arrow[d, "I_D \otimes \omega"] \\
D \otimes C \arrow[rr, "\Delta_D \otimes I_C"] & & D \otimes D \otimes C,
\end{tikzcd}
\qquad
\begin{tikzcd}
C \otimes D \arrow[r, "I_C \otimes \varepsilon_D"] \arrow[d, "\omega"'] & C \\
D \otimes C \arrow[ur, "\varepsilon_D \otimes I_C"'] &
\end{tikzcd},$$

$$\begin{tikzcd}
C \otimes D \arrow[r, "\Delta_C \otimes I_D"] \arrow[d, "\omega"'] & C \otimes C \otimes D \arrow[r, "I_C \otimes \omega"] & C \otimes D \otimes C \arrow[d, "\omega \otimes I_C"] \\
D \otimes C \arrow[rr, "I_D \otimes \Delta_C"] & & D \otimes C \otimes C,
\end{tikzcd}
\qquad
\begin{tikzcd}
C \otimes D \arrow[r, "\varepsilon_C \otimes I_D"] \arrow[d, "\omega"'] & D \\
D \otimes C \arrow[ur, "I_D \otimes \varepsilon_C"'] &
\end{tikzcd}.$$

Proof. (a)⇒(c) was shown above.

(c)⇒(a) To show this it is helpful to write out the formulas explicitely (e.g. [15, 2.14]).

(c)⇒(b) The proof is similar to the proof of 7.3, (c)⇒(b). □

In the above proposition we have shown how the tensor product of coalgebras can be made to a coalgebra. Given a coalgebra D, one may ask which (more general) conditions on C still allow for a coalgebra structure for $C \otimes_R D$, that is, the functor $C \otimes_R -$ can be lifted to a comonad $\widehat{C \otimes_R} - : {}^D\mathbb{M} \to {}^D\mathbb{M}$.

We observe that in the diagrams of 8.2 we find the maps

$$\delta := \Delta_C \otimes I_D : C \otimes_R D \to C \otimes_R C \otimes_R D \quad \text{and} \quad \zeta := \varepsilon_C \otimes I_D : C \otimes_R D \to D$$

with $(\zeta \otimes I_D) \circ \delta = I_{C \otimes D}$. These maps are sufficient to define a coproduct on $C \otimes_R D$ and we have to find out which conditions are needed to yield a coassociative and counital coalgebra.

8.3. Proposition. *Let $(D, \Delta_D, \varepsilon_D)$ be a coalgebra and C an R-module with maps $\delta : C \otimes_R D \to C \otimes_R C \otimes_R D$, $\zeta : C \otimes_R D \to D$, and $\omega : C \otimes_R D \to D \otimes_R C$. These lead to the map*

$$\Delta_{CD} : \ C \otimes D \xrightarrow{\delta} C \otimes C \otimes D \xrightarrow{I \otimes I \otimes \Delta_D} C \otimes C \otimes D \otimes D \xrightarrow{I \otimes \omega \otimes I} C \otimes D \otimes C \otimes D.$$

Then the following are equivalent:

(a) $(C \otimes_R D, \Delta_{CD}, \varepsilon_D \circ \zeta)$ *is a counital coalgebra;*

(b) $(C, D)_\omega$ *is a cowreath.*

Proof. Left to the reader. $\square$

8.4. Tensor product of three coalgebras. *Let* $(C, \Delta_C, \varepsilon_C)$, $(D, \Delta_D, \varepsilon_D)$ *and* $(E, \Delta_E, \varepsilon_E)$ *be* R*-coalgebras with coalgebra entwinings*

$$\omega_{CD} : C \otimes_R D \to D \otimes_R C, \quad \omega_{CE} : C \otimes_R E \to E \otimes_R C, \quad \omega_{DE} : D \otimes_R E \to E \otimes_R D.$$

The following statements are equivalent:

(a) $(\omega_{CD}, \omega_{CE}, \omega_{DE})$ *satisfies the Yang-Baxter equation;*

(b) $C \otimes_{\omega_{CD}} D \otimes -$ *lifts to a comonad* $C \widehat{\otimes_{\omega_{CD}}} D \otimes - : \mathbb{M}^E \to \mathbb{M}^E$*;*

(c) $C \otimes_R D \otimes_R E$ *has a coalgebra structure with coproduct*

$$(I_C \otimes I_D \otimes \omega_{CE} \otimes I_D \otimes I_E \circ (I_C \otimes \omega_{CD} \otimes \omega_{DE} \otimes I_E) \circ (\Delta_C \otimes \Delta_D \otimes \Delta_E)$$

and counit $\varepsilon_C \otimes \varepsilon_D \otimes \varepsilon_E$.

(d) $(I_C \otimes \omega_{CE}) \circ (\omega_{CD} \otimes I_E) : C \otimes_R (D \otimes_R E) \to (D \otimes_R E) \otimes_R C$ *is a coalgebra entwining.*

Proof. Left to the reader. $\square$

9 Entwining algebras and coalgebras

Having investigated tensor products of two algebras and two coalgebras, respectively, the question arises which structures can be expected when algebras are tensored with coalgebras.

9.1. Tensor product of algebras and coalgebras. Let (A, m_A, ι_A) be an R-algebra and $(C, \Delta_C, \varepsilon_C)$ an R-coalgebra. An R-linear map

$$\psi : A \otimes_R C \to C \otimes_R A$$

is called a *mixed entwining* provided it induces commutativity of the diagrams

$$\begin{tikzcd}
A \otimes A \otimes C \arrow[rr, "m_A \otimes I_C"] \arrow[d, "I_A \otimes \psi"] & & A \otimes C \arrow[d, "\psi"] \\
A \otimes C \otimes A \arrow[r, "\psi \otimes I_A"] & C \otimes A \otimes A \arrow[r, "I_C \otimes m_A"] & C \otimes A\,,
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "\iota_A \otimes I_C"] \arrow[dr, "I_C \otimes \iota_A"'] & A \otimes C \arrow[d, "\psi"] \\
& C \otimes A,
\end{tikzcd} \tag{9.1}$$

$$\begin{tikzcd}
A \otimes C \arrow[r, "I_A \otimes \Delta_C"] \arrow[d, "\psi"'] & A \otimes C \otimes C \arrow[r, "\psi \otimes I_C"] & C \otimes A \otimes C \arrow[d, "I_C \otimes \psi"] \\
C \otimes A \arrow[rr, "\Delta_C \otimes I_A"'] & & C \otimes C \otimes A,
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes C \arrow[r, "I_A \otimes \varepsilon_C"] \arrow[d, "\psi"'] & A \\
C \otimes A \arrow[ur, "\varepsilon_C \otimes I_A"'] & .
\end{tikzcd} \tag{9.2}$$

The triple $(A, C)_\psi$ is called an *entwining structure (over R)*, the map ψ is known as an *entwining map*, and we say that C and A are *entwined* by ψ.

9.2. Notation. To describe elements we write

$$\psi(a \otimes c) = \sum c^\psi \otimes a_\psi = \sum\nolimits_\alpha c^\alpha \otimes a_\alpha,$$

or use other summation indices if necessary. This leads to formulas like

$$(I_A \otimes \psi) \circ (\psi \otimes I_A)(a \otimes c \otimes d) = \sum\nolimits_{\alpha,\beta} c^\alpha \otimes d^\beta \otimes a_{\alpha\beta},$$

for all $a \in A$, $c, d \in C$. Thus the conditions on entwinings take the form

$$\begin{aligned}
\text{rectangle in (9.1):} \quad & \sum\nolimits_\alpha c^\alpha \otimes (ab)_\alpha c^\alpha = \sum\nolimits_{\alpha,\beta} c^{\alpha\beta} \otimes a_\beta b_\alpha, \\
\text{rectangle in (9.2) :} \quad & \sum\nolimits_\alpha c^\alpha{}_{\underline{1}} \otimes c^\alpha{}_{\underline{2}} \otimes a_\alpha = \sum\nolimits_{\alpha,\beta} c_{\underline{1}}{}^\alpha \otimes c_{\underline{2}}{}^\beta \otimes a_{\alpha\beta}.
\end{aligned}$$

The entwining defined should be called more precisely a left-left entwining. One can define right-left, left-right and left-left entwining structures by replacing the pair (C, A) with the pairs (C^{cop}, A), (C, A^{op}) and (C^{cop}, A^{op}), respectively, in the diagrams (9.1), (9.2). These combinations lead to equivalent theories hence we mainly concentrate on left-left entwining structures.

9.3. Entwined modules. Associated to any entwining structure $(C, A)_\psi$ is the category of $(C, A)_\psi$-*entwined modules* denoted by ${}^C_A\mathbb{M}(\psi)$. An object $M \in {}^C_A\mathbb{M}(\psi)$ is a left A-module $\varrho_M : A\otimes_R M \to M$ and a left C-comodule $\varrho^M : M \to C\otimes_R M$ inducing a commutative diagram

$$\begin{tikzcd}
A\otimes M \arrow[r, "\varrho_M"] \arrow[d, "I_A\otimes\varrho^M"'] & M \arrow[r, "\varrho^M"] & C\otimes M \\
A\otimes C\otimes M \arrow[rr, "\psi\otimes I_M"] & & C\otimes A\otimes M. \arrow[u, "I_C\otimes\varrho_M"']
\end{tikzcd}$$

A morphism in ${}^C_A\mathbb{M}(\psi)$ is a left A-module map that is at the same time a left C-comodule map.

9.4. Entwining structures. *Let (A, m_A, ι_A) be an R-algebra and $(C, \Delta_C, \varepsilon_C)$ a coalgebra. For an R-linear map $\psi : A\otimes_R C \to C\otimes_R A$ the following are equivalent:*

(a) *ψ is an entwining;*

(b) *$A\otimes_R -$ induces a monad $\widehat{A\otimes_R}- : {}^C\mathbb{M} \to {}^C\mathbb{M}$ by*

$$N \xrightarrow{\varrho^N} C\otimes N \quad\mapsto\quad A\otimes N \xrightarrow{\varrho^N\otimes I_A} A\otimes C\otimes N \xrightarrow{\psi\otimes I_N} A\otimes N;$$

(c) *$C\otimes_R -$ induces a comonad on $\widehat{C\otimes_R}- : {}_A\mathbb{M} \to {}_A\mathbb{M}$ by*

$$A\otimes M \xrightarrow{\varrho_M} M \quad\mapsto\quad A\otimes C\otimes M \xrightarrow{\psi\otimes I_M} C\otimes A\otimes M \xrightarrow{I_C\otimes\varrho_M} C\otimes M.$$

If this conditions are satisfied we have equivalent categories

$$({}_A\mathbb{M})^{\widehat{C\otimes}-} \simeq ({}^C\mathbb{M})_{\widehat{A\otimes}-} \simeq {}^C_A\mathbb{M}(\psi).$$

Proof. E.g., see [59, 5.4]. □

We have studied when the tensor product of two algebras yields an algebra and also when the tensor product of two coalgebras yields a coalgebra. The tensor product of an algebra and a coalgebra leads to a new structure.

9.5. Corings. Let A be a ring. An (A, A)-bimodule $\mathcal{C}$ is called an A-*coring* if it is a coalgebra in the category ${}_A\mathbb{M}_A$ of (A, A)-bimodules, that is, there are (A, A)-bilinear maps

$$\underline{\Delta} : \mathcal{C} \to \mathcal{C}\otimes_A \mathcal{C} \quad\text{and}\quad \underline{\varepsilon} : \mathcal{C} \to A,$$

called *(coassociative) coproduct* and *counit*, with the properties

$$(I_\mathcal{C}\otimes\underline{\Delta})\circ\underline{\Delta} = (\underline{\Delta}\otimes I_\mathcal{C})\circ\underline{\Delta}, \text{ and } (I_\mathcal{C}\otimes\underline{\varepsilon})\circ\underline{\Delta} = I_\mathcal{C} = (\underline{\varepsilon}\otimes I_\mathcal{C})\circ\underline{\Delta},$$

expressed by commutativity of the diagrams

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\underline{\Delta}"] \arrow[d, "\underline{\Delta}"'] & \mathcal{C}\otimes_A\mathcal{C} \arrow[d, "I_\mathcal{C}\otimes\underline{\Delta}"] \\
\mathcal{C}\otimes_A\mathcal{C} \arrow[r, "\underline{\Delta}\otimes I_\mathcal{C}"] & \mathcal{C}\otimes_A\mathcal{C}\otimes_A\mathcal{C},
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C} \arrow[r, "\underline{\Delta}"] \arrow[d, "\underline{\Delta}"'] \arrow[dr, "I_\mathcal{C}"] & \mathcal{C}\otimes_A\mathcal{C} \arrow[d, "\underline{\varepsilon}\otimes I_\mathcal{C}"] \\
\mathcal{C}\otimes_A\mathcal{C} \arrow[r, "I_\mathcal{C}\otimes\underline{\varepsilon}"] & \mathcal{C}.
\end{tikzcd}$$

For any A-coring $\mathcal{C}$, induces a comonad $\mathcal{C} \otimes_A - : {}_A\mathbb{M} \to {}_A\mathbb{M}$ and the $\mathcal{C} \otimes_A -$-modules are called *left $\mathcal{C}$-comodules*; they are left A-modules M, with a coassociative and counital *left $\mathcal{C}$-coaction*, that is, an A-linear map ${}^M\varrho : M \to \mathcal{C} \otimes_A M$ for which

$$(\underline{\Delta} \otimes I_M) \circ {}^M\varrho = (I_{\mathcal{C}} \otimes {}^M\varrho) \circ {}^M\varrho, \quad (\underline{\varepsilon} \otimes I_M) \circ {}^M\varrho = I_M.$$

Left $\mathcal{C}$-comodules and their morphisms form a preadditive category ${}^{\mathcal{C}}\mathbb{M} = \mathbb{M}^{\mathcal{C}\otimes_A -}$.

The free functor $\mathcal{C} \otimes_A - : {}_A\mathbb{M} \to {}^{\mathcal{C}}\mathbb{M}$ is right adjoint to the forgetful functor $U^{\mathcal{C}} : {}^{\mathcal{C}}\mathbb{M} \to {}_A\mathbb{M}$.

9.6. Corings associated to entwining structures. *View $C \otimes_R A$ as a right A-module with multiplication $(c \otimes a')a = c \otimes a'a$, for all $a, a' \in A, c \in C$. Then:*

(1) *For an entwining structure $(C, A)_\psi$, $\mathcal{C} = C \otimes_R A$ is an (A, A)-bimodule with left multiplication $a(c \otimes a') = \psi(a \otimes c)a'$, and it is an A-coring with the coproduct and counit*

$$\underline{\Delta} := \Delta \otimes I_A : \mathcal{C} \to C \otimes_R C \otimes_R A \simeq \mathcal{C} \otimes_A \mathcal{C}, \quad \underline{\varepsilon}_C := \varepsilon \otimes I_A : \mathcal{C} \to A.$$

(2) *If $\mathcal{C} = C \otimes_R A$ is an A-coring with coproduct $\underline{\Delta} = \Delta \otimes I_A$ and counit $\underline{\varepsilon}_C = \varepsilon \otimes I_A$, then $(C, A)_\psi$ is an entwining structure by*

$$\psi : A \otimes_R C \to C \otimes_R A, \quad a \otimes c \mapsto a(c \otimes 1).$$

(3) *If $\mathcal{C} = C \otimes_R A$ is the A-coring associated to $(C, A)_\psi$ as in (1), then the category of $(C, A)_\psi$-entwined modules is isomorphic to the category of left $\mathcal{C}$-comodules.*

9.7. Entwining two algebras with a coalgebra. *Let (A, m_A, ι_A), (B, m_B, ι_B) be an R-algebras and $(C, \Delta_C, \varepsilon_C)$ an R-coalgebra with an algebra entwining $\varphi_{BA} : B \otimes_R A \to A \otimes_R B$ and mixed entwinings $\psi_{AC} : A \otimes_R C \to C \otimes_R A$, $\psi_{BC} : B \otimes_R C \to C \otimes_R B$. Then the following are equivalent:*

(a) *$(\varphi_{BA}, \psi_{BC}, \psi_{AC})$ satisfy the Yang-Baxter condition;*

(b) *$(\psi_{AC} \otimes I_B) \circ (I_A \otimes \psi_{BC}) : A \otimes_{\varphi_{BA}} B \otimes_R C \to C \otimes_R A \otimes_{\varphi_{BA}} B$ is a mixed entwining;*

(c) *$A \otimes_R B \otimes_R C$ is an $A \otimes_{\varphi_{BA}} B$-coring.*

Proof. (a)⇒(b) To prove commutativity of the rectangle in (9.1), consider the diagram

$$\begin{tikzcd}
A \otimes B \otimes A \otimes B \otimes C \arrow[r, "\varphi_{BA}"] \arrow[d, "\psi_{BC}"'] & A \otimes A \otimes B \otimes B \otimes C \arrow[r, "m_B"] \arrow[d, "\psi_{BC}"'] \arrow[dr, phantom, "(1)"] & A \otimes A \otimes B \otimes C \arrow[r, "m_A"] \arrow[d, "\psi_{BC}"'] & A \otimes B \otimes C \arrow[d, "\psi_{BC}"'] \\
A \otimes B \otimes A \otimes C \otimes B \arrow[r, "\varphi_{BA}"] \arrow[d, "\psi_{BC}"'] \arrow[dr, phantom, "(2)"] & A \otimes A \otimes B \otimes C \otimes B \arrow[d, "\psi_{BC}"'] & A \otimes A \otimes C \otimes B \arrow[r, "m_A"] \arrow[d, "\psi_{AC}"] \arrow[dr, phantom, "(3)"] & A \otimes C \otimes B \arrow[d, "\psi_{AC}"] \\
A \otimes B \otimes C \otimes A \otimes B \arrow[d, "\psi_{BC}"'] & A \otimes A \otimes C \otimes B \otimes B \arrow[ur, "m_B"] \arrow[d, "\varphi_{CB}"'] & A \otimes C \otimes A \otimes B \arrow[dr, "\psi_{AC}"] & C \otimes A \otimes B \\
A \otimes C \otimes B \otimes A \otimes B \arrow[r, "\varphi_{BA}"] \arrow[dr, "\psi_{AC}"'] & A \otimes C \otimes A \otimes B \otimes B \arrow[ur, "m_B"'] \arrow[drr, "\psi_{AC}"] & & C \otimes A \otimes A \otimes B \arrow[u, "m_A"'] \\
 & C \otimes A \otimes B \otimes A \otimes B \arrow[rr, "\varphi_{BA}"'] & & C \otimes A \otimes A \otimes B \otimes B, \arrow[u, "m_B"']
\end{tikzcd}$$

in which (1) and (3) are commutative because ψ_{AC} and ψ_{BC} are mixed entwinings; (2) is commutative because of the Yang-Baxter equation, and the other inner diagrams are commutative because of naturality of the transformations involved. The outer morphisms yield the rectangle in (9.1) for the mixed entwining between $A \otimes_{\varphi_{BA}} B$ and C.

The triangle in (9.1) requires commutativity of the outer triangle in

$$\begin{tikzcd}
C \arrow[r, "\iota_A\otimes\iota_B\otimes I_C"] \arrow[dr, "I_A\otimes\iota_B\otimes I_C"] \arrow[ddr, "I_C\otimes\iota_B\otimes\iota_B"'] & A\otimes B\otimes C \arrow[d, "\psi_{BC}"] \\
 & A\otimes C\otimes B \arrow[d, "\psi_{AC}"] \\
 & C\otimes A\otimes B.
\end{tikzcd}$$

It follows from the commutativity of the inner triangles.

For commutativity of the rectangle in (9.2) consider the diagram

$$\begin{tikzcd}
A\otimes B\otimes C \arrow[r, "I\otimes I\otimes\Delta_C"] \arrow[d, "\psi_{BC}"'] & A\otimes B\otimes C\otimes C \arrow[r, "\psi_{BC}"] & A\otimes C\otimes B\otimes C \arrow[r, "\psi_{AC}"] \arrow[d, "\psi_{BC}"] & C\otimes A\otimes B\otimes C \arrow[d, "\psi_{BC}"'] \\
A\otimes C\otimes B \arrow[rr, "I_A\otimes\Delta_C\otimes I_B"] \arrow[d, "\psi_{AC}"] & & A\otimes C\otimes C\otimes B \arrow[r, "\psi_{AC}"] & C\otimes A\otimes C\otimes B \arrow[d, "\psi_{AC}"] \\
C\otimes A\otimes B \arrow[rrr, "\Delta_C\otimes I_A\otimes I_B"] & & & C\otimes C\otimes A\otimes B.
\end{tikzcd}$$

The inner rectangles are commutative since ψ_{AC} and ψ_{BC} are mixed entwinings and the inner square is commutative by naturality.

Commutativity of the triangle in (9.2) follows from the diagram

$$\begin{tikzcd}
A\otimes B\otimes C \arrow[r, "I\otimes I\otimes\varepsilon_C"] \arrow[d, "\psi_{BC}"'] & A\otimes B \\
A\otimes C\otimes B \arrow[ur, "I\otimes\varepsilon_C\otimes I"] \arrow[d, "\psi_{AC}"'] & \\
C\otimes A\otimes B, \arrow[uur, "\varepsilon_C\otimes I\otimes I"'] &
\end{tikzcd}$$

in which the inner triangles are commutative by the conditions on ψ_{AC} and ψ_{BC}. $\square$

10 Relations between functors

To study the relationship between various module categories, the following definition is of interest. It was formulated in Johnstone [29] for monads but we also consider it for arbitrary endofunctors.

10.1. Lifting for monads. Let $\mathbb{F} = (F, \mu F, \eta F)$ and $\mathbb{G} = (G, \mu G, \eta G)$ be monads on the categories $\mathbb{A}$ and $\mathbb{B}$, respectively, and let $T : \mathbb{A} \to \mathbb{B}$ be a functor. A functor $\overline{T} : \mathbb{A}_F \to \mathbb{B}_G$, is called a *lifting of* T provided the diagram

$$\begin{tikzcd} \mathbb{A}_F \arrow[r, "\overline{T}"] \arrow[d, "U_F"'] & \mathbb{B}_G \arrow[d, "U_G"] \\ \mathbb{A} \arrow[r, "T"] & \mathbb{B}, \end{tikzcd}$$

is commutative, where the U's denote the forgetful functors.

Liftings are obtained by certain natural transformations between the functors involved.

10.2. Theorem. (Applegate) *Let $\mathbb{F} = (F, \mu_F, \eta_F)$ and $\mathbb{G} = (G, \mu_G, \eta_G)$ be monads on the categories $\mathbb{A}$ and $\mathbb{B}$, respectively, and let $T : \mathbb{A} \to \mathbb{B}$ be any functor.*

The liftings $\overline{T} : \mathbb{A}_F \to \mathbb{B}_G$ of T are in bijective correspondence with the natural transformations $\lambda : GT \to TF$ inducing commutative diagrams

$$\begin{tikzcd} GGT \arrow[rr, "\mu_G T"] \arrow[d, "G\lambda"'] & & GT \arrow[d, "\lambda"] \\ GTF \arrow[r, "\lambda F"] & TFF \arrow[r, "T\mu_F"] & TF, \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\eta_G T"] \arrow[dr, "T\eta_F"'] & GT \arrow[d, "\lambda"] \\ & TF. \end{tikzcd}$$

The correspondence associates to any $\lambda : GT \to TF$ the functor

$$\begin{array}{rcl} \overline{T}_\lambda : \mathbb{A}_F \to \mathbb{B}_G, \quad F(A) \xrightarrow{\varrho_A} A & \mapsto & GT(A) \xrightarrow{\lambda_A} TF(A) \xrightarrow{T(\varrho_A)} T(A) \\ A \xrightarrow{f} A' & \mapsto & T(A) \xrightarrow{T(f)} T(A'), \end{array}$$

Proof. By the condition $U_G \circ \overline{T} = T \circ U_F$, any lifting functor $\overline{T}$ must act on objects $A \in \mathbb{A}_F$ like T, that is, $\overline{T}(A) = T(A)$.

Any natural transformation $\lambda : GT \to TF$ assigns to an F-module $\rho_A : F(A) \to A$ a G-action on $T(A)$,

$$GT(A) \xrightarrow{\lambda_A} TF(A) \xrightarrow{T\rho_A} T(A).$$

To make $T(A)$ a G-module, associativity of the G-action implies for $A = T(X)$, $X \in \mathbb{A}$, commutativity of the inner rectangle in the diagram

$$
\begin{tikzcd}
GGT \arrow[rr, "G\lambda"] \arrow[dr, "GGT\eta_F"] \arrow[ddd, "\mu_G T"'] & & GTF \arrow[d, "GTF\eta_F"'] \arrow[dr, "="] & \\
& GGTF \arrow[r, "G\lambda F"] \arrow[dd, "\mu_G TF"] & GTFF \arrow[r, "GT\mu_F"'] & GTF \arrow[d, "\lambda F"] \\
& & & TFF \arrow[d, "T\mu_F"] \\
GT \arrow[r, "GT\eta_F"] \arrow[drr, "\lambda"'] & GTF \arrow[r, "\lambda F"] & TFF \arrow[r, "T\mu_F"] & TF \\
& & TF \arrow[u, "TF\eta_F"] \arrow[ur, "="'] &
\end{tikzcd}
$$

The inner diagrams are commutative by functoriality of composition and hence the outer diagram yields the commutative diagram

$$
\begin{tikzcd}
GGT \arrow[r, "G\lambda"] \arrow[d, "\mu_G T"'] & GTF \arrow[r, "\lambda F"] & TFF \arrow[d, "T\mu_F"] \\
GT \arrow[rr, "\lambda"] & & TF.
\end{tikzcd}
$$

For the relation between λ and the units of F and G, consider the diagram

$$
\begin{tikzcd}
T \arrow[r, "\eta_G T"] \arrow[d, "T\eta_F"'] & GT \arrow[r, "\lambda"] \arrow[d, "GT\eta_F"'] & TF \arrow[d, "TF\eta_F"'] \arrow[dr, "="] & \\
TF \arrow[r, "\eta_G TF"'] & GTF \arrow[r, "\lambda F"'] & TFF \arrow[r, "T\mu_F"'] & TF,
\end{tikzcd}
$$

in which the inner rectangles are obviously commutative. The unitality condition for the module structure on TF implies that the composition of the maps in the bottom line yields the identity and hence we obtain the commutative triangle required in our statement.

To show that our conditions imply a G-module structure on TF, consider the diagram

$$
\begin{tikzcd}
GGTF \arrow[r, "G\lambda F"] \arrow[dd, "\mu_G TF"'] & GTFF \arrow[r, "GT\mu_F"] \arrow[d, "\lambda FF"'] \arrow[dr, phantom, "(2)"] & GTF \arrow[d, "\lambda F"] \\
\arrow[r, phantom, "(1)"] & TFFF \arrow[r, "TF\mu_F"] \arrow[d, "T\mu_F F"'] \arrow[dr, phantom, "(3)"] & TFF \arrow[d, "T\mu_F"] \\
GTF \arrow[r, "\lambda F"'] & TFF \arrow[r, "T\mu_F"] & TF,
\end{tikzcd}
$$

in which diagram (1) is commutative since it is just the given diagram (applied to F), (2) is commutative by functoriality of composition, and (3) is commutative by associativity of μ_F. Thus the outer diagram is commutative, showing the associativity of the G-action.

Now assume that a lifting is given, that is, TF is a G-module with structure map $\beta : GTF \to TF$; we show that the map

$$\lambda :\; GT \xrightarrow{GT\eta_F} GTF \xrightarrow{\beta} TF$$

has the properties observed under the lifting condition. In the commutative diagram

$$\begin{tikzcd}
T \arrow[r, "T\eta_F"] \arrow[d, "\eta_G T"'] & TF \arrow[dr, "="] & \\
GT \arrow[r, "GT\eta_F"'] & GTF \arrow[r, "\beta"] & TF,
\end{tikzcd}$$

the bottom line yields the identity by unitality and hence $T\eta_F = \lambda \circ \eta_G T$.

The lifting functor has to transfer the morphism $\mu_F : FF \to F$ in $\mathbb{A}_F$ into a morphism in $\mathbb{A}_G$, that is commutativity of the diagram

$$\begin{tikzcd}
GTFF \arrow[r, "GT\mu_F"] \arrow[d, "\beta F"'] & GTF \arrow[d, "\beta"] \\
FTF \arrow[r, "T\mu_F"] & TF.
\end{tikzcd} \tag{10.1}$$

The remaining condition on λ is commutativity of the outer part of the diagram

$$\begin{tikzcd}
GGT \arrow[r, "GGT\eta_F"] \arrow[dd, "\mu_G T"'] & GGTF \arrow[r, "G\beta"] \arrow[dd, "\mu_G TF"] & GTF \arrow[r, "GT\eta_F F"] \arrow[d, "="] & GTFF \arrow[dl, "GT\mu_F"] \arrow[dr, "\beta F"] \arrow[d, phantom, "(1)"] & \\
 & & GTF \arrow[dr, "\beta"] & & TFF \arrow[dl, "T\mu_F"] \\
GT \arrow[r, "GT\eta_F"] & GTF \arrow[rr, "\beta"] & & TF. &
\end{tikzcd}$$

This follows since the inner diagrams are commutative: the left rectangle by naturality, the middle diagram since β is a G-morphism, and the right trapezium is just (10.1). □

10.3. Lifting of a tensor product to modules. The above proposition can be applied to characterise monads (S, μ, η) on a monoidal category $(\mathcal{C}, \otimes, E)$ (see [33]) for which the monoidal structure from $\mathcal{C}$ lifts to the category $\mathcal{C}_S$ of S-modules. The situation is described by the diagram

$$\begin{tikzcd}
\mathcal{C}_S \times \mathcal{C}_S \arrow[r, "-\overline{\otimes}-"] \arrow[d, "U_S \times U_S"'] & \mathcal{C}_S \arrow[d, "U_S"] \\
\mathcal{C} \times \mathcal{C} \arrow[r, "-\otimes-"] & \mathcal{C},
\end{tikzcd}$$

By 10.2(1), for the existence of some $-\overline{\otimes}-$ making the diagram commute one needs a natural transformation

$$\lambda : S(X \otimes Y) \to S(X) \otimes S(Y)$$

yielding commutative diagrams

$$\begin{tikzcd} SS(X\otimes Y) \arrow[rr, "\mu_{X\otimes Y}"] \arrow[d, "S\lambda"'] & & S(X\otimes Y) \arrow[d, "\lambda"] \\ S(S(X)\otimes S(Y)) \arrow[r, "\lambda_S"] & SS(X)\otimes SS(Y) \arrow[r, "\mu_X\otimes\mu_Y"] & S(X)\otimes S(Y), \end{tikzcd}$$

$$\begin{tikzcd} X\otimes Y \arrow[r, "\eta_{X\otimes Y}"] \arrow[dr, "\eta_X\otimes\eta_Y"'] & S(X\otimes Y) \arrow[d, "\lambda"] \\ & S(X)\otimes S(Y). \end{tikzcd}$$

This corresponds to the first three diagrams in [39, Section 7].

To make $\mathcal{C}_S$ a unital tensor category one has to require that E is an S-module for some morphism $S(E)\to E$ in $\mathcal{C}$ and that the coherence conditions are respected (diagrams (4) to (7) in [39, Section 7]). Such monads are called *Hopf monads* in Moerdijk [39] (see introduction).

10.4. Lifting for comonads. Let $\mathbb{F}=(F,\delta_F,\varepsilon_F)$ and $\mathbb{G}=(G,\delta_G,\varepsilon_G)$ be comonads on the categories $\mathbb{A}$ and $\mathbb{B}$, respectively, and let $T:\mathbb{A}\to\mathbb{B}$ be a functor.

A functor $\widehat{T}:\mathbb{A}^F\to\mathbb{B}^G$ is called a lifting of T if the diagram

$$\begin{tikzcd} \mathbb{A}^F \arrow[r, "\widehat{T}"] \arrow[d, "U^F"'] & \mathbb{B}^G \arrow[d, "U^G"] \\ \mathbb{A} \arrow[r, "T"] & \mathbb{B}, \end{tikzcd}$$

is commutative, where the U's denote the forgetful functors.

10.5. Theorem. *The liftings $\widehat{T}:\mathbb{A}^F\to\mathbb{B}^G$ of T are in bijective correspondence with the natural transformations $\omega:TF\to GT$ inducing commutativity of the diagrams*

$$\begin{tikzcd} TF \arrow[r, "T\delta_F"] \arrow[d, "\omega"'] & TFF \arrow[r, "\omega_F"] & GTF \arrow[d, "G\omega"] \\ GT \arrow[rr, "\delta_G T"] & & GGT, \end{tikzcd} \qquad \begin{tikzcd} TF \arrow[r, "T\varepsilon_F"] \arrow[d, "\omega"'] & T \\ GT. \arrow[ur, "\varepsilon_G T"'] & \end{tikzcd}$$

The correspondence associates to any natural transformation $\omega:TF\to GT$, the functor

$$\begin{aligned} \widehat{T}:\mathbb{A}^F\to\mathbb{B}^G,\quad A\xrightarrow{\varrho^A}F(A) &\mapsto T(A)\xrightarrow{T(\varrho^A)}TF(A)\xrightarrow{\omega_A}GT(A),\\ A\xrightarrow{f}A' &\mapsto T(A)\xrightarrow{T(f)}T(A'). \end{aligned}$$

Proof. The necessary constructions and proofs are dual to those of 10.2. □

10.6. Lifting of a tensor product to comodules. Let (T, δ, ε) be a comonad on a monoidal category $(\mathcal{C}, \otimes, E)$ for which the monoidal structure from $\mathcal{C}$ lifts to the category $\mathcal{C}^T$ of T-comodules. The situation is described by the diagram

$$\begin{tikzcd} \mathcal{C}^T \times \mathcal{C}^T \arrow[r, "-\widehat{\otimes}-"] \arrow[d, "U^T \times U^T"'] & \mathcal{C}^T \arrow[d, "U^T"] \\ \mathcal{C} \times \mathcal{C} \arrow[r, "-\otimes-"] & \mathcal{C}, \end{tikzcd}$$

By 10.5(1) for the existence of some $-\widehat{\otimes}-$ making the diagram commute one needs a natural transformation

$$\varphi : T(X) \otimes T(Y) \to T(X \otimes Y)$$

and a morphism $E \to T(E)$ yielding the commutative diagrams as required by 10.5 plus appropriate conditions to assure the coherence conditions.

11 Relations between endofunctors

In this section we will specialise the preceding observations to $\mathbb{A} = \mathbb{B}$ and endofunctors.

11.1. Lifting of the identity. Let $\mathbb{F} = (F, \mu, \eta)$, $\mathbb{F}' = (F', \mu', \eta')$ be monads and $\mathbb{G} = (G, \delta, \varepsilon)$, $\mathbb{G}' = (G', \delta', \varepsilon')$ be comonads on the category $\mathbb{A}$. Then $\overline{I} : \mathbb{A}_F \to \mathbb{A}_{F'}$ or $\widehat{I} : \mathbb{A}^G \to \mathbb{A}^{G'}$ are liftings of the identity if the corresponding diagrams commute:

$$\begin{tikzcd} \mathbb{A}_F \arrow[r, "\overline{I}"] \arrow[d, "U_F"'] & \mathbb{A}_{F'} \arrow[d, "U_{F'}"] \\ \mathbb{A} \arrow[r, "I"] & \mathbb{A}, \end{tikzcd} \qquad \begin{tikzcd} \mathbb{A}^G \arrow[r, "\widehat{I}"] \arrow[d, "U^G"'] & \mathbb{A}^{G'} \arrow[d, "U^{G'}"] \\ \mathbb{A} \arrow[r, "I"] & \mathbb{A}. \end{tikzcd}$$

(1) *There is a bijection between the liftings $\overline{I}$ of the identity functor and the monad morphisms $\alpha : F' \to F$.*

(2) *There is a bijection between the liftings $\widehat{I}$ of the identity functor and the comonad morphisms $\alpha : G \to G'$.*

Proof. The assertions follow from 10.2 and 10.5. □

In what follows we will consider the lifting of endofunctors to the category of some modules or comodules.

11.2. Lifting of endofunctors. Let F, G and T be endofunctors of the category $\mathbb{A}$. For the functors $\overline{T} : \mathbb{A}_F \to \mathbb{A}_F$ and $\widehat{T} : \mathbb{A}^G \to \mathbb{A}^G$ we have the diagrams

$$\begin{tikzcd} \mathbb{A}_F \arrow[r, "\overline{T}"] \arrow[d, "U_F"'] & \mathbb{A}_F \arrow[d, "U_F"] \\ \mathbb{A} \arrow[r, "T"] & \mathbb{A}, \end{tikzcd} \qquad \begin{tikzcd} \mathbb{A}^G \arrow[r, "\widehat{T}"] \arrow[d, "U^G"'] & \mathbb{A}^G \arrow[d, "U^G"] \\ \mathbb{A} \arrow[r, "T"] & \mathbb{A}, \end{tikzcd}$$

and we say that $\overline{T}$ or $\widehat{T}$ are liftings of T provided the corresponding diagrams are commutative.

Besides the situations considered before we may now also ask when the liftings of a monad T are again monads.

11.3. Lifting of monads to monads. *Let $\mathbb{F} = (F, \mu, \eta)$ be a monad and $T : \mathbb{A} \to \mathbb{A}$ any functor on the category $\mathbb{A}$.*

(1) *The liftings $\overline{T} : \mathbb{A}_F \to \mathbb{A}_F$ of T are in bijective correspondence with the natural transformations $\lambda : FT \to TF$ inducing commutativity of the diagrams*

$$\begin{tikzcd} FFT \arrow[rr, "\mu T"] \arrow[d, "F\lambda"'] & & FT \arrow[d, "\lambda"] \\ FTF \arrow[r, "\lambda F"] & TFF \arrow[r, "T\mu"] & TF, \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\eta T"] \arrow[dr, "T\eta"'] & FT \arrow[d, "\lambda"] \\ & TF. \end{tikzcd}$$

(2) *If $\mathbb{T} = (T, \mu', \eta')$ is a monad, then the lifting $\overline{T} : \mathbb{A}_F \to \mathbb{A}_F$ of T with natural transformation $\lambda : FT \to TF$ is a monad if and only if we have the commutative diagrams*

$$\begin{tikzcd} FTT \arrow[rr, "F\mu'"] \arrow[d, "\lambda T"'] & & FT \arrow[d, "\lambda"] \\ TFT \arrow[r, "T\lambda"] & TTF \arrow[r, "\mu' F"] & TF, \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "F\eta'"] \arrow[dr, "\eta' F"'] & FT \arrow[d, "\lambda"] \\ & TF. \end{tikzcd}$$

(3) *For a monad $\mathbb{T} = (T, \mu', \eta')$, a natural transformation $\lambda : FT \to TF$ induces a canonical monad structure on TF with unit $\eta'\eta : I \to TF$ if and only if the diagrams in* (1) *and* (2) *are commutative.*

In this case λ - more precisely (F, T, λ) - is called a monad entwining.

Proof. The assertion in (1) follows immediately from 10.2(1).

(2) The diagram in (2) is derived from the requirement that $\mu'_{F(A)}$ and $\eta'_{F(A)}$ are to be F-module morphisms for any $A \in \mathrm{Obj}(\mathbb{A})$. The first of these conditions corresponds to commutativity of the rectangle $(\star)$ in the diagram

$$\begin{tikzcd} FTT \arrow[rr, "\lambda T"] \arrow[dr, "FTT\eta"] \arrow[dd, "F\mu'"'] & & TFT \arrow[r, "T\lambda"] \arrow[d, "TFT\eta"'] & TTF \arrow[d, "TTF\eta"'] \arrow[dr, "="] & \\ & FTTF \arrow[r, "\lambda TF"] \arrow[d, "F\mu' F"] & TFTF \arrow[r, "T\lambda F"] \arrow[r, phantom, "(\star)"', shift right=3] & TTFF \arrow[r, "TT\mu"] & TTF \arrow[d, "\mu' F"] \\ FT \arrow[r, "FT\eta"] \arrow[d, "\lambda"'] & FTF \arrow[rr, "\lambda F"] \arrow[d, "\lambda F"'] & & TFF \arrow[r, "T\mu"] & TF \\ TF \arrow[r, "TF\eta"'] & TFF \arrow[urrr, "T\mu"'] & & & \end{tikzcd} .$$

The new inner diagrams are commutative by naturality or functoriality of composition. Since $\mu \circ (F\eta)$ is the identity, the outer morphisms yield the commutative rectangle in (2).

Now assume the diagram in (2) to be given. Applying this to F, we obtain the big rectangle in the diagram

$$\begin{tikzcd} FTTF \arrow[rr, "F\mu' F"] \arrow[d, "\lambda TF"'] & & FTF \arrow[d, "\lambda F"] \\ TFTF \arrow[r, "T\lambda F"] & TTFF \arrow[r, "\mu' FF"] \arrow[d, "TT\mu"'] & TFF \arrow[d, "T\mu"] \\ & TTF \arrow[r, "\mu' F"] & TF, \end{tikzcd}$$

in which also the small rectangle is commutative. The outer morphisms yield the commutative diagram $(\star)$.

(3) A product on TF is defined by

$$\tilde{\mu} : TFTF \to TF, \qquad \begin{tikzcd} & & TFF \arrow[dr, "T\mu"] & \\ TFTF \arrow[r, "T\lambda F"] & TTFF \arrow[ur, "\mu' FF"] \arrow[dr, "TT\mu"'] & & TF \\ & & TTF \arrow[ur, "\mu' F"'] & \end{tikzcd} \tag{11.1}$$

We show that (1) and (2) imply associativity of this product. We alread know that (2) implies commutativity of the diagram $(\star)$ in the proof of (2). Applying T from the left and F from the right to $(\star)$, we get a commutative diagram $(\star\star)$ in the diagram

$$\begin{tikzcd} TFTFTF \arrow[r, "T\lambda FTF"] \arrow[d, "TFT\lambda F"'] & TTFFTF \arrow[rr, "TT\mu TF"] \arrow[d, "TTF\lambda F"'] & {} \arrow[d, phantom, "(1)"] & TTFTF \arrow[r, "\mu' FTF"] \arrow[d, "TT\lambda F"'] & TFTF \arrow[d, "T\lambda F"] \\ TFTTFF \arrow[r, "T\lambda TFF"'] \arrow[d, "TF\mu' FF"'] & TTFTFF \arrow[r, "TT\lambda FF"] \arrow[r, phantom, "(\star\star)"', shift right=3] & TTTFFF \arrow[r, "TTT\mu F"] & TTTFF \arrow[r, "\mu' TFF"] \arrow[d, "T\mu' FF"] & TTFF \arrow[d, "\mu' FF"] \\ TFTFF \arrow[rr, "T\lambda FF"] \arrow[d, "TFT\mu"'] & & TTFFF \arrow[r, "TT\mu F"] \arrow[d, "TTF\mu"'] & TTFF \arrow[r, "\mu' FF"] \arrow[d, "TT\mu"] & TFF \arrow[d, "T\mu"] \\ TFTF \arrow[rr, "T\lambda F"] & & TTFF \arrow[r, "TT\mu"] & TTF \arrow[r, "\mu' F"] & TF. \end{tikzcd}$$

Moreover, diagram (1) is commutative by condition (1) and the remaining diagrams are commutative by functoriality of composition or associativity properties of μ and μ'. Now the outer morphisms show associativity of multiplication of TF. $\square$

Obviously TF being a monad need not imply that T and F both are monads.

In 11.3(2), conditions are given for the lifting of a monad to be a monad. More generally one may ask how the lifted functor $\overline{T}$ becomes a monad without T being required to be a monad. Then of course some other data must be given.

In the definition of the product on TF in (11.1), the product μ' of T is only used in the form $\mu' F : TTF \to TF$ and the unit η' of T is used for the unit of TF in the form $\eta' F : F \to TF$.

So we may consider more general natural transformations, for example,

$$\nu : TTF \to TF, \quad \xi : F \to TF,$$

to define a multiplication and a unit on TF. Of course, associativity and unitality of TF will lead to special conditions on the maps involved. This leads to the notion of a *wreath* which was introduced by Lack and Street [32] to describe monads in certain 2-categories. In L. Koautit [23] the maps ν and ξ mentioned above are used to describe wreaths. In [32] the transformations

$$\bar{\nu} : TT \to TF, \quad \sigma : I \to TF,$$

were considered to define a monad structure on TF (see also [59]).

11.4. Liftings as monads. Let $\mathbb{F} = (F, \mu, \eta)$ be a monad and $T : \mathbb{A} \to \mathbb{A}$ any functor. Assume T can be lifted to $\mathbb{A}_F \to \mathbb{A}_F$ by the entwining $\lambda : FT \to TF$ and that there are given natural transformations $\bar{\nu} : TT \to TF$ and $\sigma : I \to TF$.

Then the lifting $\widehat{T}$ induces a monad on $\mathbb{A}_F$ provided TF has a monad structure $(TF, \overline{\mu}, \sigma)$ with

$$\overline{\mu} : \ TFTF \xrightarrow{T\lambda F} TTFF \xrightarrow{TT\mu} TTF \xrightarrow{\bar{\nu}F} TFF \xrightarrow{T\mu} TF,$$

provided the data induce commutativity of the diagrams (cocycle condition and twisted condition)

$$\begin{tikzcd}
TTT \arrow[r, "\bar{\nu}T"] \arrow[d, "T\bar{\nu}"'] & TFT \arrow[r, "T\lambda"] & TTF \arrow[r, "\bar{\nu}_F"] & TFF \arrow[d, "T\mu"] \\
TTF \arrow[r, "\bar{\nu}F"] & TFF \arrow[rr, "T\mu"] & & TF,
\end{tikzcd}$$

$$\begin{tikzcd}
FTT \arrow[r, "\lambda T"] \arrow[d, "F\bar{\nu}"'] & TFT \arrow[r, "T\lambda"] & TTF \arrow[r, "\bar{\nu}F"] & TFF \arrow[d, "T\mu"] \\
FTF \arrow[r, "\lambda F"] & TFF \arrow[rr, "T\mu"] & & TF,
\end{tikzcd}$$

The unitality conditions come out as

$$\begin{tikzcd}
F \arrow[rr, "\sigma F"] \arrow[d, "F\sigma"'] & & TFF \arrow[d, "T\mu"] \\
FTF \arrow[r, "\lambda F"] & TFF \arrow[r, "T\mu"] & TF,
\end{tikzcd}$$

$$\begin{tikzcd}
T \arrow[r, "T\sigma"] \arrow[d, "T\eta"'] & TTF \arrow[d, "\bar{\nu}F"] \\
TF & TFF, \arrow[l, "T\mu"]
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, "\sigma T"] \arrow[d, "T\eta"'] & TFT \arrow[r, "T\lambda"] & TTF \arrow[d, "\bar{\nu}F"] \\
TF & & TFF. \arrow[ll, "T\mu"]
\end{tikzcd}$$

As a special case one may take F to be the identity functor. Then the conditions reduce to T being a monad.

Given λ and $\bar{\nu}$, the multiplication $\overline{\mu}$ is obtained by the diagram

$$\begin{tikzcd}
TFTF \arrow[rr, "\overline{\mu}"] \arrow[d, "T\lambda_F"'] & & TF \\
TTFF \arrow[r, "\bar{\nu}_{FF}"] & TFFF \arrow[r, "T\mu_F"] & TFF. \arrow[u, "T\mu"']
\end{tikzcd}$$

If the monad $(TF, \overline{\mu}, \overline{\eta})$ is given, suitable λ and $\bar{\nu}$ are defined by the diagrams

$$\begin{tikzcd}
FT \arrow[r, "\lambda"] \arrow[d, "\iota_{FT}"'] & TF \\
TFT \arrow[r, "TFT\eta"] & TFTF, \arrow[u, "\overline{\mu}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
TT \arrow[r, "\bar{\nu}"] \arrow[d, "T\eta_T"'] & TF \\
TFT \arrow[r, "TFT\eta"] & TFTF. \arrow[u, "\overline{\mu}"']
\end{tikzcd}$$

11.5. Liftings as monads II. Let $\mathbb{F} = (F, \mu, \eta)$ be a monad and $T : \mathbb{A} \to \mathbb{A}$ any functor. Assume T can be lifted to $\mathbb{A}_F \to \mathbb{A}_F$ by the entwining $\lambda : FT \to TF$ and that there are given natural (F, F)-bimodule transformations

$$\nu : TTF \to TF, \quad \xi : F \to TF,$$

Then the lifting $\widehat{T}$ induces a monad on $\mathbb{A}_F$ provided TF has a monad structure $(TF, \overline{\mu}, \sigma)$

$$\tilde{\mu} : \ TFTF \xrightarrow{T\lambda F} TTFF \xrightarrow{TT\mu} TTF \xrightarrow{T\nu} TF,$$

provided the data induce commutativity of the diagrams

$$\begin{tikzcd}
TTF \arrow[r, "\nu"] & TF \\
TF, \arrow[u, "T\xi"] \arrow[ur, "="'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
TTF \arrow[r, "\nu"] & TF \\
TFT \arrow[u, "T\lambda"] & FT \arrow[l, "\xi T"'] \arrow[u, "\lambda"']
\end{tikzcd}$$

$$\begin{tikzcd}
TTFT \arrow[r, "\nu T"] \arrow[d, "TT\lambda"'] & TFT \arrow[r, "T\lambda"] & TTF \arrow[d, "\nu"] \\
TTTF \arrow[r, "T\nu"] & TTF \arrow[r, "\nu"] & TF.
\end{tikzcd}$$

Proof.

$$\begin{tikzcd}
TFTFTF \arrow[r, "T\lambda FTF"] \arrow[d, "TFT\lambda F"'] & TTFFTF \arrow[rr, "TT\mu TF"] \arrow[d, "TTF\lambda F"'] \arrow[drr, phantom, "(1)"] & & TTFTF \arrow[r, "\nu TF"] \arrow[d, "TT\lambda F"'] \arrow[dr, phantom, "(2)"] & TFTF \arrow[d, "T\lambda F"] \\
TFTTFF \arrow[r, "T\lambda TFF"'] \arrow[d, "TF\nu F"'] & TTFTFF \arrow[r, "TT\lambda FF"] \arrow[dr, phantom, "(\star\star)"] & TTTFFF \arrow[r, "TTT\mu F"] & TTTFF \arrow[d, "T\nu F"] & TTFF \arrow[d, "\nu F"] \\
TFTFF \arrow[rr, "T\lambda FF"] \arrow[d, "TFT\mu"'] & & TTFFF \arrow[r, "TT\mu F"] \arrow[d, "TTF\mu"'] & TTFF \arrow[r, "\nu F"] \arrow[d, "TT\mu"'] \arrow[dr, phantom, "(3)"] & TFF \arrow[d, "T\mu"] \\
TFTF \arrow[rr, "T\lambda F"] & & TTFF \arrow[r, "TT\mu"] & TTF \arrow[r, "\nu"] & TF.
\end{tikzcd}$$

Diagram (1) is commutative by the entwining property, diagram $(\star\star)$ is commutative since ν is a left F-module morphism (compare proof of 11.3(2)), diagram (2) is commutative by assumption (applied to F), and commutativity of diagram (3) follows since ν is a right F-module morphism. The outer morphisms show associativity of the multiplication $\tilde{\mu}$.

From the two commutative diagram

$$\begin{tikzcd}
TTF \arrow[d, "T\eta TF"'] \arrow[dr, "TT\eta"'] \arrow[drr, "="] & & & \\
TFTF \arrow[r, "T\lambda F"'] & TTFF \arrow[r, "TT\mu"'] & TTF \arrow[r, "\nu"] & TF \\
TFT \arrow[u, "TFT\eta"] \arrow[r, "T\lambda"'] & TTF, \arrow[u, "TTF\eta"] \arrow[ur, "="'] & &
\end{tikzcd}$$

we obtain $\tilde{\mu} \circ T\eta TF = \nu$ and $\tilde{\mu} \circ TFT\eta = \nu \circ T\lambda$. □

Dual to the constructions considered in 11.3 one obtains

11.6. Lifting of comonads to comonads. *Let $\mathbb{G} = (G, \delta, \varepsilon)$ be a comonad and $T : \mathbb{A} \to \mathbb{A}$ any functor on the category $\mathbb{A}$.*

(1) *The liftings $\widehat{T} : \mathbb{A}^G \to \mathbb{A}^G$ of T are in bijective correspondence with the natural transformations $\varphi : TG \to GT$ inducing the commutative diagrams*

$$\begin{tikzcd} TG \arrow[r, "T\delta"] \arrow[d, "\varphi"'] & TGG \arrow[r, "\varphi G"] & GTG \arrow[d, "G\varphi"] \\ GT \arrow[rr, "\delta T"] & & GGT, \end{tikzcd} \qquad \begin{tikzcd} TG \arrow[r, "T\varepsilon"] \arrow[d, "\varphi"'] & T \\ GT. \arrow[ur, "\varepsilon_T"'] & \end{tikzcd}$$

(2) *If $\mathbb{T} = (T, \delta', \varepsilon')$ is a comonad, then the lifting $\widehat{T} : \mathbb{A}^G \to \mathbb{A}^G$ of T with natural transformation $\varphi : TG \to GT$ is a comonad if and only if we have the commutative diagrams*

$$\begin{tikzcd} TG \arrow[r, "\delta' G"] \arrow[d, "\varphi"'] & TTG \arrow[r, "T\varphi"] & TGT \arrow[d, "\varphi_T"] \\ GT \arrow[rr, "G\delta'"] & & GTT, \end{tikzcd} \qquad \begin{tikzcd} TG \arrow[r, "\varepsilon'_G"] \arrow[d, "\varphi"'] & G \\ GT. \arrow[ur, "G\varepsilon'"'] & \end{tikzcd}$$

(3) *For a comonad $\mathbb{T} = (T, \delta', \varepsilon')$, a natural transformation $\varphi : TG \to GT$ induces a canonical comonad structure on TG if and only if the diagrams in* (1) *and* (2) *are commutative.*

Proof. (1) is a special case of 10.5 and the diagram shows that φ_A is a G-comodule morphism for any $A \in \text{Obj}(\mathbb{A})$.

(2) The diagrams are derived from the conditions that $\delta'_{G(A)}$ and $\varepsilon'_{G(A)}$ must be G-comodule morphisms for all $A \in \text{Obj}(\mathbb{A})$. This is seen by arguments dual to those of the proof of 11.3.

(3) This goes back to Barr [2, Theorem 2.2]. □

Similar to the composition for monads, a canonical comonad structure on TG need not imply that T and G are comonads.

11.7. Definition. Given two comonads $\mathbb{G} = (G, \delta, \varepsilon)$ and $\mathbb{T} = (T, \delta', \varepsilon')$ on a category $\mathbb{A}$, a natural transformation $\varphi : TG \to GT$ is said to be *comonad distributive* provided the diagrams in 11.6(1) and (2) are commutative.

11.8. Tensor product of coalgebras. Given two R-coalgebras C, D, and an R-linear map

$$\varphi : C \otimes_R D \to D \otimes_R C,$$

the tensor product $C \otimes_R D$ can be made into a coalgebra by putting

$$\Delta = (I_C \otimes \varphi \otimes I_D) \circ (\Delta_C \otimes \Delta_D).$$

If C and D are coassociative, the functors $-\otimes_R C$ and $-\otimes_R D$ are comonads on the category of R-modules. Then the coproduct defined on $C\otimes_R D$ is coassociative and counital if and only if $-\otimes_R C\otimes_R D$ is a comonad for the R-modules, that is, φ has to induce commutativity of the corresponding diagrams in 11.6. For this special case the conditions are formulated in Caenepeel, Ion, Militaru and Zhu [16, Theorem 3.4] and also in [15, 2.14].

Similar to the case of algebras (see **??**), for a prebraiding τ on ${}_R\mathbf{M}$ and R-coalgebras C, D, the natural morphism

$$-\otimes_R \tau_{C,D} : -\otimes_R C\otimes_R D \to -\otimes_R D\otimes_R C$$

is comonad distributive (the diagrams in 11.6 commute) and thus induces a coassociative coproduct on $C\otimes_R D$.

In particular the twist map $\mathrm{tw}: C\otimes_R D\to D\otimes_R C$ satisfies the conditions imposed yielding the standard coproduct on $C\otimes_R D$.

11.9. Liftings as comonads. In 11.6(2), conditions are given for the lifting of a comonad to be a comonad. Dual to the case of monads one may ask how the lifted functor $\widehat{T}$ of a comonad $\mathbb{G}=(G,\delta,\varepsilon)$ becomes a comonad without T being a comonad. This can be handled similar to the constructions considered in 11.4. In particular, based on a natural transformation $\varphi: TG\to GT$ satisfying 11.6(1), natural transformations $\bar\nu: TG\to TT$ and $\varepsilon: T\to I$ are needed satisfying appropriate conditions.

In this section we consider relationships between monads and comonads.

11.10. Lifting of monads for comonads. *Let $\mathbb{G}=(G,\delta,\varepsilon)$ be a comonad and $T:\mathbb{A}\to\mathbb{A}$ any functor on the category $\mathbb{A}$.*

(1) *The liftings $\widehat{T}:\mathbb{A}^G\to\mathbb{A}^G$ of T are in bijective correspondence with the natural transformations $\varphi: TG\to GT$ inducing the commutative diagrams*

$$\begin{tikzcd} TG \arrow[r, "T\delta"] \arrow[d, "\varphi"'] & TGG \arrow[r, "\varphi G"] & GTG \arrow[d, "G\varphi"] \\ GT \arrow[rr, "\delta T"] & & GGT, \end{tikzcd} \qquad \begin{tikzcd} TG \arrow[r, "T\varepsilon"] \arrow[d, "\varphi"'] & T \\ GT. \arrow[ur, "\varepsilon T"'] & \end{tikzcd}$$

(2) *If $\mathbb{T}=(T,\mu,\eta)$ is a monad, then the lifting $\widehat{T}:\mathbb{A}^G\to\mathbb{A}^G$ of T with associated natural transformation $\varphi: TG\to GT$ is a monad if and only if we have the commutative diagrams*

$$\begin{tikzcd} TTG \arrow[rr, "\mu G"] \arrow[d, "T\varphi"'] & & TG \arrow[d, "\varphi"] \\ TGT \arrow[r, "\varphi T"] & GTT \arrow[r, "G\mu"] & GT, \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, "G\eta"'] & TG \arrow[d, "\varphi"] \\ & GT. \end{tikzcd}$$

Proof. (1) follows from 10.5 and the diagrams are induced by the requirement that the φ_A are G-comodule morphisms for all $A\in\mathrm{Obj}(\mathbb{A})$.

(2) These diagrams are consequences of the condition that μ_A and η_A are G-comodule morphisms but they can also be read as condition for φ_A being a T-module morphism for any $A \in \mathrm{Obj}(\mathbb{A})$. $\square$

11.11. Lifting of comonads for monads. *Let $\mathbb{F} = (F, \mu, \eta)$ be a monad and $T : \mathbb{A} \to \mathbb{A}$ any functor on the category $\mathbb{A}$.*

(1) *The liftings $\overline{T} : \mathbb{A}_F \to \mathbb{A}_F$ of T are in bijective correspondence with the natural transformations $\lambda : FT \to TF$ inducing the commutative diagrams*

$$\begin{tikzcd} FFT \arrow[rr, "\mu_T"] \arrow[d, "F\lambda"'] & & FT \arrow[d, "\lambda"] \\ FTF \arrow[r, "\lambda_F"] & TFF \arrow[r, "T\mu"] & TF, \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\eta_T"] \arrow[dr, "T\eta"'] & FT \arrow[d, "\lambda"] \\ & TF. \end{tikzcd}$$

(2) *If $\mathbb{T} = (T, \delta, \varepsilon)$ is a comonad, then the lifting $\overline{T} : \mathbb{A}_F \to \mathbb{A}_F$ of T with associated natural transformation $\lambda : FT \to TF$ is a comonad if and only if we have the commutative diagrams*

$$\begin{tikzcd} FT \arrow[r, "F\delta"] \arrow[d, "\lambda"'] & FTT \arrow[r, "\lambda_T"] & TFT \arrow[d, "T\lambda"] \\ TF \arrow[rr, "\delta_F"] & & TTF, \end{tikzcd} \qquad \begin{tikzcd} FT \arrow[r, "F\varepsilon"] \arrow[d, "\lambda"'] & F \\ TF. \arrow[ur, "\varepsilon_F"'] \end{tikzcd}$$

Proof. (1) follows from 10.2 and the diagrams are induced by the requirement that the λ_A are F-module morphisms for any $A \in \mathrm{Obj}(\mathbb{A})$.

(2) These diagrams are consequences of the condition that δ_A and ε_A are F-module morphisms but they can also be interpreted as the condition that λ_A is a T-comodule morphism for any $A \in \mathrm{Obj}(\mathbb{A})$. $\square$

We observe that in 11.10 and 11.11 essentially the same diagrams arise.

11.12. Mixed distributive laws. Let $\mathbb{F} = (F, \mu, \eta)$ be a monad and $\mathbb{G} = (G, \delta, \varepsilon)$ a comonad on the category $\mathbb{A}$. Then a natural transformation

$$\lambda : FG \to GF$$

is said to be *mixed distributive* or *entwining* provided it induces commutative diagrams

$$\begin{tikzcd} FFG \arrow[rr, "\mu_G"] \arrow[d, "F\lambda"'] & & FG \arrow[d, "\lambda"] \\ FGF \arrow[r, "\lambda_F"] & GFF \arrow[r, "G\mu"] & GF, \end{tikzcd} \qquad \begin{tikzcd} FG \arrow[r, "F\delta"] \arrow[d, "\lambda"'] & FGG \arrow[r, "\lambda_G"] & GFG \arrow[d, "G\lambda"] \\ GF \arrow[rr, "\delta_F"] & & GGF, \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r, "\eta_G"] \arrow[dr, "G\eta"'] & FG \arrow[d, "\lambda"] \\ & GF, \end{tikzcd} \qquad \begin{tikzcd} FG \arrow[r, "F\varepsilon"] \arrow[d, "\lambda"'] & F \\ GF. \arrow[ur, "\varepsilon_F"'] \end{tikzcd}$$

The suggestion to consider distributive laws of mixed type goes back to Beck [6, page 133] (see Remarks 11.14). The interest in these structures is based on the following theorem which follows from 11.10 and 11.11.

11.13. Characterisation of entwinings. *For a monad $\mathbb{F} = (F, \mu, \eta)$ and a comonad $\mathbb{G} = (G, \delta, \varepsilon)$ on the category $\mathbb{A}$, consider the diagrams*

$$\begin{tikzcd} \mathbb{A}_F \arrow[r, "\overline{G}"] \arrow[d, "U_F"'] & \mathbb{A}_F \arrow[d, "U_F"] \\ \mathbb{A} \arrow[r, "G"] & \mathbb{A}, \end{tikzcd} \qquad \begin{tikzcd} \mathbb{A}^G \arrow[r, "\widehat{F}"] \arrow[d, "U^G"'] & \mathbb{A}^G \arrow[d, "U^G"] \\ \mathbb{A} \arrow[r, "F"] & \mathbb{A}. \end{tikzcd}$$

The following conditions are equivalent:

(a) *There is an entwining natural transformation $\lambda : FG \to GF$;*
(b) *$\overline{G} : \mathbb{A}_F \to \mathbb{A}_F$ is a lifting of G and has a comonad structure;*
(c) *$\widehat{F} : \mathbb{A}^G \to \mathbb{A}^G$ is a lifting of F and has a monad structure.*

11.14. Remarks. The preceding theorem was first formulated 1973 by van Osdol in [53, Theorem IV.1]. It was extended to $\mathcal{V}$-categories in Wolff [61, Theorem 2.4] and was rediscovered in 1997 by Turi and Plotkin in the context of operational semantics in [52, Theorem 7.1]. In the same year the corresponding notion for tensor functors was considered by Brzeziński and Majid who coined the name *entwining structure* for a mixed distributive law for an algebra A and a coalgebra C over a commutative ring R in [14, Definition 2.1] (see 11.17). The connection between this notions is also mentioned in Hobst and Pareigis [28].

It was observed by Takeuchi that these structures are closely related to corings (see [12, Proposition 2], [15, 32.6]). This is a special case of 11.13(b) since the coring $A \otimes_R C$ is just a comonad on the category of right A-modules. The comultiplication is a special case of the constructions considered in the next section. Similarly, by 11.13(c), $C \otimes_R A$ can be seen as a monad on the category of right C-comodules.

11.15. Comultiplication induced by units. Let F, G be endofunctors on a category $\mathbb{A}$ and $\eta : I \to F$ a natural transformation. Then we have natural transformations

$$\eta G : G \to FG, \quad G\eta : G \to GF,$$

and naturality of η implies commutativity of the diagrams

$$\begin{tikzcd} G \arrow[r, "\eta G"] \arrow[d, "\eta G"'] & FG \arrow[d, "F\eta G"] \\ FG \arrow[r, "\eta FG"] & FFG, \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "G\eta"] \arrow[d, "G\eta"'] & GF \arrow[d, "G\eta F"] \\ GF \arrow[r, "GF\eta"] & GFF. \end{tikzcd}$$

For $F = G$ the diagrams show that both ηF and $F\eta$ induce coassociative comultiplications on F.

If there is a coassociative comultiplication $\delta : G \to GG$, then we can define a comultiplication on FG by

$$\bar{\delta} : FG \xrightarrow{F\delta} FGG \xrightarrow{FG\eta G} FGFG,$$

which is coassociative by commutativity of the diagram

$$\begin{tikzcd}
FG \arrow[r, "F\delta"] \arrow[d, "F\delta"'] & FGG \arrow[r, "FG\eta G"] \arrow[d, "FG\delta"] & FGFG \arrow[d, "FGF\delta"] \\
FGG \arrow[r, "F\delta G"] \arrow[d, "FG\eta G"'] & FGGG \arrow[r, "FG\eta GG"] \arrow[d, "FGG\eta G"] & FGFGG \arrow[d, "FGFG\eta G"] \\
FGFG \arrow[r, "F\delta FG"] & FGGFG \arrow[r, "FG\eta GFG"] & FGFGFG.
\end{tikzcd}$$

The left top rectangle commutes by coassociativity of δ, the right top rectangle by naturality of η, the left bottom rectangle by naturality of δ and the right bottom rectangle again by naturality of η.

For a monad $\mathbb{F} = (F, \mu, \eta)$ the comultiplication on FG can also be derived from general properties of adjoint functors.

Symmetrically, a coassociative comultiplication for GF is defined by

$$\tilde{\delta} : GF \xrightarrow{\delta_F} GGF \xrightarrow{G\eta GF} GFGF.$$

In case a natural transformation $\varepsilon : G \to I$ is given, we have natural transformations $\varepsilon_F : GF \to F$ and $F\varepsilon : FG \to F$ allowing to dualise the above constructions. Then an associative multiplication $\mu : FF \to F$ induces associative multiplications on GF and FG.

Now let $\mathbb{F} = (F, \mu, \eta)$ be a monad and $\mathbb{G} = (G, \delta, \varepsilon)$ a comonad on $\mathbb{A}$ with a natural transformation $\lambda : FG \to GF$ satisfying $\lambda \circ \eta G = G\eta$ (left triangle in 11.12). Then we have the commutative diagram

$$\begin{tikzcd}
 & FGG \arrow[dr, "F\eta GG"'] \arrow[drr, "FG\eta G"] & & \\
FG \arrow[ur, "F\delta"] \arrow[dr, "F\eta G"'] & & FFGG \arrow[r, "F\lambda G"'] & FGFG \\
 & FFG, \arrow[ur, "FF\delta"'] & &
\end{tikzcd}$$

showing that the coproduct on FG induced by an entwining λ is the same as the one considered above.

11.16. Mixed bimodules. Given a monad $\mathbb{F} = (F, \mu, \eta)$ and a comonad $\mathbb{G} = (G, \delta, \varepsilon)$ on the category $\mathbb{A}$ with an entwining $\lambda : FG \to GF$, *λ-bimodules* or *mixed bimodules* are defined as those $A \in \mathrm{Obj}(\mathbb{A})$ with morphisms

$$F(A) \xrightarrow{h} A \xrightarrow{k} G(A)$$

such that (A, h) is an $\mathbb{F}$-module and (A, k) is a $\mathbb{G}$-comodule satisfying the pentagonal law

$$\begin{tikzcd}
F(A) \arrow[r, "h"] \arrow[d, "F(k)"'] & A \arrow[r, "k"] & G(A) \\
FG(A) \arrow[rr, "\lambda_A"] & & GF(A). \arrow[u, "G(h)"']
\end{tikzcd}$$

A morphism $f : A \to A'$ between two λ-bimodules is a *bimodule morphism* provided it is both an F-module and a G-comodule morphism.

These notions yield the category of λ-bimodules which we denote by $\mathbb{A}_F^G$. This category can also be considered as the category of $\widehat{G}$-comodules for the comonad $\widehat{G} : \mathbb{A}_F \to \mathbb{A}_F$ and also as the category of $\overline{F}$-modules for the monad $\overline{F} : \mathbb{A}^G \to \mathbb{A}^G$ (e.g. [52, 7.1]). For every F-module A, $G(A)$ is a λ-bimodule and for any G-comodule A', $F(A')$ is a λ-bimodule canonically. In particular, for every $A \in \mathrm{Obj}(\mathbb{A})$, $FG(A)$ and $GF(A)$ are λ-bimodules.

As a sample we draw the diagram showing that, for any F-module $\varrho_A : F(A) \to A$, $G(A)$ is a λ-bimodule with module structure given by the composition $G\rho_A \circ \lambda_A : FG(A) \to G(A)$:

$$\begin{tikzcd}
FG(A) \arrow[r, "\lambda_A"] \arrow[dd, "F\delta_A"'] & GF(A) \arrow[r, "G\varrho_A"] \arrow[drr, "\delta_{F(A)}"'] & G(A) \arrow[r, "\delta_A"] & GG(A) \\
 & & & GGF(A) \arrow[u, "GG\varrho_A"'] \\
FGG(A) \arrow[rrr, "\lambda_{G(A)}"] & & & GFG(A). \arrow[u, "G\lambda_A"']
\end{tikzcd}$$

The triangle is commutative by naturality of δ, the pentagon is commutative by one of the mixed distributive laws.

11.17. Entwined algebras and coalgebras. Given an R-algebra (A, μ, η) and an R-coalgebra (C, Δ, ε), the functor $- \otimes_R A$ is a monad and $- \otimes_R C$ is a comonad on the category of R-modules.

If the functor $- \otimes_R C$ can be lifted to the A-modules (equivalently $- \otimes_R A$ can be lifted to the C-comodules) then there is an R-linear map

$$\psi : C \otimes_R A \to A \otimes_R C,$$

and the diagrams in 11.12 yield the conditions for an *entwining structure* introduced by Brzeziński and Majid in [14] (see *bow-tie diagram* in [15, 32.2]):

$$\begin{tikzcd}
C \otimes A \otimes A \arrow[rr, "I\otimes\mu"] \arrow[d, "\psi\otimes I"'] & & C \otimes A \arrow[r, "\Delta\otimes I"] \arrow[d, "\psi"] & C \otimes C \otimes A \arrow[r, "I\otimes\psi"] & C \otimes A \otimes C \arrow[d, "\psi\otimes I"] \\
A \otimes C \otimes A \arrow[r, "I\otimes\psi"] & A \otimes A \otimes C \arrow[r, "\mu\otimes I"] & A \otimes C \arrow[rr, "I\otimes\Delta"] & & A \otimes C \otimes C,
\end{tikzcd}$$

$$\begin{tikzcd}
C \arrow[r, "I\otimes\eta"] \arrow[dr, "\eta\otimes I"'] & C \otimes A \arrow[r, "\varepsilon\otimes I"] \arrow[d, "\psi"] & A \\
 & A \otimes C. \arrow[ur, "I\otimes\varepsilon"'] &
\end{tikzcd}$$

A comultiplication on $A \otimes_R C$ is defined by the general formalism considered in 11.15 making $A \otimes_R C$ an A-coring.

Let M be an R-module with an A-module structure $\varrho_M : M \otimes_R A \to M$ and a C-comodule structure $\varrho^M : M \to M \otimes_R C$. Then M is an *entwined module* if the diagram

$$\begin{tikzcd}
M \otimes A \arrow[r, "\varrho_M"] \arrow[d, "\varrho^M \otimes I_A"'] & M \arrow[r, "\varrho^M"] & M \otimes C \\
M \otimes C \otimes A \arrow[rr, "I \otimes \psi"] & & M \otimes A \otimes C, \arrow[u, "\varrho_M \otimes I"']
\end{tikzcd}$$

is commutative (e.g. [15, 32.4]). This means that ϱ_M is a comodule morphism when $M \otimes_R A$ is considered as a C-comodule with structure map $(I_M \otimes \psi) \circ (\varrho^M \otimes I_A)$, and ϱ^M is an A-module morphism when $M \otimes_R C$ is an A-module with structure map $(\varrho_M \otimes I_C) \circ (I_M \otimes \psi)$. Observe the interplay between these structures: given an entwining ψ the diagram imposes conditions on ϱ_M or ϱ^M. On the other hand, if these two morphisms are given the problem is to find a suitable ψ.

Notice that A need not be a C-comodule unless it has a *group like element*. For more details the reader may consult [15, Section 32].

A *braiding* on the category of entwined modules induced by a morphism $C \otimes_R C \to A \otimes_R A$ is considered by Hobst and Pareigis in [28, Theorem 5.5].

11.18. Galois comodules. Let $\mathcal{C}$ be a coring over a ring A and $P \in \mathbb{M}^{\mathcal{C}}$ with $S := \mathrm{End}^G P$. Then there is an adjoint pair of functors

$$- \otimes_S P : \mathbb{M}_S \to \mathbb{M}^{\mathcal{C}}, \quad \mathrm{Hom}^{\mathcal{C}}(P, -) : \mathbb{M}^{\mathcal{C}} \to \mathbb{M}_S,$$

with counit $\mathrm{ev} : \mathrm{Hom}^{\mathcal{C}}(P, -) \otimes_S P \to I_{\mathbb{M}^{\mathcal{C}}}$, and, by **??**, there is a functorial morphism

$$\mathrm{ev}_{\mathcal{C}} : \mathrm{Hom}_A(P, -) \otimes_S P \to - \otimes_A \mathcal{C}.$$

P is called a *Galois comodule* provided $\mathrm{ev}_{\mathcal{C}}$ is an isomorphism. For further details about these comodules we refer to [58].

11.19. Bialgebras and Hopf modules. Consider an R-module B which is both an R-algebra $\mu : B \otimes_R B \to B$, $\eta : R \to B$, and an R-coalgebra $\Delta : B \to B \otimes_R B$, $\varepsilon : B \to R$. Define a linear map ψ by commutativity of the diagram

$$\begin{tikzcd}
B \otimes B \arrow[r, "\psi"] \arrow[d, "I \otimes \Delta"'] & B \otimes B \\
B \otimes B \otimes B \arrow[r, "\mathsf{tw} \otimes I"] & B \otimes B \otimes B \arrow[u, "I \otimes \mu"']
\end{tikzcd}$$

which produces

$$\psi : B \otimes_R B \to B \otimes_R B, \quad a \otimes b \mapsto (1 \otimes a)\Delta(b).$$

To make B a *bialgebra*, μ and η must be coalgebra maps (equivalently, Δ and ε are to be algebra maps) with respect to the obvious product and coproduct on $B \otimes_R B$ (induced by tw). This can be expressed by commutativity of the set of diagrams

$$
\begin{tikzcd}
B \otimes B \arrow[r, "\mu"] \arrow[d, "\Delta\otimes I"'] & B \arrow[r, "\Delta"] & B \otimes B \\
B \otimes B \otimes B \arrow[rr, "I\otimes\psi"] & & B \otimes B \otimes B, \arrow[u, "\mu\otimes I"']
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "\eta"] \arrow[d, "\eta"'] & B \arrow[d, "\eta\otimes I"] \\
B \arrow[r, "\Delta"] & B \otimes B,
\end{tikzcd}
$$

$$
\begin{tikzcd}
B \otimes B \arrow[r, "\varepsilon\otimes I"] \arrow[d, "\mu"'] & R \otimes B \arrow[d, "I\otimes\varepsilon"] \\
B \arrow[r, "\varepsilon"] & R,
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "\eta"] \arrow[dr, "="'] & B \arrow[d, "\varepsilon"] \\
 & R.
\end{tikzcd}
$$

These show that ε is a monad morphism and η is a comonad morphism, and μ is a right B-comodule morphism when $B \otimes_R B$ is considered as right B-comodule by $(I \otimes \psi) \circ (\Delta \otimes I)$. They also imply that every R-module M is a B-module and B-comodule trivially by $I \otimes \varepsilon : M \otimes_R B \to M$ and $I \otimes \eta : M \to M \otimes_R B$.

If the above conditions hold then it is easily checked that the given ψ is an entwining and B is called a *(ψ-)bialgebra*. Similarly, for any entwining $\psi' : B \otimes_R B \to B \otimes_R B$ one may define ψ'-bialgebras. Certainly, the twist tw is an entwining but B is only a tw-bialgebra provided Δ is trivial, that is, $\Delta(b) = b \otimes 1$ for any $b \in B$.

An R-module M which is both a B-module $\varrho_M : M \otimes_R B \to M$ and a B-comodule $\varrho^M : M \to M \otimes_R B$ is called a *(ψ-)B-bimodule* or a *B-Hopf module* if the diagram

$$
\begin{tikzcd}
M \otimes B \arrow[r, "\varrho_M"] \arrow[d, "\varrho^M\otimes I"'] & M \arrow[r, "\varrho^M"] & M \otimes B \\
M \otimes B \otimes B \arrow[rr, "I\otimes\psi"] & & M \otimes B \otimes B, \arrow[u, "\varrho_M\otimes I"']
\end{tikzcd}
$$

is commutative. In this case B is a right B-bimodule and we have the commutative diagram

$$
\begin{tikzcd}
M \otimes B \arrow[r, "\varrho_M"] \arrow[d, "\varrho^M\otimes\Delta"'] & M \arrow[r, "\varrho^M"] & M \otimes B \\
M \otimes B \otimes B \otimes B \arrow[rr, "I\otimes\mathsf{tw}\otimes I"] & & M \otimes B \otimes B \otimes B \arrow[u, "\varrho_M\otimes\mu"']
\end{tikzcd}
$$

which holds in particular for $M = B$.

Here we have derived our constructions from the twist tw but the same pattern can be followed starting with a (pre-)braiding on ${}_R\mathbf{M}$ (or on a monoidal category, e.g. [46]).

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