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Conjugate Connections on Statistical Manifolds

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1. INTRODUCTION

1.1. The theory of conjugate connections constructed by A. P. Norden and developed by his disciples (see, e.g., [1–3]) experiences now a new birth in connection with the interest growing in the last decades in the research works of geometricians abroad to affine differential geometry (see, e.g., [4, 5]) and the rapid development of the theory which N. N. Chentsov, its founder, called the "*geometrostatistics*" ([6], p. 5). This paper is devoted to the latter aspect of the application of conjugate connections.

1.2. The starting point for the "*geometrostatistics*" was paper [7]. In this paper, on the base of Fisher information matrix, a Riemannian tensor on the manifold of probability distributions has been defined which turned it into a Riemannian manifold. In addition, the equations of geodesics were derived.

In the fundamental monograph [6], N. N. Chentsov developed the geometry of statistical solutions with the category of Markov morphisms on the manifold of probability distributions and demonstrated that the Fisher information tensor generates a unique up to a constant factor invariant Riemannian metric in this category ([6], pp. 176–179). He also constructed a one-parameter family of invariant linear connections ${}^{\gamma}\nabla$ on the manifold of probability distributions and revealed that the exponential families are geodesic with respect to the connections ${}^{\gamma}\nabla$ ([6], pp. 189–201, 284–300).

S. Lauritzen (see [8], pp. 163–216) generalized the notion of a Riemannian manifold of probability distributions with one-parameter family of Chentsov's connections ${}^{\gamma}\nabla$ and gave the definition of an *abstract statistical manifold* as a triple (M, g, D). Here, according to the author's idea, a smooth n-dimensional ($n \geq 2$) manifold M symbolizes a manifold of probability distributions, a metric tensor g plays the role of a Fisher information tensor, and a family of linear connections ${}^{\gamma}\nabla = \nabla + \gamma D$, where ∇ is the Levi-Civita connection and γ is an arbitrary real parameter, is interpreted as a one-parameter family of linear Chentsov's connections. The key points in his theory are the indication of conjugate pairs of connections ${}^{\gamma}\nabla$ and ${}^{-\gamma}\nabla$ and the introduction of a *conjugate symmetric statistical manifold*, for which, by the definition, for all γ, the curvature tensors ${}^{\gamma}R$ and ${}^{-\gamma}R$ of the conjugate connections ${}^{\gamma}\nabla$ and ${}^{-\gamma}\nabla$ coincide.

The theory of statistical manifolds was reflected in several tens of articles and a series of monographs [6], [8]–[12].

1.3. Section 2 of this paper introduces the reader into the range of concepts of the "geometrostatistics". These concepts will be necessary for the study of invariant Chentsov's connections ${}^{\gamma}\nabla$, they are taken from the monograph [6] and the paper [13]. At the end of the section we deduce simple consequences from the introduced definitions.

Section 3 is devoted to application of the theory of A. P. Norden's conjugate connections to the study of invariant Chentsov's connections ${}^{\gamma}\nabla$ on a manifold of probability distributions. In Section 4, we describe the geometry of a conjugate Ricci-symmetric statistical manifold, which is defined as an immediate generalization of a conjugate symmetric statistical manifold of S. Lauritzen.

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