Russian Mathematics (Iz. VUZ) *Izvestiya VUZ. Matematika*
Vol. 41, No. 10, pp.59–63, 1997 *UDC 517.956*

ON THE RELATIONSHIP BETWEEN SOME LIOUVILLE THEOREMS ON RIEMANNIAN MANIFOLDS OF A SPECIAL TYPE

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The classical Liouville theorem asserts that any bounded solution of the Laplace equation

$$\Delta u = 0 \tag{1}$$

is identically constant in R^n. Note that the set of manifolds, on which nontrivial bounded harmonic functions do exist, is rather wide. Moreover, it is known that any bounded solution of the equation

$$\Delta u - \mu u = 0 \tag{2}$$

in R^n with $\mu = \mathrm{const} > 0$ is the null equation.

In the present article we shall consider the bounded solutions of equations (1) and (2) on some Riemannian manifolds, in this case Δ is the Laplace–Beltrami operator. A.A. Grigor'yan has called the author's attention to the following problem: What are the manifolds with which the fulfillment of the Liouville theorem for equation (1) implies the fulfillment of the Liouville theorem for equation (2)? Note (see [1]) that the existence of a nonzero bounded solution of (2) is equivalent to the stochastic uncompleteness of the manifold under consideration (a manifold is stochastically complete if the Wiener process on this manifold is unique; this is equivalent to the fact that the solution of the Cauchy problem for the heat conduction equation $\frac{\partial u}{\partial t} - \Delta u = 0$ is unique in the class of bounded functions).

In the present article we study the relationship between the Liouville theorems for equations (1), (2) on some specific Riemannian manifolds.

Let M be a complete Riemannian manifold without boundary, being such that $M = B \bigcup D$, where B is a compact set, and D is isometric to the direct product $\mathrm{R}_+ \times S$ (where $\mathrm{R}_+ = (0, \infty)$, and S is a compact Riemannian manifold) endowed by a metric of the form

$$ds^2 = h^2(r)dr^2 + g^2(r)d\theta^2.$$

Here $h(r)$ and $g(r)$ are positive smooth functions on R_+, $d\theta^2$ is a metric on S. Examples of the manifolds of that kind are: the Euclidean space ($h(r) = 1$, $g(r) = r$), the Lobachevskiĭ space ($h(r) = 1$, $g(r) = \mathrm{sh}(r)$), the surface obtained via rotation of the graph of a function $f(r)$ about the ray Or in R^n ($h(r) = \sqrt{1 + |f'(r)|^2}$, $g(r) = f(r)$), etc.

Remark 1. In [2] it was proved that the following conditions are equivalent:

a) on M a nontrivial bounded solution of equation (2) exists, where $\mu \geq 0$;

b) on $M \setminus B$ there exists a nontrivial bounded solution of the boundary problem

$$\Delta u - \mu u = 0 \text{ on } M \setminus B \ (\mu \geq 0), \quad \left.\frac{\partial u}{\partial \nu}\right|_{\partial B} = 0. \tag{3}$$

First, let us find conditions for the Liouville theorem to hold true for equation (2) on M.