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BOUNDARY OF CLOSE-TO-CONVEXITY AT POINT FOR UNIVALENT FUNCTIONS

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In this article we determine the exact boundary of close-to-convexity of a given order at a point for regular and univalent (schlicht) functions in the unit disk.

Let R be a class of all regular in unit disk $E = \{z : |z| < 1\}$ functions, β be an arbitrarily fixed number, $0 \le \beta \le 1$.

As is known (see, e. g., [1], [2]), a function $f(z) \in R$, normed by the conditions $f(0) = 0$, $f'(0) = 1$, is called close-to-convex of order β in E if a convex function $g(z) \in R$, $g(0) = 0$, $g'(0) = 1$, and a complex constant ε, $|\varepsilon| = 1$, exist such that in E

$$\left|\arg\left\{\frac{\varepsilon f'(z)}{g'(z)}\right\}\right| \le \frac{\beta\pi}{2}.$$

If the norm of functions $f(z)$ and $g(z)$ (or one of them) is absent, then it is assumed $\varepsilon = 1$.

The functions close-to-convex of order 0 in E compose a class of convex functions from R, and the close-to-convex functions of order $\beta = 1$ form the introduced in [3] class of close-to-convex functions. All close-to-convex of order β, $0 \le \beta \le 1$, functions in E are univalent in E (see [3]) and form special subclasses of classes of functions introduced in [4].

Let c be an arbitrarily fixed point in the disk E and $E(c, \rho) = \{z : |z - c| < \rho\}$ be a disk lying in E.

Definition. A function $f(z) \in R$ is called close-to-convex of order β in the disk $E(c, \rho)$ if a regular and convex in $E(c, \rho)$ function $g(z)$ exists such that $|\arg\{f'(z)/g'(z)\}| \le \beta\pi/2$ in $E(c, \rho)$.

One can easily substantiate the following

Lemma. *A function $f(z)$ from R is close-to-convex of order β in $E(c, \rho)$ if and only if $f'(z) \ne 0$ in $E(c, \rho)$ and, for all r, $0 < r < \rho$, φ_1, φ_2, $\varphi_1 < \varphi_2$, the inequality*

$$\int_{\varphi_1}^{\varphi_2} \operatorname{Re}\left\{1 + (z - c)\frac{f''(z)}{f'(z)}\right\} d\varphi > -\beta\pi, \quad z = c + re^{i\varphi} \tag{1}$$

is fulfilled.

1. The lower estimate of $\arg\{(z_2 - c)f'(z_2)/(z_1 - c)f'(z_1)\}$ on class S

Let us denote by S the class of all univalent functions $f(z) \in R$ normed by the conditions $f(0) = 0$, $f'(0) = 1$. Using the technique first given in [5], we will prove the following theorem which will be used in what follows.