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GEOMETRY OF GRASSMANN BUNDLE. II [1]

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Let a Grassmann bundle $G^l(M)$ of a Riemannian manifold (M, g) be endowed with the Levi-Civita connection of Riemannian metric $\widetilde{g}$, where $\widetilde{g}$ is associated with g and the canonical metric of Grassmannian (see [1]). In the present part of the article we shall establish a connection between the geometry of $G^l(M)$ and the geometry of M. In the first part of the article, on $G^l(M)$ we constructed a generating system of vector fields $\widetilde{S}^v$, $\widetilde{Y}^h$, and found their properties, which here will be consistently used in Section 3. In Theorem 3.1 we obtain the O'Neill equations (see [2]), where the invariant A and its covariant derivatives are expressed solely in terms of geometrical properties of the base. Similar problems for the tangent bundle, the spherical bundle, the normal bundle, and the frame bundle have been already solved in various papers. We refer the reader to the reviews [3] and [4] for detailed exposition of the results concerning this field.

3. Geometry of Grassmann bundle

In this Section we obtain formulas expressing geometrical objects of Grassmann bundle, like the covariant derivative $\widetilde{\nabla}$, the curvature operator $\widetilde{R}(\cdot,\cdot)$, and the sectional curvature $\widetilde{K}$, in terms of the similar geometrical objects ∇, $R(\cdot,\cdot)$, K, ∇R on the Riemannian manifold M (on the base) together with the geometry of the Grassmannian G^l_n (of the standard fibre). In what follows we denote by Q, S, P, and T skew-symmetric tensor fields, and by X, Y, Z, and U — vector fields on M.

Lemma 3.1. *At each point* Π *in* $G^l(M)$ *the following equalities hold*:

$$\text{a) } \widetilde{\nabla}_{\widetilde{Q}^v}\widetilde{S}^v = -[\widetilde{\{Q\},|S|}]^v = -[\widetilde{\{Q\},S}]^v, \quad \text{b) } \widetilde{\nabla}_{\widetilde{Q}^v}\widetilde{Y}^h = -\frac{1}{4}(\widetilde{\rho(\{Q\})Y})^h,$$

$$\text{c) } \widetilde{\nabla}_{\widetilde{X}^h}\widetilde{S}^v = -\frac{1}{4}(\widetilde{\rho(\{S\})X})^h + (\widetilde{\nabla_X S})^v, \quad \text{d) } \widetilde{\nabla}_{\widetilde{X}^h}\widetilde{Y}^h = (\widetilde{\nabla_X Y})^h - \frac{1}{2}\widetilde{\rho(X\wedge Y)}^v,$$

where both the inner and outer tensor components are taken with respect to the subspace Π.

Proof. Since the covariant derivative of a Riemannian manifold M is determined by the equation

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]), \quad (48)$$

we can carry out the following computation.

a) Using (48), for the Riemannian manifold $G^l(M)$, we get

$$2\widetilde{g}(\widetilde{\nabla}_{\widetilde{Q}^v}\widetilde{S}^v, \widetilde{X}^h) = -\widetilde{X}^h\widetilde{g}(\widetilde{Q}^v, \widetilde{S}^v) - \widetilde{g}(\widetilde{S}^v, [\widetilde{Q}^v, \widetilde{X}^h]) - \widetilde{g}(\widetilde{Q}^v, [\widetilde{S}^v, \widetilde{X}^h]), \quad (49)$$

because each horizontal vector is orthogonal to each vertical vector and identity a) of Lemma 2.6 holds. Recall that a function f is said to be basic with respect to a submersion R_H if on the base manifold a function h exists such that f is the pullback of h, i.e. $R^*_H h = f$. Obviously, if on

[1] This article is a continuation, see Part I in "Izv. VUZ. Matematika", no. 9, pp. 54–67, 1997.