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CONVEX SETS IN NONCOMMUTATIVE L_1-SPACES, CLOSED IN THE TOPOLOGY OF LOCAL CONVERGENCE IN MEASURE

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Introduction

In 1973 A.V. Bukhvalov and G.Ya. Lozanovskiĭ (see [1]) demonstrated that the convex sets, bounded in norm and closed in the topology of the local convergence in measure in some "good" Banach lattices of measurable functions, possess a series of properties which are close to those of compact sets (see also [2] and [3], Chap. X, § 5). These results obtained various useful applications in the theory of optimal control, the minimax theorems, the geometry of Banach spaces, etc. (see [2] and the survey [4]). In 1992 A.V. Bukhvalov posed the question on the possibility of creation of "noncommutative" version of this theory. The present article gives a positive answer to this question. Namely, we prove that the results by Bukhvalov and Lozanovskiĭ can be transferred to the case of the space of selfadjoint operators, integrable by Segal, i. e., the real L_1-space associated with an exact normal semifinite trace on the Neumann algebra. For the case of a finite trace this assertion was announced in [5].

It turns that to the noncommutative case the principal scheme of the proof of [2] can be transferred. Some obvious modifications were necessary due to the specificity of the situation under consideration. For instance, instead of the Iosida–Hewitt theorem, we use the Takesaki theorem on decomposition of a functional on the Neumann algebra into the normal and singular components. In Section 1 we prove a number of assertions for ordered spaces and Neumann algebras, which represent an autonomous interest in our opinion. In Section 2 we cite some concepts and definitions of the theory of integration with respect to a trace on the Neumann algebra. Sections 3 and 4 are devoted to deduction of the basic result. Let us note that the proof of the theorem in Section 4 in its essence is analogous to the proof of theorem 1.1 in [2].

1. Preliminary results

Let X be an ordered real linear space with the cone of positive elements X^+, generated by that cone. We shall denote by X^{al} the space algebraically conjugate to X, $(X^{\mathrm{al}})^+$ means the cone of positive functionals, which is dual to X^+. For $x \in X$ and $f \in (X^{\mathrm{al}})^+$, we put

$$r_f(x) = \inf\{f(x_1) + f(x_2) \mid x_1, x_2 \in X^+,\ x = x_1 - x_2\}.$$

Clearly, r_f is a seminorm on X. Let $B_f = \{x \in X \mid r_f(x) \le 1\}$ and B_f° be a polar B_f in X^{al}.

Proposition 1. $B_f^\circ = \{g \in X^{\mathrm{al}} \mid -f \le g \le f\}$.

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