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Decomposability of Low 2-Computably Enumerable Degrees and Turing Jumps in the Ershov Hierarchy

M. Kh. Faizrakhmanov[1*]

[1]*Kazan State University, 18 Kremlevskaya str, Kazan, 420008 Russia*

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Abstract—In this paper we prove the following theorem: For every notation of a constructive ordinal there exists a low 2-computably enumerable degree that is not splittable into two lower 2-computably enumerable degrees whose jumps belong to the corresponding Δ-level of the Ershov hierarchy.

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Let $\langle O, <_O \rangle$ be the Kleene system for enumerating constructive ordinals and let $a \in O$. A set X is located in the *class* Σ_a^{-1} *of the Ershov hierarachy* if there exists a partial computable function of two arguments ψ such that for all x,

$$x \in X \Leftrightarrow \exists t <_O a \, \forall t' <_O a \, \left[\psi(t,x)\downarrow = 1 \,\&\, [\psi(t',x)\downarrow \Rightarrow [t <_O t' \;\vee\; t = t']]\right].$$

Here the notation $\psi(t,x)\downarrow$ means that the value $\psi(t,x)$ is defined. We say that such a function ψ *defines* X *as a* Σ_a^{-1}*-set*. We define the Π_a^{-1}*-class* and the Δ_a^{-1}*-class* in the hierarchy as follows:

$$\Pi_a^{-1} = \{X : \overline{X} \in \Sigma_a^{-1}\},$$
$$\Delta_a^{-1} = \Sigma_a^{-1} \cap \Pi_a^{-1}.$$

It is clear that X is contained in Δ_a^{-1} if and only if there exists a partial function ψ that defines X as a Σ_a^{-1}-set, and for every x there exists $t <_O a$ such that $\psi(t,x)\downarrow$. In such cases we say that the function ψ defines X as a Δ_a^{-1}-set. In papers [1–3] Yu. L. Ershov proves that $\Delta_2^0 = \bigcup_{a \in O} \Delta_a^{-1}$. See papers [1–3] and the paper [4] by M. M. Arslanov for alternate definitions of the Ershov hierarchy. If a denotes a finite ordinal n, i.e., $|a|_O = n$, then sets from the class Σ_a^{-1} are called n*-computably enumerable* (n*-c. e.*) *sets*. A (Turing) degree is called n*-c. e.* if it contains an n-c. e. set. Instead of 1-c. e. one usually writes just c. e.

The first elementary distinction between semilattices of c. e. and 2-c. e. degrees was established by M. M. Arslanov. In [5] he proved that every nonzero 2-c. e. degree is complemented upwards (capped). In other words, if a 2-c. e. degree $d_0 > 0$ is given, then another 2-c. e. degree $d_1 < 0'$ exists such that $d_0 \cup d_1 = 0'$. According to Cooper and Yates (unpublished), the latter proposition is false in a semilattice of c. e. degrees. The next elementary distinction was established by R. Downey. In [6] he proved that the Lachlan nondiamond theorem [7] is not valid for 2-c. e. degrees. Finally, in [8] one has constructed the maximal incomplete 2-c. e. degree, which, together with the Sacks density theorem [9], gives an elementary distinction between semilattices of c. e. and 2-c. e. degrees. However, none of the above results gives an elementary distinction between the partial ordering of low c. e.degrees and the partial ordering of low 2-c. e. degrees. The reason is that in propositions on the capping of 2-c. e. degrees and those on embedding of the diamond lattice, the degree $0'$ is mentioned, but in [10] M. M. Arslanov, S. B. Cooper, and A. Li reject the existence of the maximal low 2-c. e. degree. In this paper we establish

[*]E-mail: Marat.Faizrahmanov@ksu.ru.