Reflection principles, algebras and progressions of theories

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Standard axiomatic theories

Peano arithmetic PA: formalizes 'finitary mathematics'; based on axioms for natural numbers with $+$ and $\cdot$.

Second order arithmetic PA^2: formalizes analysis; extends PA by variables for sets of numbers and assumes the schemata of full comprehension and induction.

Zermelo–Fraenkel set theory ZFC: formalizes *all* conventional mathematics; based on axioms for sets and membership relation.

Nota bene: Formal axiomatic theories are materialized in various *automatic* and *interactive theorem provers* such as Coq, Isabelle/HOL or Mizar.

Comparing axiomatic theories

Theories differ in

- the expressivity of their languages (*richness*);
- the amount of axioms (*strength*),
- speed of proofs,
- deductive mechanism, etc.

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Background formal arithmetic

Elementary arithmetic EA is formulated in the language $(0, 1, +, \cdot, 2^x, \leq, =)$ and has some minimal set of basic axioms defining these symbols plus the induction schema for bounded formulas.[1]

A formula is *bounded* if all its quantifier occurrences are of the form $\forall x \leq t$ or $\exists x \leq t$ where t is a term (not containing x).

Peano arithmetic PA is EA with full induction:

$$\varphi(0) \wedge \forall x\,(\varphi(x) \to \varphi(x+1)) \to \forall x \varphi(x),$$

where φ is any formula (possibly with parameters).

[1] EA is also known as $I\Delta_0 + \exp$ and EFA.

Example: fragments of PA

Σ_n-formulas: $\exists x_1 \forall x_2 \ldots Q x_n \varphi(x_1, \ldots, x_n)$, with $\varphi(\vec{x})$ bounded.
Π_n-formulas: $\forall x_1 \exists x_2 \ldots Q x_n \varphi(x_1, \ldots, x_n)$

$I\Sigma_n = \mathsf{EA} +$ induction for Σ_n-formulas

$$\mathsf{EA} \subset I\Sigma_1 \subset I\Sigma_2 \cdots \subset \mathsf{PA}$$

It is known that $I\Sigma_n \equiv I\Pi_n$ for all $n \geq 1$.

Example

Let $A(x, y)$ denote the Ackermann function. How much induction is needed to prove that this function is everywhere defined?

Theorem

1. $I\Sigma_2 \vdash \forall x, y \, \exists z \, A(x, y) = z$;
2. $I\Sigma_1 \nvdash \forall x, y \, \exists z \, A(x, y) = z$ (Parsons 1970, Mints 1971).

Provably total computable functions

Let $\mathcal{F}(T)$ be the class of *provably total computable functions* of a theory T.

Definition

$g \in \mathcal{F}(T)$ iff for some $\varphi(x, y) \in \Sigma_1$,

1. $g(x) = y \Leftrightarrow \mathbb{N} \models \varphi(x, y)$
2. $T \vdash \forall x \exists y \varphi(x, y)$.

Fact

If T is r.e. and true in the standard model, then $\mathcal{F}(T) \subsetneq$ total computable functions.

Examples

$\mathcal{F}(\mathsf{EA}) = \mathcal{E}$ (elementary recursive functions)
$\mathcal{F}(I\Sigma_1) =$ primitive rec. functions (Parsons, Mints)
$\mathcal{F}(\mathsf{PA}) =\ < \varepsilon_0$-recursive functions (Ackermann 1944, Kreisel 1952)

In general,

- $\mathcal{F}(T) \supseteq \mathcal{E}$ and closed under composition
- $\mathcal{F}(T)$ only depends on the Π_2-theorems of T
- $\mathcal{F}(T) = \mathcal{F}(T + \mathbf{Th}_{\Pi_1}(\mathbb{N}))$

Theories $\rightsquigarrow$ *Ordinals*

Π^1_1-*analysis:* Provable well-orderings

$$|S|_{\Pi^1_1} := \sup\{|\prec| : S \vdash WF(\prec)\},$$

where $WF(\prec)$ expresses that $\prec$ is a well-ordering.

G. Gentzen (1936) considered a conservative second-order extension S of PA and showed that $|S|_{\Pi^1_1} = \varepsilon_0$, where

$$\varepsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots\}.$$

Since then, Π^1_1-ordinals have been characterized for a large number of systems of second order arithmetic and set theory.

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$$|S|_{\Pi^0_2} := \min\{\alpha : S \nvdash \forall x \exists y\, F_\alpha(x) = y\}, \text{where}$$

functions F_α are defined by extension of the Ackermannian construction.

For $\alpha = |S|_{\Pi^0_2}$ it usually happens that $\mathcal{F}(T) = \mathcal{E}_\alpha$, where $\mathcal{E}_\alpha := \mathbf{E}(\{F_\beta : \beta < \alpha\})$ is the class of functions computable in time bounded by some F_β for $\beta < \alpha$.

NB: Characterizations of $\mathcal{F}(T)$ are closely related to unprovable in T combinatorial statements such as Paris–Harrington principle, the Hydra and Goodstein's principles for PA.

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Examples

Definition

$\varepsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \dots\}$

Example

$|\mathsf{ACA}_0 + PH|_{\Pi^1_1} = |\mathsf{ACA}_0|_{\Pi^1_1} = \varepsilon_0$

Example

$|\mathsf{PA} + \mathsf{Con}(\mathsf{PA})|_{\Pi^0_2} = |\mathsf{PA}|_{\Pi^0_2} = \varepsilon_0$

Π^1_1-ordinal is insensitive to true Σ^1_1 axioms.
Π^0_2-ordinal is insensitive to true Π^0_1 axioms.

Gödel's 2nd Incompleteness Theorem

Definition

A theory T is Gödelian if

- Natural numbers and operations $+$ and $\cdot$ are definable in T;
- T proves basic properties of these operations (contains EA);
- There is an algorithm (and a Σ_1-formula) recognizing the axioms of T.

$\Box_T(x)$ = 'x is the Gödel number of a T-provable formula'
Con(T) = 'T is consistent'

K. Gödel (1931): If a Gödelian theory T is consistent, then Con(T) is true but unprovable in T.

Turing vs. Gödel

A natural response to Gödel: add $\mathrm{Con}(T)$ to T as a new axiom.
Is $T + \mathrm{Con}(T)$ complete? No, because it is Gödelian.

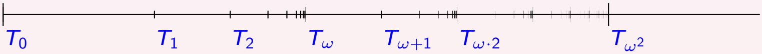
A. Turing (1939) suggested to continue the process:

$T_0 = T$
$T_1 = T + \mathrm{Con}(T)$
$T_2 = T + \mathrm{Con}(T) + \mathrm{Con}(T + \mathrm{Con}(T))$
...
$T_{n+1} = T_n + \mathrm{Con}(T_n)$
...

Is $\cup_{n\geq 0} T_n$ complete?

No: $T_\omega := \cup_{n\geq 0} T_n$ is Gödelian. Hence, T_ω does not prove $\mathrm{Con}(T_\omega)$ and the process continues:

$$T_{\omega+1} = T_\omega + \mathrm{Con}(T_\omega)$$
$$T_{\omega+2} = T_{\omega+1} + \mathrm{Con}(T_{\omega+1})$$
$$\ldots$$



Turing's classification program

Turing hoped to obtain a classification of all true arithmetical statements according to the stages of this (and similar) processes – but encountered difficulties.

A.M. Turing 1939 *System of logics based on ordinals*:

> *We might also expect to obtain an interesting classification of number-theoretic theorems according to "depth". A theorem which required an ordinal α to prove it would be deeper than one which could be proved by the use of an ordinal β less than α. However, this presupposes more than is justified.*

Turing progressions

The difficulties are:

- Logical complexity restriction;
- The problem of canonicity of ordinal notations.

Ordinal notations

Orderings can be represented in T, for example, by assigning rational numbers to points. The resulting set of numbers must be recognizable by an algorithm. (Otherwise, the axioms of T_α would not be recognizable.)

A problem: theories T_α depend on a particular way the ordering is computed rather than on the isomorphism type (the ordinal) of α.

Turing, Feferman, Kreisel: the whole classification idea breaks down because of this problem.

Turing's theorem

Theorem

For each true Π_1-sentence π there is a ordinal notation α such that $|\alpha| = \omega + 1$ and $T_{\alpha+1}$ proves π.

A. Turing:

> *This completeness theorem as usual is of no value. Although it shows, for instance, that it is possible to prove Fermat's last theorem with Λ_P (if it is true) yet the truth of the theorem would really be assumed by taking a certain formula as an ordinal formula*[2].

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Restricted Turing's program

> *We can still give a certain meaning to the classification into depths with highly restricted kinds of ordinals. Suppose that we take a particular ordinal logic Λ and a particular ordinal formula Ψ representing the ordinal α say (preferably a large one), and that we restrict ourselves to ordinal formulae of the form $\mathrm{Inf}(\Psi, a)$.[3] We then have a classification into depths, but the extents of all the logics which we so obtain are contained in the extent of a single logic.*

A partial way out: careful selection of 'canonical' or 'natural' ordinal notations. This is possible for very large constructive ordinals, but we lack a general understanding of what is a natural ordinal notation system.

[3] These formulas define initial segments of α.

Logical complexity restriction

Fact: There are true statements that cannot be proved at any stage of a Turing progression. Let

$$T' = T + \{\mathrm{Con}(S) : S \text{ any consistent Gödelian theory}\}.$$

T' obviously contains any T_α.

Is T' Gödelian? No: there is no algorithm to recognize the consistency of an arbitrary given system S.

Nonetheless, Gödel theorem holds for T': $\mathrm{Con}(T')$ is expressible but not provable in T'. Since $T_\alpha \subseteq T'$, T_α does not prove $\mathrm{Con}(T')$.

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Reflection principles

Let T be a Gödelian theory.

- *Reflection principles* $R_n(T)$ for T are arithmetical sentences expressing *"every* Σ_n*-sentence provable in* T *is true"*.

$R_n(T)$ can be seen as a relativization of the consistency assertion $Con(T) = R_0(T)$.

$R_n(T)$ is expressible as a Π_{n+1}-sentence.

Semilattice of Gödelian theories

Def. $\mathfrak{G}_{\mathsf{EA}}$ is the set of all Gödelian extensions of EA mod $=_{\mathsf{EA}}$.

$S \leq_{\mathsf{EA}} T \iff \mathsf{EA} \vdash \forall x\,(\Box_T(x) \to \Box_S(x))$;

$S =_{\mathsf{EA}} T \iff (S \leq_{\mathsf{EA}} T \text{ and } T \leq_{\mathsf{EA}} S)$.

Then $(\mathfrak{G}_{\mathsf{EA}}, \wedge_{\mathsf{EA}})$ is a lower semilattice with
$S \wedge_{\mathsf{EA}} T := S \cup T$
(defined by the disjunction of the formulas defining sets of axioms of S and T)

Monotone operators

Each of R_n correctly defines a monotone operator $R : \mathfrak{G}_S \to \mathfrak{G}_S$ on the semilattice of Gödelian extensions of S.

An operator R is:

- *monotone* if $x \leq y$ implies $R(x) \leq R(y)$;
- *semi-idempotent* if $R(R(x)) \leq R(x)$;
- *closure* if R is m., s.i. and $x \leq R(x)$.

All R_n are monotone and semi-idempotent, but not closure.

Iteration theorem

Def. $R : \mathfrak{G}_T \to \mathfrak{G}_T$ is *computable* if it can be defined by a computable map on the Gödel numbers of formulas defining the extensions of T.

Suppose $(\Omega, \prec)$ is an elementary recursive well-ordering and R is a computable m.s.i. operator on $\mathfrak{G}_T$.

Theorem
There exist theories $R^\alpha(S)$ (where $\alpha \in \Omega$):
$R^0(S) =_T S$ and, if $\alpha \succ 0$,

$$R^\alpha(S) =_T \bigcup\{R(R^\beta(S)) : \beta \prec \alpha\}.$$

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Remarks

- Theorem holds for any of the reflection schemata R_n, in particular, for $R = \mathsf{Con}$.
- *Turing progression over S* is equivalent to iterations of the operator $R(S) := (S \wedge \mathsf{Con}(S))$.
- If $\alpha \in \mathsf{Lim}$ then $R^\alpha(S)$ is, in general, not a finitely axiomatized theory, e.g., $\mathsf{Con}^\omega(S) = \{\mathsf{Con}^n(S) : n < \omega\}$.

Conservative approximation

Definition

T is Π_{n+1}-*conservative* over U if U proves all Π_{n+1}-theorems of T. Denoted $T \subseteq_n U$.

We write $T \equiv_n U$ if both $T \subseteq_n U$ and $U \subseteq_n T$.

T is a *conservative extension* of U if $U \subseteq T$ and $T \subseteq_n U$.

Proof-theoretic Π^0_{n+1}*-ordinals*

Let S be a Gödelian extension of EA and $(\Omega, <)$ a (natural) elementary recursive well-ordering.

- Π^0_{n+1}*-ordinal of* S, denoted $ord_n(S)$, is the sup of all $\alpha \in \Omega$ such that $S \vdash R^\alpha_n(\mathsf{EA})$;
- S is Π^0_{n+1}*-regular* if S is Π^0_{n+1}-conservative over $R^\alpha_n(\mathsf{EA})$, for some $\alpha \in \Omega$;
- *Conservativity spectrum of* S is the sequence $(\alpha_0, \alpha_1, \alpha_2, \ldots)$ such that $\alpha_i = ord_i(S)$.

Examples of spectra:

$I\Sigma_1$: $(\omega^\omega, \omega, 1, 0, 0, \ldots)$

PA : $(\varepsilon_0, \varepsilon_0, \varepsilon_0, \ldots)$

PA + PH : $(\varepsilon_0^2, \varepsilon_0 \cdot 2, \varepsilon_0, \varepsilon_0, \ldots)$

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Regular theories

The best we can hope with a Turing progression $R_n^\alpha(\mathrm{EA})$ is to approximate a given theory U from below up to $\equiv_n$. If U is *natural* it is likely that *natural* ordinal notations would suffice and at some point α we would obtain:

$$U \equiv_n R_n^\alpha(\mathrm{EA}).$$

Theorem (U. Schmerl, 1979): For all n,

$$\mathrm{PA} \equiv_n R_n^{\varepsilon_0}(\mathrm{EA}).$$

Information from Π^0_2-ordinals

With the elementary well-ordering $(\Omega, <)$ we can associate a hierarchy of fast-growing functions F_α for all $\alpha < \Omega$:

$$F_\alpha(x) := \max\{2^x_x + 1\} \cup \{F_\beta^{(m)}(n) + 1 : \beta \prec \alpha, \beta, m, n \leq x\}.$$

We let $\mathcal{F}_\alpha := \mathbf{E}(\{F_\beta : \beta \prec \alpha\})$.

Theorem

$\mathcal{F}(R_1^\alpha(\mathsf{EA})) = \mathcal{F}_\alpha$;

Corollary

If $S \equiv_1 R_1^\alpha(\mathsf{EA})$ *then* $\mathcal{F}(S) = \mathcal{F}_\alpha$.

Information from Π^0_1-ordinals

- $S \equiv_0 T$ yields equiconsistency. Hence, if $S \equiv_0 T$ is provable in a weak system, so is $\mathsf{Con}(S) \leftrightarrow \mathsf{Con}(T)$.
- $S \equiv_0 T$ is equivalent to mutual interpretability of S and T (the existence of an internal model), provided S and T are reflexive (Orey–Hájek–Guaspari–Lindström theorem).

A theory is *reflexive* if it proves the consistency of each finite subtheory of itself. Every extension of PA in the language of PA and of *ZFC* in the language of *ZFC* is reflexive.

Conclusions

- Proof-theoretic ordinals by iterated reflection principles provide the finest of the existing ordinal classifications of arithmetical theories.
- They also capture the standard classification of provably recursive functions via fast growing hierarchies.
- Pakhomov and Walsh recently adapted the definition to also capture Π^1_1-ordinals in the context of second order arithmetic.

In the next lecture we will see how to use reflection algebras to compute such ordinals.