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Exact Differential Equations Over Matrix Skew Series

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Abstract—In this paper we establish sufficient conditions for the reducibility of first-order matrix differential equations stated in terms of differentials to exact scalar differential equations.

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1. The problem. The generalization of basic methods developed for integrating scalar differential equations to a matrix case is an actual problem in many branches of modern mathematics, both theoretical and applied ones. Its complexity can be explained by the fact that the matrix multiplication operation is not commutative, which does not allow one to use the well-developed apparatus of the scalar theory in the case of a matrix manifold. Exact differential equations (EDE) may serve as an example illustrating the difficulty of the transition from scalar equations to matrix ones ([1], P. 30). EDE are often used in the classical theory. Usually an EDE is stated in the following differential form:

$$a(x,y)dy + b(x,y)dx = 0, \tag{1}$$

where both functions are continuous in x and y. If, in addition, $a = \frac{\partial f}{\partial y}$ and $b = \frac{\partial f}{\partial x}$, where the function $f \equiv f(x,y)$ is continuously differentiable in both variables, then we can write the left-hand side of Eq. (1) as the exact differential $\frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial x}dx = df = 0$. This allows us to write the integral in Eq. (1) as $f(x,y) = c = \text{const}$. If functions $a(x,y)$ and $b(x,y)$ are continuously differentiable, then Eq. (1) is solvable (in the mentioned sense) if and only if ([1], P. 30)

$$\frac{\partial a}{\partial x} \equiv \frac{\partial b}{\partial y}. \tag{2}$$

Trying to use this method developed for integrating ordinary differential equations of the first order (ODE.1) for the matrix set, we encounter certain difficulties. Indeed, let us write a matrix analog of Eq. (1) in the form

$$A(X,t)dX + B(X,t)dt = 0, \tag{3}$$

where $A(X,t)$, $B(X,t)$, and $X \equiv X(t)$ are elements of the set M_n of square matrices of the nth order over the real number field R containing a continuous variable t, while components of the desired matrix $X \equiv (x_{ij}(t))$ are continuously differentiable in t. The following questions arise. First, how can we calculate derivatives of matrix functions with respect to matrix arguments? Second, how can we find elements of the desired solution from the integral $F(X,t) = C = \text{const}$? In a general case, there are no answers to these questions. Therefore we have to restrict the algebraic structure of M_n so as to reduce the solution of Eq. (3) to the solution of some set of scalar ODE.1. This scalarization is possible owing to introduced in [2] skew series $A_{\pm\alpha}$ of the matrix $A = (a_{ij})$, whose nonzero elements are only $a_{i\,i\pm\alpha}$. They allow us to define several types of subalgebras of M_n, including $H_{n,0}$, $T_{n,0}$, $T_{n,q}$, and $M_{n/k}$, i.e., subalgebras of diagonal, upper triangular, upper niltriangular, and striped matrices, respectively. The first three types are "solvable" ([3], P. 33), while the latter one by a renumbering of indices can be reduced to a block-diagonal form [2]. However, in this paper we introduce dual-skew series $A^*_{\pm\alpha}$, whose nonzero elements are only $a_{i\,n+1\pm\alpha-i}$. We use them for defining sets of dual-diagonal and dual-striped

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