Some remarks on discretization

Vladimir Temlyakov

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Two settings

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- Functions belong to an N-dimensional subspace X_N. We call such results the Bernstein-type and Marcinkiewicz-type discretization theorems.
- Functions belong to a given function class.
  There are different settings and different ingredients, which play important role in this problem.

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Sampling discretization with absolute error

Let $W \subset L_q(\Omega,\mu)$, $1 \le q < \infty$, be a class of continuous on Ω functions. We are interested in estimating the following optimal errors of discretization of the L_q norm of functions from W

$$er_m(W,L_q) := \inf_{\xi^1,\dots,\xi^m} \sup_{f\in W} \left| \|f\|_q^q - \frac{1}{m}\sum_{j=1}^m |f(\xi^j)|^q \right|,$$

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$$er_m^o(W, L_q) := \inf_{\xi^1,\dots,\xi^m;\lambda_1,\dots,\lambda_m} \sup_{f\in W} \left| \|f\|_q^q - \sum_{j=1}^m \lambda_j |f(\xi^j)|^q \right|.$$

Entropy numbers

For a compact subset Θ of a Banach space B we define the entropy numbers as follows

$$\varepsilon_n(\Theta, B) := \inf\{\varepsilon : \exists f_1, \dots, f_{2^n} \in \Theta : \Theta \subset \cup_{j=1}^{2^n}(f_j + \varepsilon U(B))\}$$

where $U(B)$ is the unit ball of a Banach space B.

General theorem

Theorem (T1; VT, 2018)

Assume that a class of real functions W is such that for all $f \in W$ we have $\|f\|_\infty \le M$ with some constant M. Also assume that the entropy numbers of W in the uniform norm L_∞ satisfy the condition

$$\varepsilon_n(W, L_\infty) \le Cn^{-r}, \qquad r \in (0, 1/2).$$

Then

$$er_m(W) := er_m(W, L_2) \le Km^{-r}.$$

Comments

Theorem T1 is a rather general theorem, which connects the behavior of absolute errors of discretization with the rate of decay of the entropy numbers. This theorem is derived from known results in supervised learning theory. It is well understood in learning theory that the entropy numbers of the class of priors (regression functions) is the right characteristic in studying the regression problem.

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- We impose a restriction $r < 1/2$ in Theorem T1 because the probabilistic technique from the supervised learning theory has a natural limitation to $r \leq 1/2$.

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Theorem T1 is a rather general theorem, which connects the behavior of absolute errors of discretization with the rate of decay of the entropy numbers. This theorem is derived from known results in supervised learning theory. It is well understood in learning theory that the entropy numbers of the class of priors (regression functions) is the right characteristic in studying the regression problem.

- We impose a restriction $r < 1/2$ in Theorem T1 because the probabilistic technique from the supervised learning theory has a natural limitation to $r \le 1/2$.
- It would be interesting to understand if Theorem T1 holds for $r \ge 1/2$.

Sampling discretization with relative error. Bernstein problem

Let W be the unit ball $X_N^q := \{f \in X_N : \|f\|_q \le 1\}$. Then in the case, say, $er_m(W, L_q) < \epsilon < 1$ we obtain for all $f \in X_N^q$

$$(1-\epsilon)\|f\|_q^q \le \frac{1}{m}\sum_{\nu=1}^{m} |f(\xi^\nu)|^q \le (1+\epsilon)\|f\|_q^q.$$

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Bernstein problem. In the case $q = \infty$ we define L_∞ as the space of continuous on Ω functions and ask for

$$C_1\|f\|_\infty \le \max_{1\le\nu\le m} |f(\xi^\nu)| \le \|f\|_\infty. \tag{1}$$

Let Ω be a compact subset of $\mathbb{R}^d$ with the probability measure μ. We say that a linear subspace X_N of the $L_q(\Omega)$, $1 \le q < \infty$, admits the Marcinkiewicz-type discretization theorem with parameters m, q and constants C_1, C_2 if there exists a set $\{\xi^\nu \in \Omega, \nu = 1, \dots, m\}$ such that for any $f \in X_N$ we have

$$C_1\|f\|_q^q \le \frac{1}{m}\sum_{\nu=1}^{m} |f(\xi^\nu)|^q \le C_2\|f\|_q^q. \tag{2}$$

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We will also use a brief way to express the above property: $X_N \in \mathcal{M}(m,q)$ or $X_N \in \mathcal{M}(m,q,C_1,C_2)$.

Entropy. Conditional theorem for L_q

Theorem (T4, DPSTT, 2020)

Let $1 \le q < \infty$. Suppose that a real N-dimensional subspace X_N satisfies the following condition on the entropy numbers of the unit ball $X_N^q := \{f \in X_N : \|f\|_q \le 1\}$

$$\varepsilon_k(X_N^q, L_\infty) \le B \begin{cases} (N/k)^{1/q}, & k \le N, \\ 2^{-k/N}, & k \ge N, \end{cases}$$

with $B \ge 1$. Then there exists a set of $m \le C(q)NB^q(\log_2(2BN))^2$ points $\xi^j \in \Omega$, $j = 1, \dots, m$, with large enough constant $C(q)$, such that for any $f \in X_N$ we have

$$\frac{1}{2}\|f\|_q^q \le \frac{1}{m}\sum_{j=1}^{m} |f(\xi^j)|^q \le \frac{3}{2}\|f\|_q^q.$$

Entropy. Conditional theorem for L_q continues

Very recently E. Kosov, 2020, proved a theorem with the conditions imposed on the entropy numbers in a weaker metric than the uniform norm. We now formulate his result.

Let $Y_s := \{y_j\}_{j=1}^s \subset \Omega$ be a set of sample points from the domain Ω. Introduce a semi-norm

$$\|f\|_{Y_s} := \|f\|_{L_\infty(Y_s)} := \max_{1\le j\le s} |f(y_j)|.$$

Clearly, for any Y_s we have $\|f\|_{Y_s} \le \|f\|_\infty$.

Entropy. Conditional theorem for L_q continues

Theorem (T5, K, 2020)

Let $1 \le q < \infty$. There exists a number $C_1(q) > 0$ such that for m and B satisfying $m \ge C_1(q) N B^q (\log N)^{w(q)}$,

$$w(1) := 2, \quad w(q) := \max(q,2) - 1, \quad 1 < q < \infty,$$

and for a subspace X_N satisfying the condition: for any set $Y_m \subset \Omega$

$$\varepsilon_k(X_N^q, L_\infty(Y_m)) \le B(N/k)^{1/q}, \quad 1 \le k \le N$$

there are points $\xi^j \in \Omega$, $j = 1, \dots, m$, such that for any $f \in X_N$ we have

$$\frac{1}{2}\|f\|_q^q \le \frac{1}{m}\sum_{j=1}^{m} |f(\xi^j)|^q \le \frac{3}{2}\|f\|_q^q.$$

Two important directions

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- Universal Bernstein-type and Marcinkiewicz-type discretization theorems.
- Constructive sets for good discretization.

Universal discretization problem

Let $\mathcal{X}_N := \{X_N^j\}_{j=1}^k$ be a collection of linear subspaces X_N^j of the $L_q(\Omega)$, $1 \le q \le \infty$. We say that a set $\{\xi^\nu \in \Omega, \nu = 1, \dots, m\}$ provides universal discretization for the collection $\mathcal{X}_N$ if,

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Let $\mathcal{X}_N := \{X_N^j\}_{j=1}^k$ be a collection of linear subspaces X_N^j of the $L_q(\Omega)$, $1 \le q \le \infty$. We say that a set $\{\xi^\nu \in \Omega, \nu = 1, \dots, m\}$ provides universal discretization for the collection $\mathcal{X}_N$ if, in the case $1 \le q < \infty$, there are two positive constants $C_i(d,q)$, $i = 1, 2$, such that for each $j \in [1,k]$ and any $f \in X_N^j$ we have

$$C_1(d,q)\|f\|_q^q \le \frac{1}{m}\sum_{\nu=1}^{m} |f(\xi^\nu)|^q \le C_2(d,q)\|f\|_q^q. \tag{3}$$

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$$C_1(d,q)\|f\|_q^q \le \frac{1}{m}\sum_{\nu=1}^m |f(\xi^\nu)|^q \le C_2(d,q)\|f\|_q^q. \qquad (3)$$

In the case $q = \infty$ for each $j \in [1,k]$ and any $f \in X_N^j$ we have

$$C_1(d)\|f\|_\infty \le \max_{1\le\nu\le m} |f(\xi^\nu)| \le \|f\|_\infty. \qquad (4)$$

A known result

We begin with the universal discretization for the collection of subspaces of trigonometric polynomials with frequencies from parallelepipeds (rectangles). For $\mathbf{s} \in \mathbb{Z}_+^d$ define

$$R(\mathbf{s}) := \{\mathbf{k} \in \mathbb{Z}^d : |k_j| < 2^{s_j}, \quad j = 1, \dots, d\}.$$

Clearly, $R(\mathbf{s}) = \Pi(\mathbf{N})$ with $N_j = 2^{s_j} - 1$. Consider the collection $\mathcal{C}(n, d) := \{\mathcal{T}(R(\mathbf{s})), \|\mathbf{s}\|_1 = n\}$.

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Theorem (1; VT, 2017)

For every $1 \le q \le \infty$ there exists a large enough constant $C(d,q)$, which depends only on d and q, such that for any $n \in \mathbb{N}$ there is a set $\xi(m) := \{\xi^\nu\}_{\nu=1}^m \subset \mathbb{T}^d$, with $m \le C(d,q)2^n$ that provides universal discretization in L_q for the collection $\mathcal{C}(n,d)$.

Dispersion

Let $d \geq 2$ and $[0,1)^d$ be the d-dimensional unit cube. For $\mathbf{x}, \mathbf{y} \in [0,1)^d$ with $\mathbf{x} = (x_1, \dots, x_d)$ and $\mathbf{y} = (y_1, \dots, y_d)$ we write $\mathbf{x} < \mathbf{y}$ if this inequality holds coordinate-wise.

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$$\mathcal{B} := \{[\mathbf{x}, \mathbf{y}) : \mathbf{x}, \mathbf{y} \in [0,1)^d, \mathbf{x} < \mathbf{y}\}.$$

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$$\mathcal{B} := \{[\mathbf{x}, \mathbf{y}) : \mathbf{x}, \mathbf{y} \in [0,1)^d, \mathbf{x} < \mathbf{y}\}.$$

For $n \geq 1$ let T be a set of points in $[0,1)^d$ of cardinality $|T| = n$. The volume of the largest empty (from points of T) axis-parallel box, which can be inscribed in $[0,1)^d$, is called the dispersion of T:

$$\mathrm{disp}(T) := \sup_{B \in \mathcal{B}: B \cap T = \emptyset} vol(B).$$

Theorem (2; VT, 2017)

Let a set T with cardinality $|T| = 2^r =: m$ have dispersion satisfying the bound $disp(T) < C(d)2^{-r}$ with some constant $C(d)$. Then there exists a constant $c(d) \in \mathbb{N}$ such that the set $2\pi T := \{2\pi \mathbf{x} : \mathbf{x} \in T\}$ provides the universal discretization in L_∞ for the collection $\mathcal{C}(n, d)$ with $n = r - c(d)$.

Universal discretization in L_∞

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Theorem (3; VT, 2017)

Assume that $T \subset [0, 1)^d$ is such that the set $2\pi T$ provides universal discretization in L_∞ for the collection $\mathcal{C}(n, d)$. Then there exists a positive constant $C(d)$ with the following property $disp(T) \leq C(d)2^{-n}$.

Difinition of the (t,r,d)-net

Definition

A (t,r,d)-net (in base 2) is a set T of 2^r points in $[0,1)^d$ such that each dyadic box
$[(a_1-1)2^{-s_1},a_1 2^{-s_1})\times\cdots\times[(a_d-1)2^{-s_d},a_d 2^{-s_d})$, $1\le a_j\le 2^{s_j}$, $j=1,\dots,d$, of volume 2^{t-r} contains exactly 2^t points of T.

Arbitrary trigonometric polynomials

For $n \in \mathbb{N}$ denote $\Pi_n := \Pi(\mathbf{N}) \cap \mathbb{Z}^d$ with $\mathbf{N} = (2^{n-1}-1, \dots, 2^{n-1}-1)$, where, as above, $\Pi(\mathbf{N}) := [-N_1, N_1] \times \cdots \times [-N_d, N_d]$. Then $|\Pi_n| = (2^n-1)^d < 2^{dn}$. Let $v \in \mathbb{N}$ and $v \le |\Pi_n|$. Consider

$$\mathcal{S}(v,n) := \{Q \subset \Pi_n : |Q| = v\}.$$

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$$\mathcal{S}(v, n) := \{Q \subset \Pi_n : |Q| = v\}.$$

Then it is easy to see that

$$|\mathcal{S}(v, n)| = \binom{|\Pi_n|}{v} < 2^{dnv}.$$

Universal discretization problem

We are interested in solving the following problem of universal discretization. For a given $\mathcal{S}(v,n)$ and $q\in[1,\infty)$ find a condition on m such that there exists a set $\xi=\{\xi^\nu\}_{\nu=1}^m$ with the property: for any $Q\in\mathcal{S}(v,n)$ and each $f\in\mathcal{T}(Q)$ we have

$$C_1(q,d)\|f\|_q^q \le \frac{1}{m}\sum_{\nu=1}^{m}|f(\xi^\nu)|^q \le C_2(q,d)\|f\|_q^q.$$

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$$C_1(q, d)\|f\|_q^q \leq \frac{1}{m}\sum_{\nu=1}^{m} |f(\xi^\nu)|^q \leq C_2(q, d)\|f\|_q^q.$$

We present results for $q = 2$ and $q = 1$.

We begin with a general construction. Let $X_N = \mathrm{span}(u_1, \dots, u_N)$, where $\{u_j\}_{j=1}^N$ is a real orthonormal system on $\mathbb{T}^d$. With each $\mathbf{x} \in \mathbb{T}^d$ we associate the matrix $G(\mathbf{x}) := [u_i(\mathbf{x})u_j(\mathbf{x})]_{i,j=1}^N$. Clearly, $G(\mathbf{x})$ is a symmetric matrix. For a set of points $\xi^k \in \mathbb{T}^d$, $k = 1, \dots, m$, and $f = \sum_{i=1}^N b_i u_i$ we have

$$\frac{1}{m}\sum_{k=1}^{m} f(\xi^k)^2 - \int_{\mathbb{T}^d} f(x)^2 d\mu = \mathbf{b}^T \left(\frac{1}{m}\sum_{k=1}^{m} G(\xi^k) - I\right)\mathbf{b},$$

where $\mathbf{b} = (b_1, \dots, b_N)^T$ is the column vector. Therefore,

$$\left|\frac{1}{m}\sum_{k=1}^{m} f(\xi^k)^2 - \int_{\mathbb{T}^d} f(x)^2 d\mu\right| \le \left\|\frac{1}{m}\sum_{k=1}^{m} G(\xi^k) - I\right\| \|\mathbf{b}\|_2^2.$$

Probability bound

We recall that the system $\{u_j\}_{j=1}^N$ satisfies Condition **E** if there exists a constant t such that

$$w(x) := \sum_{i=1}^{N} u_i(x)^2 \leq Nt^2.$$

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Let points $\mathbf{x}^k$, $k = 1, \dots, m$, be independent uniformly distributed on $\mathbb{T}^d$ random variables. Then with a help of deep results on random matrices it was proved that

$$\mathbb{P}\left\{\left\|\sum_{k=1}^{m}(G(\mathbf{x}^k) - I)\right\| \geq m\eta\right\} \leq N \exp\left(-\frac{m\eta^2}{ct^2 N}\right)$$

with an absolute constant c.

Consider real trigonometric polynomials from the collection $\mathcal{S}(v,n)$. Using the union bound for the probability we get that the probability of the event

$$\left\|\sum_{k=1}^{m}(G_Q(\mathbf{x}^k)-I)\right\| \le m\eta \quad \text{for all} \quad Q\in\mathcal{S}(v,n)$$

is bounded from below by

$$1-|\mathcal{S}(v,n)|v\exp\left(-\frac{m\eta^2}{cv}\right).$$

For any fixed $\eta\in(0,1/2]$ the above number is positive provided $m\ge C(d)\eta^{-2}v^2n$ with large enough $C(d)$. The above argument proves the following result.

Main result for $q = 2$

Theorem (Dai, Prymak, VT, Tikhonov, 2018.)

There exist three positive constants $C_i(d)$, $i = 1, 2, 3$, *such that for any* $n, v \in \mathbb{N}$ *and* $v \leq |\Pi_n|$ *there is a set* $\xi = \{\xi^\nu\}_{\nu=1}^m \subset \mathbb{T}^d$, *with* $m \leq C_1(d)v^2 n$, *which provides universal discretization in* L_2 *for the collection* $\mathcal{S}(v, n)$*: for any* $f \in \cup_{Q \in \mathcal{S}(v,n)} \mathcal{T}(Q)$

$$C_2(d)\|f\|_2^2 \leq \frac{1}{m}\sum_{\nu=1}^{m} |f(\xi^\nu)|^2 \leq C_3(d)\|f\|_2^2.$$

Similar to the case $q = 2$ a result on the universal discretization for the collection $\mathcal{S}(v, n)$ will be derived from the probabilistic result on the Marcinkiewicz-type theorem for $\mathcal{T}(Q)$, $Q \subset \Pi_n$. However, the probabilistic technique used in the case of $q = 1$ is different from the probabilistic technique used in the case $q = 2$. The proof from VT, 2017, gives the following result.

Probability bound

Theorem (VT, 2017)

Let points $\mathbf{x}^j \in \mathbb{T}^d$, $j = 1, \dots, m$, *be independently and uniformly distributed on* $\mathbb{T}^d$. *There exist positive constants* $C_1(d)$, C_2, C_3, *and* $\kappa \in (0,1)$ *such that for any* $Q \subset \Pi_n$ *and* $m \ge yC_1(d)|Q|n^{7/2}$, $y \ge 1$,

$$\mathbb{P}\left\{\forall f \in \mathcal{T}(Q), \quad C_2\|f\|_1 \le \frac{1}{m}\sum_{j=1}^{m}|f(\mathbf{x}^j)| \le C_3\|f\|_1\right\} \ge 1 - \kappa^y.$$

The union bound

Therefore, using the union bound for probability we obtain the Marcinkiewicz-type inequalities for all $Q \in \mathcal{S}(v,n)$ with probability at least $1 - |\mathcal{S}(v,n)|\kappa^y$. Choosing $y = y(v,n) := C(d)vn$ with large enough $C(d)$ we get

$$1 - |\mathcal{S}(v,n)|\kappa^{y(v,n)} > 0.$$

This argument implies the following result on universality in L_1.

Main result for $q = 1$

Theorem (Dai, Prymak, VT, Tikhonov, 2018.)

There exist three positive constants $C_1(d)$, C_2, C_3, *such that for any* $n, v \in \mathbb{N}$ *and* $v \leq |\Pi_n|$ *there is a set* $\xi = \{\xi^\nu\}_{\nu=1}^m \subset \mathbb{T}^d$, *with* $m \leq C_1(d) v^2 n^{9/2}$, *which provides universal discretization in* L_1 *for the collection* $\mathcal{S}(v, n)$: *for any* $f \in \cup_{Q \in \mathcal{S}(v,n)} \mathcal{T}(Q)$

$$C_2 \|f\|_1 \leq \frac{1}{m} \sum_{\nu=1}^m |f(\xi^\nu)| \leq C_3 \|f\|_1.$$

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- Equal weights.
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A probabilistic technique is used for the equal weights discretization. This helps us to obtain the corresponding theorems on the universal discretization.

We don't know how to use weighted discretization for the universal discretization.

Sampling discretization of the uniform norm (Bernstein-type problem) is very different from sampling discretization of the integral norms (Marcinkiewicz-type problem).

Sampling discretization of the uniform norm. Lower bound

Theorem (Kashin, VT, 2018.)

Let $\Lambda_n = \{k_j\}_{j=1}^n$ be a lacunary sequence: $k_1 = 1$, $k_{j+1} \geq bk_j$, $b > 1$, $j = 1, \dots, n-1$. Assume that a finite set $\xi = \{\xi^\nu\}_{\nu=1}^m \subset \mathbb{T}$ has the following property

$$\forall f \in \mathcal{T}(\Lambda_n) \qquad \|f\|_\infty \leq L \max_\nu |f(\xi^\nu)|. \tag{5}$$

Then

$$m \geq \frac{|\Lambda_n|}{e} e^{Cn/L^2}$$

with a constant $C > 0$ which may only depend on b.

Upper bound

Theorem (Kashin, Konyagin, VT, 2021.)

Let X_N *be an* N*-dimensional subspace of* $\mathcal{C}(\Omega)$*. There exists a set* $\xi = \{\xi^\nu\}_{\nu=1}^m$ *of* $m \le 9^N$ *points such that for any* $f \in X_N$ *we have*

$$\|f\|_\infty \le 2 \max_\nu |f(\xi^\nu)|. \tag{6}$$

Let Ω be a compact subset of $\mathbb{R}^d$ and let μ be a probability measure on Ω. Let as above X_N be an N-dimensional subspace of $\mathcal{C} := \mathcal{C}(\Omega)$. Assume that there exists an orthonormal basis $\{u_j\}_{j=1}^N$ with respect to measure μ of the subspace X_N. Denote

$$\mathcal{D}_{X_N}(x,y) := \sum_{j=1}^{N} u_j(x)\bar{u}_j(y), \quad x,y \in \Omega,$$

the corresponding Dirichlet kernel. Then for any $f \in X_N$ we have

$$f(x) = \int_\Omega \mathcal{D}_{X_N}(x,y)f(y)d\mu.$$

Consider the following problem of constrained best M-term approximation with respect to the bilinear dictionary. Define

$$\mathcal{B}_M(X_N^\perp) := \{\mathcal{W} : \mathcal{W}(x,y) = \sum_{i=1}^{M} w_i(x)v_i(y),\ w_i \in \mathcal{C},\ v_i \in L_1\},$$

satisfying the condition: For any $f \in X_N$ and each $x \in \Omega$ we have

$$\int_\Omega \mathcal{W}(x,y)f(y)d\mu = 0\}.$$

Consider

$$\sigma_M^c(\mathcal{D}_{X_N})_{(\infty,1)} := \inf_{\mathcal{W}\in\mathcal{B}_M(X_N^\perp)} \sup_{x\in\Omega} \|\mathcal{D}_{X_N}(x,\cdot) - \mathcal{W}(x,\cdot)\|_1.$$

Under a certain condition (see Condition D below) on the subspace X_N (see Theorem below) we proved in Kashin, Konyagin, VT, 2021, that there exists a set of points $\{\xi^\nu\}_{\nu=1}^m$ such that for any $f\in X_N$

$$\|f\|_\infty \le 6\sigma^c_M(\mathcal{D}_{X_N})_{(\infty,1)}\left(\max_\nu |f(\xi^\nu)|\right). \tag{7}$$

We point out that inequality (7) is a conditional result – it is proved under Condition D, which is a condition on the L_1 norm discretization of functions from a special subspace related to the subspace X_N. Therefore, in order to apply inequality (7) we need to establish the corresponding discretization theorem. Here we use known results on the L_1 discretization.

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Let $\mathcal{W}_M = \sum_{i=1}^M w_i^* v_i^* \in \mathcal{B}_M(X_N^\perp)$ be such that

$$\sup_{x\in\Omega} \|\mathcal{D}_{X_N}(x,\cdot) - \mathcal{W}_M(x,\cdot)\|_1 \le 2\sigma_M^c(\mathcal{D}_{X_N})_{(\infty,1)}.$$

Assume as above that $\{u_j\}_{j=1}^N$ is an orthonormal basis with respect to measure μ of the subspace X_N. Consider a subspace of $L_1(\Omega,\mu)$

$$Y_S := \operatorname{span}\{v_j^*(y)f(y), \bar{u}_i(y)f(y) \,:\, f \in X_N,$$

$$j = 1,\dots,M,\ i = 1,\dots,N\}.$$

Then $S := \dim Y_S \le (M+N)N$.

Condition D. Suppose that X_N is such that there exists a set of points $\{\xi^\nu\}_{\nu=1}^m$ and a set of positive weights $\{\lambda_\nu\}_{\nu=1}^m$ such that for any $g \in Y_S$

$$\frac{1}{2}\|g\|_1 \le \sum_{\nu=1}^{m} \lambda_\nu |g(\xi^\nu)| \le \frac{3}{2}\|g\|_1. \tag{8}$$

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Theorem (Kashin, Konyagin, VT, 2021.)

Let $X_N \subset \mathcal{C}(\Omega)$ be an N-dimensional subspace. Assume that function 1 belongs to X_N (if not, we include it, which results in increase of dimension by 1). Assume that X_N satisfies Condition D. Then for the set of points $\{\xi^\nu\}_{\nu=1}^m$ from Condition D we have: For any $f \in X_N$

$$\|f\|_\infty \le 6\sigma^c_M(\mathcal{D}_{X_N})_{(\infty,1)} \left(\max_\nu |f(\xi^\nu)|\right).$$