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ON SMOOTH PARTITIONS OF UNITY OVER BANACH MANIFOLDS

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As it is known (see [1]–[3]), there are Banach spaces which do not admit smooth partitions of unity. For example, $C_{[0,1]}$ does not admit a C^1-partition of unity; L^p and l^p with $p \geq 1$ being non-even do not admit a C^r-partition of unity when $r > p$. In the present article we separate out a class of the Banach manifolds (in part, the Banach spaces) which admit a C^r-partition of unity ($r \geq 2$) of a special type. A practical need to construct those special partitions of unity arose in the author's attempt to transfer the theory of boundary indices of sets of generalized eigenvectors of a pair of nonlinear operators in Hilbert space by Yu.G. Borisovich and the author (see [4], [5]) to Banach spaces. Well-known Eells–Lang's theorem (see [6]) on existence of a C^r-partition of unity for a paracompact Hausdorff C^r-manifold, where $r \geq 1$, which is modeled by a separable Hilbert space, here is deduced as a corollary of a more general theorem. We use the standard terminology (see [6]–[9]). Our results were partially announced in [10]–[13].

1. SC^r-functions on Banach manifolds

Let E be a real Banach space and U be an open set in E. Let us recall that a real-valued function $f : E \to \mathbf{R}$ of the class C^2 is called a *Fredholm functional* (see [14]) if the derivative $Df : E \to E^*$ is a Fredholm map. Let us introduce a more general concept.

Definition 1.1. A real-valued function $f : U \to \mathbf{R}$ of class C^r, $r \geq 2$, will be called a *Fredholm C^r-function* (or a *ΦC^r-function*) if the derivative mapping $Df : U \to E^*$ is Fredholm.

We denote by $\Phi C^r(U)$ the class of all ΦC^r-functions on U.

Definition 1.2. A function $g : U \to \mathbf{R}$ of class C^r, $r \geq 2$, will be called an *elementary SC^r-function* (or, briefly, *SC^r_e-function*) if it C^∞-smoothly depends on a finite number of ΦC^r-functions $f_1, \dots, f_k : U \to \mathbf{R}$, i.e., there exists an open set W in $\mathbf{R}^k$ such that $W \supset F(U)$, $F = (f_1, \dots, f_k)$, and a C^∞-function $\varphi : W \to \mathbf{R}$ such that $g = \varphi \circ F$.

We denote by $SC^r_e(U)$ the associative commutative $\mathbf{R}$-algebra of all SC^r_e-functions on U. The $SC^r_e(U)$ is a subalgebra of the algebra $C^r(U)$ of all C^r-functions on U and it contains a subalgebra of constant functions on U. Moreover,

$$\Phi C^r(U) \subset SC^r_e(U). \tag{1}$$

One easily sees that the following three statements are equivalent: 1) $\Phi C^r(U) \neq \emptyset$; 2) $SC^r(U) \neq \emptyset$; 3) any constant function on U is an SC^r_e-function. Therefore, if one of these three conditions holds for a given open subset U in a Banach space E, (1) is the strict inclusion.

Definition 1.3. A Banach space E is said to be *SC^r-smooth* if there exists an SC^r_e-function $g : E \to \mathbf{R}$, $r \geq 2$, with a bounded nonempty support.