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SUPERPOSITION OPERATORS IN SOBOLEV SPACES

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Introduction

Consideration of superposition operators in the Sobolev spaces is related to the generalization of the problem by Yu.G. Reshetnyak about description of all isomorphisms φ^* of homogeneous Sobolev spaces L^1_n, generated by quasiconformal mappings φ of the Euclidean space $\mathbb{R}^n$ by the rule $\varphi^*(u) = u \circ \varphi$; the problem was posed in 1968 at the First Donetsk Colloquium on the theory of mappings. In [1] it was proved that the desired isomorphisms are the structure isomorphisms of the spaces L^1_n and only these isomorphisms. An approach suggested in [1] to the Reshetnyak problem can be naturally considered in the context of the preceding results (see [2], pp. 419–420). In theorems by Banach, Stone, Eilenberg, Arens and Kelly, Hewitt, Gel'fand, and Kolmogorov, conditions were obtained for various structures of the space of continuous functions $C(S)$, whose isomorphism defines the topological space S up to a homeomorphism. Let us note a result by Stone, which states that, treated as a structurally ordered group, $C(S)$ defines S. On the other hand, M. Nakai (see [3]) and L. Lewis (see [4]) established that the isomorphy of the Royden algebras is equivalent to the quasiconformal equivalence of domains of definition. Now, by choosing in the homogeneous Sobolev space L^1_n the two structures: that of a vector lattice and that of a seminormed space, we obtain a situation which is algebraically close to the Stone's work, while, in the metrical sense, close to Nakai's work. Such a point of view on the problem is most natural, because it still gives us a possibility to restore a mapping in spite of a minimal "matter" for finding this, to prove its continuity, and to establish its metrical properties.

In the frameworks of an approach to Rechetnyak problem, which was found in [1], the following problem arises: What metrical and analytic properties does possess a measurable mapping φ inducing the isomorphism φ^* by the rule $\varphi^*(f) = f \circ \varphi$, $f \in L^1_n$? By varying the functional space L^1_n, we arrive every time at a new problem: the Sobolev spaces W^1_p, $p > n$, were considered in [5], the homogeneous Besov spaces $b^l_p(\mathbb{R}^n)$, $n > 1$, $lp = n$, in [6] for $p = n+1$ and in [7] for $p > n+1$, the Sobolev spaces W^1_p, $n-1 < p < n$, in [8], the Sobolev spaces W^1_p, $1 \le p < n$ (and spaces of potentials) in [9], the three-index scales of Nikol'skiĭ–Besov and Lizorkin–Triebel spaces (and their anisotropic analogs) in [10]. In [11], to the problem of change of variable in the Sobolev spaces the theory of multipliers was applied. The conclusion from [5]–[10] is that the isomorphy of the operator φ^* implies (it depends on the relation between the exponents of smoothness, integrability, and the dimension) the mapping's properties of quasiconformity or quasi-isometry in the metric of the domain of definition adequate to the geometry of the functional space.

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