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Canonical Frame of a Curve on a Conformal Plane

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Abstract—We show how differential geometry of smooth curves on the conformal plane can be studied by Élie Cartan's method of exterior forms and moving frames. We find the canonical form of the derivation equations of a curve (which is not a circle) in the case of a semi-isotropic frame. We give a new proof of the theorem that states that curves of constant (in particular, zero) conformal curvature are loxodromes. We integrate the system of structure equations of the isotropy subgroup of a point.

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As is known, Élie Cartan's method of exterior forms and moving frames is a universal method for the study of differential geometry of homogeneous spaces. The object under study is described by a system of structure equations, i.e., a system of differential equations whose form does not depend on the choice of local coordinates and changes of variables. Then the study of the geometry is reduced to some standard procedures. On the other hand, Cartan's method provides a geometric approach to the study of differential equations arising in various applications.

In this paper, we show how Cartan's method can be applied to the conformal geometry of plane curves. The conformal group, which acts on the conformal plane, is the subgroup of the group of projective transformations that send circles to circles (they are called circle transformations or Möbius transformations).

The problem is not a new one, but we return to it for the following reasons. First of all, recently the interest to the conformal geometry raised in connection with its various applications, e.g., to the geometry of human eye.

Recall that the structure equations of the conformal space were obtained by É. Cartan (for example, [1], P. 153, where these equations are written in terms of a semi-isotropic frame; in the same book the structure of the group of spherical transformations is considered).

The conformal differential geometry of a curve in the n-dimensional space, $n > 2$, was studied in detail in [2] with the use of classical tensor method. The differential geometry of curves and families of circles in two-dimensional and three-dimensional spaces was studied in detail in [3–6]. In [6], a curve was studied as the envelope of a family of circles, in derivation equations of which the authors preserve relative invariants (in contrast to the Frenet equations of a Euclidean curve, where only absolute invariants take part). These two circumstances complicate the task of bringing of derivation equations of a curve to the canonical form and make these equations awkward and nonsymmetric.

The conformal differential geometry of a hypersurface was exposed by Cartan's method in [7], but in this book the case of the least dimension (a curve in the plane) is only mentioned and does not considered in detail. We fill this gap using the general approach and the ideas of [7] which preserve their sense in the case of the minimal dimension.

In the paper, we study the isotropy subgroup of a point on the conformal plane, find a canonical form of the derivation equations of a curve in terms of a semi-isotropic frame, give a new proof of the known theorem which states that curves of constant (in particular, zero) conformal curvature are loxodromes.

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