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The methodical guide is intended for students of the Institute of Physics, Institute of Geology and Petroleum Technologies, as a support to the general physical practicum to the courses «Physics of particles and atomic nuclei», «Nuclear physics» and «Physics».

Contents

Introduction

The method of counting the number of particles registered by a detector is widely used in experimental nuclear physics. Depending on the tasks, this method can give various information about the radiation being studied and about the source of this radiation. This can be the spectral composition, the nature of the interaction of particles with matter, the regularities of nuclear reactions, etc. The energies of nuclear transformations are large enough (>10 keV) to register individual particles via products of their interaction with matter. At the same time, it is often necessary to work with sources with low activity. In this case, the statistical nature of radioactive decay becomes the factor determining the accuracy of the experiment. The stochastic nature of nuclear transformations leads to the impossibility of determining the time at which a decay process takes place for one of the source nuclei. We can speak only of the probability of decay over a certain time interval dt. For small values of dt, the decay probability of one nucleus will depend linearly on dt:

$$dP_d = \lambda dt, \tag{1}$$

where dP_d – the probability of decay, λ – the decay constant.

In the case of a large number of radioactive nuclei N, we can speak of the average number dN of nuclei decaying during the time dt:

$$dN = \lambda N dt. \tag{2}$$

Integration of this equation leads to the well-known law of radioactive decay. In the ensemble of N radioactive nuclei the probability of the decay event occurring in a short time dt:

$$dP = \lambda N dt = pdt, \tag{3}$$

where $p = \lambda N$ is the probability density with respect to the time of occurrence of the decay event. Let us find the probability distribution to which the number of particles registered by the detector (the number of counts) in time t obeys. The number of the events is directly proportional to the number of nuclear transformations in the radioactive source; hence, we can introduce, by analogy with (3), the probability density with respect to the occurrence time of the registration of the event p. We

consider the intensity of the source unchanged during the measurement. We introduce a partition into small intervals *dt* on the interval of time *t*. The number of segments of the partition *t/dt* is denoted by *m*. In this case, the probability *P(n,t)* of observing *n* decays in a time *t* will obey the binomial distribution:

$$P(n,t) = C_m^n \cdot (pdt)^n \cdot (1 - pdt)^m.$$ (4)

For $dt \to 0$, we obtain known formula for Poisson distribution:

$$P(n,t) = \lim_{dt \to 0} \left[\frac{m!}{n!(m-n)!} \cdot (pdt)^n \cdot (1 - pdt)^m \right] =$$

$$= \lim_{dt \to 0} \left[\frac{m^n}{n!} \cdot \left(pt \cdot \frac{1}{m} \right)^n \cdot (1 - pdt)^{\frac{1}{pdt} \cdot m \cdot pdt} \right],$$

$$P(n,t) = \frac{(pt)^n}{n!} e^{-pt}.$$ (5)

Thus, the random value of the number of particles *n* registered by the detector during time *t* is subject to the Poisson distribution.

A remarkable feature of the Poisson distribution is that the mathematical expectation *M[n]* and variance *D[n]* of the random variable *n* are the same (see details of derivation in Appendix A):

$$M[n] = D[n] = pt.$$ (6)

Introducing average value $\langle n \rangle = M[n] = pt$ of registered particles we can rewrite (5):

$$P(n, \langle n \rangle) = \frac{(\langle n \rangle)^n}{n!} e^{-\langle n \rangle}.$$ (7)

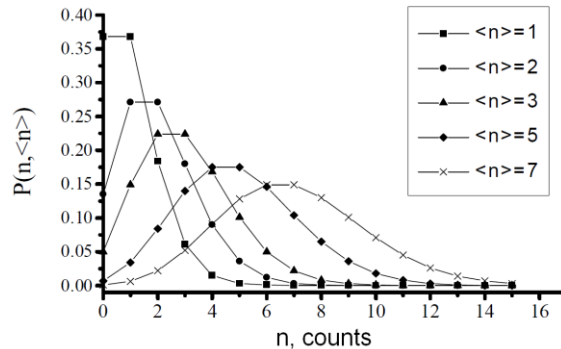Figure 1 shows the Poisson distribution for some $\langle n \rangle$ values.

Fig. 1. Poisson distribution for some average values

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Equation (6) provides easy way for estimation of statistical error in experiments through standard deviation σ:

$$\sigma[n] = \sqrt{D[n]} = \sqrt{pt} \approx \sqrt{n} . \tag{8}$$

For small values $\langle n \rangle$ the distribution is strongly asymmetric. For large $\langle n \rangle$ the distribution asymptotically tends to Gaussian distribution, which can be proved with the help of the central limit theorem of probability theory:

$$P(n, \langle n \rangle) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(n - \langle n \rangle)^2}{2\sigma^2} \right] = \frac{1}{\sqrt{2\pi\langle n \rangle}} \exp\left[-\frac{(n - \langle n \rangle)^2}{2\langle n \rangle} \right] . \tag{9}$$

If we try to show Poisson distribution (7) and corresponding Gaussian approximation (9) for $\langle n \rangle > 100$ in the manner like in Figure 1, then we will not be able to see any difference between them.

Such asymptotic behavior allows applying rules known for Gaussian distribution. For example, the so called "*1-2-3 sigma rule*" for normal distribution, or "*68-95-99.7 rule*", is applicable in many experiments with registration of ionizing radiation by counting of particles. This simple mnemonic rule defines relation between confidence interval measured in σ units and confidence probability. Thus, 1σ corresponds to ~68% of confidence probability, 2σ to ~95%, and 3σ to ~99.7%.

Finally, we can do simple semi-quantitative estimation in practice. For example, if we perform measurement in some experiment (let's imagine that we measure the activity *A* of radioactive source on a calibrated setup), and this measurement is based on the number *n* = 10000 of registered particles, we can judge about measured value (in our case the activity) with error having statistical nature and equal to:

$$\frac{\Delta A}{A} = \frac{\Delta n}{n} = \frac{\sqrt{\langle n \rangle}}{n} \cong \frac{\sqrt{n}}{n} = 0.01 . \tag{10}$$

Thus, in this example the relative error will be estimated as 1% for confidence interval 1σ.

Experimental Setup

The setup for the experiment consists of end-window Geiger counter, CassyLab2 digital acquisition unit (labeled as Cassy Sensors 2), personal computer (notebook), and the unit of shielded and collimated radioactive source ^{90}Sr with the activity of 45 kBq.

The Geiger counter can be moved manually with respect to the source.

Carrying out the experiment

Preparation

Before carrying out the experiments make sure that all units of the setup are available, and connected properly.

• Move the Geiger counter to the radioactive source as close as possible. Plastic cover of the detector may slightly touch the output window of the source unit.

• Switch on the notebook, power supply of the CassyLab2 data acquisition unit.

• Make sure that the CassyLab2 data acquisition unit is recognized and connected properly. The green LED of USB interface on CassyLab2 unit will indicate the correct connection.

• Run CassyLab2 by clicking the icon on the desktop.

• In the window "CASSYs" select "measurements channel" by the upper square selection of the right column (Geiger counter). In such a manner the Geiger counter hardware module of CassyLab2 is activated. Green LED near to the module of Geiger counter will indicate correct activation. After this, the window of registered events should appear in the CassyLab2 PC software. You must see increasing number of events if all parts of the setup work correctly.

• Close the window "CASSYs".

• Load settings file for this work from "Main Menu" → "Help" → "Experiment Examples" → "Physics" → "Poisson distribution" → "Load Settings". The window will appear which offers to save current settings. Do not save current setting when offered to do so.

• Close the window "CASSYs" again.

• Make sure the tabs "Standard" and "Frequency distribution" have appeared, and the window of the counting rate "Rate R_{A1}" shows the value in the range 100-500 [1/s].

Experiment

- In the "Settings" tree on the right side of the main window select the set of parameters for A1 input (the Geiger counter input, "Settings" → "CASSYs" → "Input A_1 (Geiger-Müller counter tube S, 5240331)" → "Rate R_{A1}"). Set the following combination of parameters: Gate Time = 1s; Meas. Time – left this field empty, empty field means measurement until user's interruption; Recording – Automatic. Put in the field "Range, 1/s" the value which is higher than the observed average count rate by a factor 1.5 − 2.5.

- Start measurement from "Main Menu" → "Measurement" → "Start/Stop" or by pressing <F9> key.

- By switching between tabs "Standard" and "Frequency distribution" make sure that consequent measurements have started properly and data collection has been started.

- Do measurement until maximum number of events in the tab "Frequency distribution" reaches 50. This will require approximately 40 min. for count rate 350[1/s] and Gate Time = 1s. The corresponding number of consecutive single-second measurements can be seen on the tab "Standard", and it should be equal to approximately 2000. The graph on the tab "Standard" demonstrates statistical fluctuations of single-second measurements.

- Stop the measurement from "Main Menu" → "Measurement" → "Start/Stop" or by pressing <F9> key.

- Save the results from "Main Menu" → "File" → "Save" or by pressing <F2> key.

- Increase the distance between the source and the Geiger counter to about 10 cm, so that the average counting speed will be roughly equal to 10[1/s].

- Repeat the measurement for this new average count rate. Do not forget to set "Range, 1/s" parameter for new lower value. This parameter affects the presentation of the results in the graph in the tab "Standard". The estimation of required time is 10-20 min.

- Save the result for low count rate.

Evaluation and results

Treat the measured data for both count rates.

Open the file and make scaling of the data first. You can do it by right-click of the mouse over each axis of the graph. In the appeared pop-up window set the minimum and maximum values to show, or press "Find Minimum and Maximum". Perform these steps for each of tabs, "Standard" and "Frequency distribution".

Open the tab "Frequency distribution" and fit the experimental histogram first by Poisson distribution and then by Gaussian distribution. To do this, make the following actions:

- do right-click of the mouse over the graph and go through menu in the pop-up window: "Other Evaluations" → "Calculate Poisson distribution"/ "Calculate Gaussian distribution";

- select the area of fitting by holding left button of mouse and moving it;

- record the results of fitting, which are shown at the bottom of the main window. Average of distribution and standard deviation is designed here as μ and σ, correspondingly.

The result of evaluations should be four pairs of values: μ and σ for two average count rates, and two distributions.

Report

Process the data by any appropriate software.

Make graphs of experimental Poisson distribution with obtained results of fitting. Note if there is no visible difference between the Gaussian distribution and the Poisson distribution.

Show the value of the sigma in the graphs using a line segment.

On the graphs of consequent single-second measurements show the "*1-2-3 sigma rule*" by means of using a line segment. Verify that the corresponding confidence probability is consistent. For example, ~68% of experimental points must be located inside of 1σ experimental confidence interval.

Present the results in printed form, including all estimations and graphs.

Safety notes

The following safety rules must be kept to:

- **Prevent access to the preparations by unauthorized persons.**

- **Before using the preparations make sure that they are intact.**

- **For the purpose of shielding, keep the preparations in their safety vessel.**

- **To ensure minimum exposure time and minimum activity, take the preparations out of the safety vessel only as long as is necessary for carrying out the experiment.**

- **To ensure maximum distance, hold the preparations only at the upper end of the metal holder and keep them away from your body as far as possible.**

Self-test problems

1.   Geiger-counter based dosimeter estimates dose rate by counting the registered particles. They count, on average, 10 particles at one 30-seconds measurement and at natural radiation background condition (0.1 µSv/h). What is the statistical accuracy of measuring natural radiation background by means of this dosimeter? What is the accuracy of the dosimeter at 100 µSv/h?

2.   The activity of radioactive source was measured on calibrated setup with statistical accuracy 10% during the time 1 hour. What time is required to reach the accuracy 1% on the same setup? Do not take into account any other experimental errors.

3.   Real detectors of particles have *dead time* parameter, which means impossibility to register two events happening during the time shorter than the dead time. So, the real distribution of counted particles is binomial, and the Poisson distribution is rather a convenient approximation. Assuming *dead time* for Geiger counter equal to 100 µsec, find the quality of this approximation. Make with Poisson and binomial distribution the estimations of the probability of registration of 10 particles at average of distribution equal to 10.

4.   The cake contains, on average, 5 raisins. Calculate the probability of finding exactly one raisin in it.

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Appendix A. Variance and average of Poisson distribution

The Poisson distribution:

$$P(n,t) = \frac{(pt)^n}{n!}e^{-pt}. \qquad (A.1)$$

Let's note that the probability to obtain any number of events n, from 0 to ∞, equals to 1, i.e.:

$$\sum_{n=0}^{\infty} P(n,t) = 1. \qquad (A.2)$$

The average of distribution $M[n] = \langle n \rangle$ is:

$$M[n] = \sum_{n=0}^{\infty} nP(n,t) = \sum_{n=1}^{\infty} n\frac{(pt)^n}{n!}e^{-pt} = pt\sum_{n=1}^{\infty}\frac{(pt)^{n-1}}{(n-1)!}e^{-pt} = pt\sum_{k=0}^{\infty}\frac{(pt)^k}{k!}e^{-pt} = pt. \qquad (A.3)$$

Here $k = n-1$.

The variance $D[n]$ is:

$$D[n] = M\left[(n - \langle n \rangle)^2\right] = \sum_{n=0}^{\infty}(n - pt)^2 P(n,t) = \sum_{n=0}^{\infty} n^2 \frac{(pt)^n}{n!}e^{-pt} - 2(pt)^2 + (pt)^2 =$$

$$= pt\sum_{n=1}^{\infty} n\frac{(pt)^{n-1}}{(n-1)!}e^{-pt} - (pt)^2 = pt\sum_{k=0}^{\infty}(k+1)\frac{(pt)^k}{k!}e^{-pt} - (pt)^2 = pt(pt+1) - (pt)^2,$$

$$D[n] = pt. \qquad (A.4)$$

Thus, the average and variance of Poisson distribution are equal to each other.