ISSN 1066-369X, Russian Mathematics (Iz. VUZ), 2009, Vol. 53, No. 6, pp. 63–66. © Allerton Press, Inc., 2009.
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Splitting in 2-Computably Enumerable Degrees with Avoiding Cones

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Received December 16, 2008

Abstract—In this paper we show that for any pair of properly 2-c. e. degrees $\mathbf{0} < \mathbf{d} < \mathbf{a}$ such that there are no c. e. degrees between $\mathbf{d}$ and $\mathbf{a}$, the degree $\mathbf{a}$ is splittable in the class of 2-c. e. degrees avoiding the upper cone of $\mathbf{d}$. We also study the possibility to characterize such an isolation in terms of splitting.

DOI: 10.3103/S1066369X09060127

Key words and phrases: *2-c. e. degrees, Turing degrees, splitting, splitting with avoiding cones, isolation.*

In the study of the degree structures of finite levels of Ershov's hierarchy the splitting properties form a wide class of problems. A degree $\mathbf{a}$ is splittable in a class of degrees $\mathcal{C}$ if there exist degrees $\mathbf{x}_0, \mathbf{x}_1 \in \mathcal{C}$ such that $\mathbf{a} \leq \mathbf{x}_0 \cup \mathbf{x}_1$ and $\mathbf{x}_0, \mathbf{x}_1 < \mathbf{a}$. Considering finite levels of Ershov's hierarchy, one usually tries to carry out the splitting in the same level, but often it is impossible. For instance, according to splitting theorems of G. E. Sacks [1] and S. B. Cooper [2], computably enumerable (further c. e.) and 2-c. e. degrees can be splitted in the class of c. e. and the class of 2-c. e. degrees, respectively. On the other hand, in the theorem of M. M. Arslanov, S. B. Cooper, and A. Li [3] any c. e. degree is splittable in the class of 2-c. e. degrees above any low 2-c. e. degree. The questions about the splitting of certain degrees above other ones present a special interest, they are not completely studied yet. The question about the splitting (hereinafter in the class of 2-c. e. degrees) of a 2-c. e. degree above an arbitrary low 2-c. e. degree is still open.

Another direction of research is splitting with avoiding cones. For arbitrary degrees $\mathbf{x}$ and $\mathbf{y}$ we say that the degree $\mathbf{x}$ avoids the upper (lower) cone of the degree $\mathbf{y}$ if $\mathbf{y} \nleq \mathbf{x}$ ($\mathbf{x} \nleq \mathbf{y}$). In this paper we study the problem of splitting a proper 2-c. e. degree $\mathbf{a}$ avoiding the upper cone of a proper 2-c. e. degree $\mathbf{d}$ that is below $\mathbf{a}$. Such splitting of the degree $\mathbf{a}$ means the existence of 2-c. e. degrees $\mathbf{x}_0$ and $\mathbf{x}_1$ such that $\mathbf{x}_0, \mathbf{x}_1 < \mathbf{a}$, $\mathbf{a} \leq \mathbf{x}_0 \cup \mathbf{x}_1$, and $\mathbf{d} \nleq \mathbf{x}_0, \mathbf{x}_1$. Without additional requirements to properties of degrees $\mathbf{a}$ and $\mathbf{d}$ there exist proper 2-c. e. degrees $\mathbf{0} < \mathbf{d} < \mathbf{a}$ such that every splitting of $\mathbf{a}$ is splitting above $\mathbf{d}$. This follows from the theorem of M. M. Arslanov, I. Sh. Kalimullin, and S. Lempp [4] about the existence of a "bubble": $\exists \mathbf{x}, \mathbf{y}\ \{[\mathbf{0} < \mathbf{x} < \mathbf{y}] \wedge \forall \mathbf{z}\ [\mathbf{z} \leq \mathbf{y} \rightarrow [\mathbf{z} \leq \mathbf{x} \vee \mathbf{x} \leq \mathbf{z}]]\}$, where all quantifiers go through the class of 2-c. e. degrees. We take $\mathbf{y}$ as the degree $\mathbf{a}$ and we do any proper 2-c. e. degree below $\mathbf{x}$ as the degree $\mathbf{d}$. Since all 2-c. e. degrees below $\mathbf{a}$ are comparable with $\mathbf{x}$, every splitting of $\mathbf{a}$ is splitting above $\mathbf{x}$ and, consequently, that above $\mathbf{d}$.

The following theorem presents sufficient conditions for a degree $\mathbf{a}$ to be splitted avoiding the upper cone of $\mathbf{d}$.

Theorem 1. *Let* $\mathbf{a}$ *and* $\mathbf{d}$ *be proper* 2*-c. e. degrees such that* $\mathbf{0} < \mathbf{d} < \mathbf{a}$; *assume that there are no c. e. degrees between* $\mathbf{a}$ *and* $\mathbf{d}$. *Then there exist* 2*-c. e. degrees* $\mathbf{x}_0$ *and* $\mathbf{x}_1$ *such that* $\mathbf{a} \leq \mathbf{x}_0 \cup \mathbf{x}_1$, $\mathbf{x}_0 < \mathbf{a}$, $\mathbf{x}_1 < \mathbf{a}$ *and* $\mathbf{d} \nleq \mathbf{x}_0$, $\mathbf{d} \nleq \mathbf{x}_1$.

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