Elliptic equations and boundary value problems on manifolds with a non-smooth boundary

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Outline

A classical pseudo-differential operator A with the symbol $A(x,\xi)$ in Euclidean space $\mathbb{R}^m$ is defined by the formula [1, 2, 3, 4]

$$(Au)(x) = \int\limits_{\mathbb{R}^m} A(x,\xi)e^{-ix\cdot\xi}\tilde{u}(\xi)d\xi,$$

where the sign $\sim$ over a function denotes its Fourier transform

$$\widetilde{u}(\xi) = \int\limits_{\mathbb{R}^m} u(x)e^{ix\cdot\xi}dx.$$

Our main goal is describing possible solvability conditions or at least Fredholm properties for a pseudo-differential equation

$$(Au)(x) = f(x),\ x \in D,$$

where D is a manifold with non-smooth boundary, A is an elliptic pseudo-differential operator with the symbol $A(x,\xi)$. such operators are defined locally by the formula

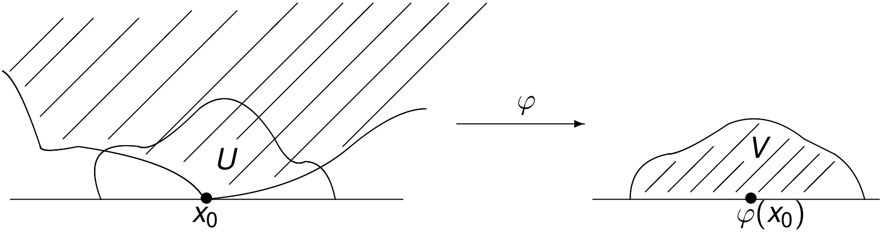
$$u(x) \longmapsto \int\limits_{\mathbf{R}^m}\int\limits_{\mathbf{R}^m} A(\cdot,\xi)u(y)e^{i(x-y)\cdot\xi}d\xi dy,$$

if D is a smooth compact manifold because one can use a freezing coefficients principle or in other words a local principle.

For a manifold with a smooth boundary one needs other local formulas for defining the operator A; more precisely for inner points of the manifold D we use the first formula but for boundary points we use another formula namely

$$u(x) \longmapsto \int\limits_{\mathbf{R}^m_+} \int\limits_{\mathbf{R}^m} A(\cdot, \xi) u(y) e^{i(x-y)\cdot\xi} d\xi dy,$$

where $\mathbf{R}^m_+ = \{x \in \mathbf{R}^m : x = (x_1, ..., x_m), \ x_m > 0\}$ is a half-space. For studying invertibility of the last operator with a symbol $A(\cdot, \xi)$ non-depending on the spatial variable x one can use the theory of classical Riemann boundary value problem [5, 6] for upper and lower half-planes with a parameter ξ'. It was done systematically in Vishik–Eskin's papers [3].

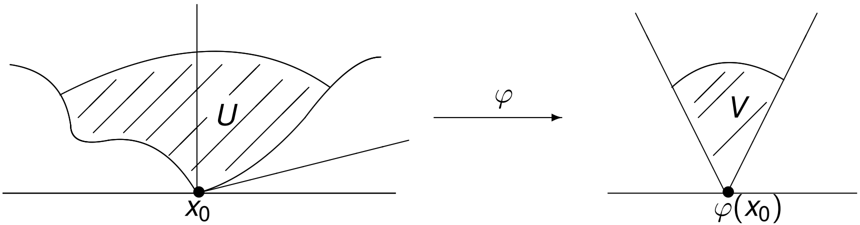

U
x_0
φ
V
$\varphi(x_0)$

But if a boundary ∂D has at least one conical point this approach is not applicable.
A conical point at a boundary is such a point for which its neighborhood is diffeomorphic to the cone

$$C_+^a = \{x \in \mathbf{R}^m : \ x_m > a|x'|, \ x' = (x_1, ..., x_{m-1}), \ a > 0\},$$

and from this the following local definition for a pseudo-differential operator in conical neighborhood arises

$$u(x) \longmapsto \int\limits_{C_+^a} \int\limits_{\mathbf{R}^m} A(\cdot, \xi) u(y) e^{i(x-y)\cdot\xi} d\xi dy.$$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The classical Riemann boundary problem in its simplest form [4]is to find piecewise analytic function, more precisely the function which is analytic in upper and lower complex plane, and which satisfies the linear relation on a straight real line

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in \mathbf{R}. \tag{1}$$

If $\Phi(z)$, $z \in \mathbf{C} \setminus \mathbf{R}$, is analytic function, then $\Phi^\pm(t)$ denote its boundary values on $\mathbf{R}$ ($z = x + iy$, $y \to 0\pm$), $G(t)$ is called coefficient of the Riemann problem, $g(t)$ and $G(t)$ are given functions on $\mathbf{R}$.
The solution of the problem (1) is constructed with the help of factorization of function G and Cauchy type integral

(one-dimensional singular integral)

$$\Phi(z) = \frac{1}{2\pi i} \int\limits_{-\infty}^{+\infty} \frac{\varphi(t)}{z-t} dt, \quad z \in \mathbf{C} \setminus \mathbf{R}. \tag{2}$$

Definition. Factorization of function $G(t)$ is called its representation in the form

$$G(t) = G_+(t)G_-(t), \tag{3}$$

where the functions $G_\pm(t)$ admit analytic continuation in upper and lower complex half-plane. Key point for solving the problem (1) takes the formulas for limit boundary values for integral (2) and Sokhotski formulas

$$\Phi_+(t) - \Phi_-(t) = \varphi(t),$$

$$\Phi_+(t) + \Phi_-(t) = \frac{1}{\pi i} \text{ v.p. } \int\limits_{-\infty}^{+\infty} \frac{\varphi(t)}{t-\tau} d\tau. \tag{4}$$

The last integral is treated in principal value sense and is called Hilbert transform of function φ :

$$(H\varphi)(t) = \frac{1}{\pi i} \text{ v.p. } \int\limits_{-\infty}^{+\infty} \frac{\varphi(t)}{t-\tau} d\tau. \tag{5}$$

This problem permits a series of multi-dimensional generalizations. I will talk on these variants step by step.

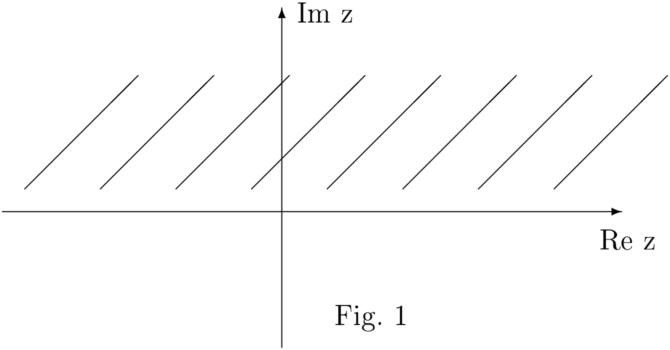

Fig. 1

For one-dimensional case the upper and lower half-plane is a set of complex numbers of the type

$$\mathbf{R} \pm \mathbf{iR}_+,$$

$Im\ z \subset \mathbf{R}_+$, and $\mathbf{R}_\pm$ is unique simplest one-dimensional cone. For two-dimensional case the situation is more complicated. The first generalizations were related to so-called bicylindrical domains [8,11].

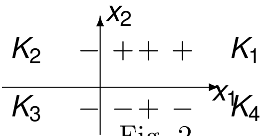

Fig. 2

In this picture the imaginary part of two-dimensional bicylindrical domains is shown, and the two-dimensional Riemann problem is stated by the following way. We seek a function $\Phi(z_1, z_2)$ which is analytic in the four domains of complex space $\mathbf{C^2}$ of type $\mathbf{R^2} + \mathbf{iK_m}$, $m = 1, 2, 3, 4$ (these domains are called radial tube domains over the cones K_m [7,19]), and for which their boundary values (there are four boundary values in this case) satisfy the linear relation

$$A(x_1,x_2)\Phi^{++}(x_1,x_2) + B(x_1,x_2)\Phi^{-+}(x_1,x_2)+$$

$$+C(x_1,x_2)\Phi^{--}(x_1,x_2) + D(x_1,x_2)\Phi^{+-}(x_1,x_2) = f(x_1,x_2), \quad (6)$$

$$(x_1,x_2) \in \mathbf{R^2}.$$

Although for one-dimensional case the problem (1) is completely solvable by the Cauchy type integral (2), the two-dimensional analogue of Cauchy type integral

$$\Phi(z_1, z_2) = \frac{1}{4\pi^2} \int_{\mathbf{R}^2} \frac{\varphi(t_1, t_2) dt_1 dt_2}{(t_1 - z_1)(t_2 - z_2)} \quad (7)$$

doesn't help for solving the problem (6).
For some special cases only it's possible constructing the solution with the help of the integral (7).
An another variant of multidimensional generalization of the Riemann problem was suggested by V.S. Vladimirov [18] (I will use the picture 2), which coincides with the problem (1) in one-dimensional case and is formulated by the following way. Finding the function $\Phi(z_1, z_2)$ which is analytic in radial tube domains [7,19] $T(K_1)$, $T(K_2)$ over the cones K_1, K_2 respectively,

and for which boundary values satisfy the linear relation

$$\Phi_{++}(x_1,x_2) = G(x_1,x_2)\Phi_{--}(x_1,x_2) + g(x_1,x_2), \quad (x_1,x_2) \in \mathbf{R}^2, \tag{8}$$

but this statement for such problem doesn't take into account the domains $T(K_2)$, $T(K_4)$.

All these problems mentioned above were solved by factorization method, namely, by function decomposition into the product of two factors admitting an analytic continuation into appropriate domain. In this way the different functional classes for solution were described, this sufficient factorization conditions were obtained, but in my point of view, no one from this multidimensional generalizations had future development and serious application.

I will describe now one variant of multidimensional Riemann problem and will show what consequences we can have stating

from this statement and existence of a special factorization.
So, for simplicity, we consider the space $L_2(\mathbf{R^2})$ and the spaces $A(\mathbf{R^2})$ which is consisting of analytic functions in radial tube domain $T(K_1)$ and satisfying the condition [7,19]

$$\sup_{y\in K_1}\int_{\mathbf{R^2}}|f(x+iy)|^2dx<+\infty,$$

$B(\mathbf{R^2})$ is an orthogonal complement of $A(\mathbf{R^2})$ in the space $L_2(\mathbf{R^2})$, so that

$$A(\mathbf{R^2})\oplus\mathbf{B}(\mathbf{R^2})=\mathbf{L_2}(\mathbf{R^2}).$$

Further, the statement of multidimensional Riemann problem is precise copy of (1)

$$\Phi^+(t)=G(t)\Phi^-(t)+g(t),\quad t\in\mathbf{R^2}\qquad(9)$$

with the one difference, the function Φ^+ is sought in $A(\mathbf{R^2})$ and the function Φ^- is sought in the space $B(\mathbf{R^2})$.
It was shown [21,22] such statement for multidimensional boundary Riemann problem must give certain meaning to studying solvability of pseudo differential equations in model non-smooth domains. If we consider the pseudo differential equation

$$(Au)(x) = f(x), \quad x \in D, \tag{10}$$

in multidimensional domain $D \subset \mathbf{R^m}$, $m \geq 2$, for which its boundary is a smooth surface, then a model problem is the equation in the half-space

$$(Au)(x) = f(x), \quad x \in \mathbf{R^m_+}. \tag{11}$$

Here A stands for a pseudo differential operator with symbol non-depending on the pole x (the equation with "frozen coefficients"). The equation (11) in Fourier images is reduced to one-dimensional singular integral equation with the parameter $\xi' = (\xi_1, \ldots, \xi_{m-1})$, i.e. to the Riemann problem (1) which is solved by factorization method. This situation is studied in details in papers' series of M.I.Vishik and G.I.Eskin [3,16,17], is fixed algebraically by L. Boutet de Monvel [1] and moved up to index theorem (see also S.Rempel, B.-W. Schulze [13]). But existence of one conical point only on a boundary forbids to use this theory.

I wrote many times on another approach to studying solvability for pseudo differential equations in domains with conical points and wedges, but now I would like to speak on principal difference of my papers from other authors (V.G. Maz'ya [5,6], B.A. Plamenevski [9,10], B.-W. Schulze [14] and many others).

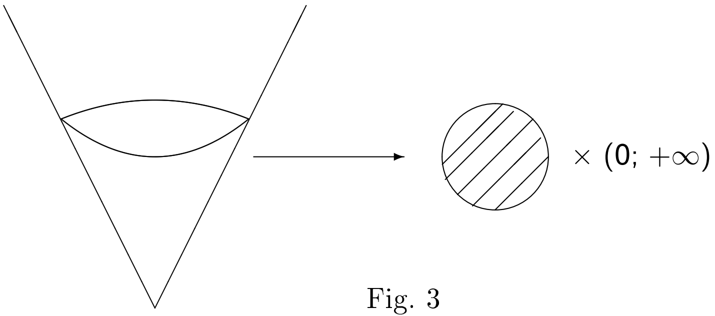

Fig. 3

In all papers the conical domain (see Fig.3) is treated as the direct product of a circle and a half-axis (but in my point of view, it is a cylinder, see, for example [12]), then they apply the Mellin transform on half-axis, and the initial problem is reduced to a problem in a domain with a smooth boundary with operator-valued symbol. That follows further it is like the generalization of well-known results on operator symbol case. Of course, the my approach is generalization also, but it is a generalization on dimension space, and the principal difference is that I don't divide the cone, and it's treated as an emergent thing.

In this way we meet the multidimensional Riemann boundary problem mentioned above; it permits to construct very interesting theory of pseudo differential equations and boundary value problems in domains, for which their boundaries have

singularities of "cone" and "wedge" type.
Let $C^a_+ = \{x \in \mathbf{R^m}: \ \mathbf{x_m} > \mathbf{a}|\mathbf{x'}|, \ \mathbf{x'} = (\mathbf{x_1}, \ldots, \mathbf{x_{m-1}}), \ \mathbf{a} > \mathbf{0}\}$ be a cone in m-dimensional space, $\overset{*}{C}{}^a_+$ be a conjugate cone, $C^a_- = -C^a_+$, $T(\overset{*}{C}{}^a_+)$ be a radical tube domain over the cone C^a_+ [7,19]. The model pseudo differential equation in the cone C^a_+ is the equation of type

$$(Au)(x) = f(x), \quad x \in C^a_+, \tag{12}$$

where A is pseudo differential operator with the symbol $A(\xi)$, $\xi \in \mathbf{R^m}$, satisfying the condition

$$c_1 \leq |A(\xi)(1 + |\xi|)^{-\alpha}| \leq c_2, \tag{13}$$

c_1, c_2 are positive constants, $\alpha \in \mathbf{R}$, is roughly speaking the order of a pseudo differential operator.

Definition. Wave factorization of a symbol $A(\xi)$ with respect to C_+^a is called its representation in the form

$$A(\xi) = A_{\neq}(\xi)A_{=}(\xi),$$

and the factors $A_{\neq}(\xi)$, $A_{=}(\xi)$ have satisfy the following conditions:
1) $A_{\neq}(\xi)$, $A_{=}(\xi)$ are defined on R^m without may be the points $\{\xi \in \mathbf{R^m} : \mathbf{a}\xi_{\mathbf{m}}^{\mathbf{2}} = |\xi'|^{\mathbf{2}}\}$;
2) $A_{\neq}(\xi)$, $A_{=}(\xi)$ admit an analytical condition into radial tube domains $T(\overset{*}{C}{}_{\pm}^a)$ over the cones $\overset{*}{C}{}_{\pm}^a$ respectively which satisfy the estimates

$$\left|A_{\neq}^{\pm 1}(\xi + i\tau)\right| \leq c(1 + |\xi| + |\tau|)^{\pm æ},$$

$$\left|A_{\neq}^{\pm 1}(\xi - i\tau)\right| \leq c(1 + |\xi| + |\tau|)^{\pm(\alpha - æ)}, \quad \tau \in \overset{*}{C}{}_{+}^a.$$

The number $æ \in \mathbf{R}$ is called index of wave factorization. Existence of wave factorization permits to obtain the solution of multidimensional Riemann problem (9) by the special integral

$$(G_m u)(x) = \lim_{\tau \to 0+} \int\limits_{\mathbf{R^m}} \frac{u(y', y_m)dy'dy_m}{(|x' - y'|^2 - a^2(x_m - y_m + i\tau)^2)^{m/2}}. \quad (14)$$

The integral G_m is multidimensional analogue of the Cauchy type integral (more precisely, its limit case corresponding to boundary values). It looks as a convolution which kernel is Fourier image of C_+^a-indicator. But this multiplier is not integrable function, and we need to go out into complex plane to destroy the divergence (see [15]). The definition (14) is one of possible definitions for such singular integral. Of course, it is very desirable to give this definition for real variables (as principal value type of Cauchy integral like one-dimensional case), but I would like to note such definition was used in classical papers [2].

So, what we can obtain for solvability of equation (12), if we have wave factorization for the symbol $A(\xi)$?

I will enumerate main conclusions which we can obtain (see [21,22] for details) starting from existence of wave factorization for the symbol $A(\xi)$. We consider Sobolev-Slobodetski space $H^s(C^a_+)$ (there are functions from $H^s(\mathbf{R^m})$ with support in C^a_+). We study the equation (3.3) in the space $H^s(C^a_+)$, and the right hand side is fixed in the space $H^{s-\alpha}_0(C^0_+)$ [21,22].

1) The index of wave factorization determines fully the solvability cases for the equation (12). If the solution is unique ($æ - s = \delta, \ |\delta| < 1/2$), then it can be written by integral (14). For the case $æ - s = n + \delta, \ n \in \mathbf{Z}, \ n > 0, \ |\delta| < 1/2$, there are many solutions, but we have the formula for a general solution

which includes $2n$ arbitrary functions from corresponding Sobolev-Slobodetski spaces. Last, $æ - s = n + \delta, \ \ n \in \mathbf{Z}, \ \ n > 0$, $|\delta| < 1/2$, the equation (3.3) is over-determined and the solvability conditions are given.

2) There are many interesting applied problems, particularly, the diffraction problem of a spatial wave on a plane screen, and the problem of indentation of a wedge-shaped punch into elastic half-space. These problems are reduced to two-dimensional equation (12) for which the wave factorization for its symbol is constructed explicitly. Earlier these problems solved approximately, or solution's construction was very hard.

3) It is very problematic that wave factorization exists for every symbol $A(\xi)$ satisfying (13). The author proved the class of such symbols is very wide. But we can't give the constructive algorithm for wave factorization in this time although in

one-dimensional case (3) such factorization can be constructed by Cauchy type integral (2).
4) Although we have pessimistic point 3) the optimistic point 2) permits to construct wave factorization needed for two-dimensional case and the Laplacian, and taking into account point 1) to consider the classical Dirichlet and Neumann problems in a plane case. By transformations series (first Fourier, then Mellin transforms) the boundary value problems were reduced to equivalent system of linear algebraic equations. The unique solvability was verified by direct calculation for determinant needed. For a general case the unique solvability condition for such system of linear algebraic equation was called angle (conical) Shapiro-Lopatinski condition.
These results from 1)-4) show the direction to more complicated singularities, so called "thin" singularities.

These cases include such situations like plane cut in a space. The first preliminary sketches are presented in [23], and the author hopes to obtain something like 1)-4) in this case also. From my point of view any singularity corresponds to a certain distribution. It will be the distribution

$$\frac{a\Gamma(m/2)}{2\pi^{\frac{m+2}{2}}} \frac{1}{\left(|\xi'|^2 - a^2(\xi_m + i0)^2\right)^{m/2}}, \tag{15}$$

for the cone C_+^a, Γ is Euler function [21,22].
If we try to find a limit under $a \to +\infty$, then we must obtain the distribution corresponding to singularity of one-dimensional cut (as a ray) in a plane. I calculated these limits for some cases both two-dimensional and multidimensional, and obtained some interesting formulas. I give some of these results (see [23] for

details). Let us consider a two-dimensional domain of out cusp point (see Fig. 4)

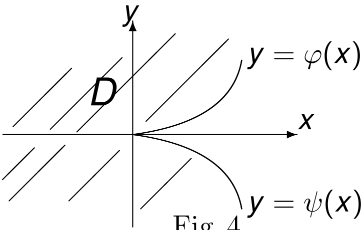

Fig. 4

with vanishing angle, so that the functions $\varphi(x), \psi(x)$ are continuously differentiable on $[0, +\infty)$ and $\varphi'(0) = \psi'(0) = 0$. Obviously such domain will be diffeomorphic to $\mathbf{R^2} \setminus [\mathbf{0}, +\infty)$. Indeed the diffeomorphism for origin's neighborhood can be defined by the formulas

$$\begin{cases} \xi = x \\ \eta = y - \varphi(x) \end{cases}$$

for the points from the first quadrant, and

$$\begin{cases} \xi = x \\ \eta = y - \psi(x) \end{cases}$$

for the points from the fourth quadrant, but the points from second and third quadrants must be in their own places. The

Jacobian for such transformation will be the following

$$\frac{D(\xi,\eta)}{D(x,y)} = \begin{vmatrix} 1 & 0 \\ -\varphi'(x) & 1 \end{vmatrix}, \qquad \frac{D(\xi,\eta)}{D(x,y)} = \begin{vmatrix} 1 & 0 \\ -\psi'(x) & 1 \end{vmatrix}$$

for the second and fourth quadrants, and equals to 1 for the second and third quadrants. The Jacobian is continuous in origin's neighborhood, and equals to 1 at the origin.
If in the origin's neighborhood we transfer to coordinates (ξ,η), then the singular integral operator with Calderon-Zygmund kernel

$$u(x) \longmapsto \int_D K(x, x-y)u(y)dy$$

is quasi-equivalent [21] to the operator

$$u(x) \longmapsto \int_{\mathbf{R}^2} K(0, \xi - \eta) u(\eta) d\eta.$$

Because for invertibility of the last operator we need nothing excluding ellipticity, if we construct for the Lebesgue integrable functions,for example $L^2(\mathbf{R}^2)$), then we conclude the question on Noether property for the operator considered is solved. According to [21] it can be shown that ellipticity condition implies the index of such operator is vanishing.

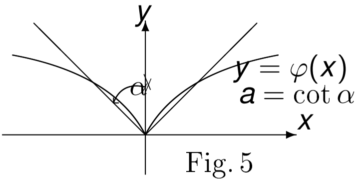

Fig. 5

If we consider the domain with singularity of inner cusp point type (see Fig.5), then obviously the previous arguments don't work, and we suggest to use some "asymptotical"ideas. Such singularity can be treated as a limit case of cone when its size tends to zero. Here we give initial conclusions and results related to this approach.

We begin from two-dimensional case. The problem is to know what is Fourier image of multiplication operator on characteristic function of positive half-axis y. Analytically this multiplier is

$$m(x, y) = \begin{cases} 1, \ x = 0, \ y > 0, \\ 0, \text{ in other cases.} \end{cases}$$

It's obviously a priori the Fourier image for such multiplier is a convolution operator for some distribution, and the distribution

must be homogeneous of order -2.
The angle of size α is the set $\{(x,y)\in \mathbf{R^2}:\ \mathbf{y}>\mathbf{a}|\mathbf{x}|\},\ \mathbf{a}=\mathbf{cot}\,\alpha$, and then we need the asymptotic $(\alpha\to 0)$, i.e. $a\to\infty$. The distribution corresponding to such multiplier is [21,22]

$$\begin{aligned}&\tfrac{1}{2}\delta\left(\xi\right)+K_a\left(\xi_1,\xi_2\right),\\ &K_a\left(\xi_1,\xi_2\right)=\tfrac{a}{2\pi^2}\tfrac{1}{\xi_1^2-a^2(\xi_2+i\,0)^2},\end{aligned}\qquad (16)$$

where $\xi=(\xi_1,\xi_2),\ \delta(\xi)$ is Dirac mass function.
We need to find

$$\lim_{a\to\infty}\frac{a}{2\,\pi^2}\frac{1}{\xi_1^2-a^2\,\xi_2^2}$$

in distribution sense. Let $\varphi\left(\xi\right)\in S\left(\mathbf{R}^2\right)$(Schwartz class of infinitely differentiable rapidly decreasing at infinity functions), and then we have [23]

Theorem

The following formula holds

$$\lim_{a\to\infty} \frac{a}{2\pi^2} \frac{1}{\xi_1^2 - a^2\xi_2^2} = \frac{i}{2\pi} \mathrm{P}\frac{1}{\xi_1} \otimes \delta(\xi_2), \qquad (17)$$

where the notation for distribution P is taken from V.S. Vladimirov's book [20], and $\otimes$ denotes the direct product of distributions.
So, the distribution (17) is that corresponds to half-infinite crack (of course with mass-supplement).
If we find another asymptotic for distribution (16) $a \to 0$, then we have

$$\lim_{a\to\infty} \frac{a}{2\pi^2} \frac{1}{\xi_1^2 - a^2\xi_2^2} = \frac{1}{2\,\pi\, i} \delta(\xi_1) \otimes \mathrm{P}\frac{1}{\xi_2}, \qquad (18)$$

and it corresponds to half-plane case (see [3]).
Now we will speak on another asymptotic related to multi-wedge angle. The simplest variant of such angle is the following: $\{x \in \mathbf{R}^3 : \ \mathbf{x_3} > \mathbf{a}\,|\mathbf{x_1}| + \mathbf{b}\,|\,\mathbf{x_2}|\}$, and we have two parameters a, b. If the parameters tend to 0 or ∞, we obtain new types of thin singularities.
The distribution corresponding to such angle is [21,22]

$$K_{a,b}(\xi_1, \xi_2, \xi_3) = \frac{4iab}{(2\pi)^3} \frac{\xi_3}{\left(\xi_1^2 - a^2\xi_3^2\right)\left(\xi_2^2 - b^2\xi_3^2\right)}.$$

We consider the different relations between a and b.

Theorem

The following formula holds

$$\lim_{b\to\infty}\frac{4\,i\,a\,b\,\xi_3}{(2\,\pi)^3\left(\xi_1^2-a^2\xi_3^2\right)\left(\xi_2^2-b^2\xi_3^2\right)}=\frac{i}{2\,\pi}\delta\left(\xi_1\right)\otimes \mathrm{P}\frac{1}{\xi_2}\otimes\delta\left(\xi_3\right).$$

Analogously one can obtain

Theorem

The equality

$$\lim_{a\to\infty}\frac{4\,i\,a\,b\,\xi_3}{(2\,\pi)^3\left(\xi_1^2-a^2\xi_3^2\right)\left(\xi_2^2-b^2\xi_3^2\right)}=\frac{i}{2\,\pi}\mathrm{P}\frac{1}{\xi_1}\otimes\delta\left(\xi_2\right)\otimes\delta\left(\xi_3\right)$$

is valid

Theorem

The equality

$$\lim_{b\to 0}\frac{4\,i\,a\,b}{(2\,\pi)^3}\frac{\xi_3}{\left(\xi_1^2-a^2\xi_3^2\right)\;\left(\xi_2^2-b^2\xi_3^2\right)}=\delta\left(\xi_2\right)\otimes K_a\left(\xi_1,\xi_3\right)$$

is valid.

(see formula (16)).

Theorem

The equality

$$\lim_{a\to 0} \frac{4\,i\,a\,b}{(2\,\pi)^3} \frac{\xi_3}{\left(\xi_1^2 - a^2\xi_3^2\right)\;\left(\xi_2^2 - b^2\xi_3^2\right)} = \delta\,(\xi_1) \otimes K_b\,(\xi_2, \xi_3)$$

holds.

Theorem

The equality

$$\lim_{\substack{a\to 0\\ b\to 0}} \frac{4\,i\,a\,b}{(2\,\pi)^3}\frac{\xi_3}{(\xi_1^2 - a^2\xi_3^2)\ \ (\xi_2^2 - b^2\xi_3^2)} == \frac{1}{2\,\pi\,i}\delta\left(\xi'\right)\otimes \mathrm{P}\frac{1}{\xi_3}, \qquad \xi' =$$

holds.

The last result corresponds to half-space case $x_3 > 0$ [3].
The author hopes such experiments will help to explain how to formulate the Noether property condition for multi-dimensional singular integral and pseudo differential equations in domains with singularities mentioned. As we see the limit operator are more simple than initial ones. It may be this point will permit

to find convenient form for these conditions.

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