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CONFORMAL TRANSFORMATIONS OF SURFACES IN MINKOWSKI SPACE WHICH PRESERVE THE GRASSMANN IMAGE

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1. Introduction

In the present paper we study two-dimensional surfaces in an n-dimensional Minkowski space M^n and their conformal G-transformations. For surfaces F^2 and $\widetilde{F}^2$ in M^n, a map $\phi : F^2 \to \widetilde{F}^2$ is called a G-transformation if the plane tangent to F^2 at $p \in F^2$ is parallel to the plane tangent to $\widetilde{F}^2$ at $\phi(p) \in \widetilde{F}^2$. In other words, a G-transformation is a transformation which pointwise preserves the Grassmann image of surface. A G-transformation $\phi : F^2 \to \widetilde{F}^2$ is said to be conformal if the first fundamental form of F^2 at $p \in F^2$ is proportional to the first fundamental form of $\widetilde{F}^2$ at $\phi(p) \in \widetilde{F}^2$. The simplest examples of conformal G-transformations are translations and homothetic transformations of the ambient space M^n.

The problem to find surfaces in the Minkowski space which admit continuous nontrivial conformal G-deformations is motivated by the fact that in the Euclidean space only minimal surfaces admit such deformations (see, e.g., [1]). In some sense this result remains true in the Minkowski space. Namely, we prove that *if a regular surface F^2 in M^n is space-like, or time-like, then F^2 admits continuous nontrivial G-deformations if and only if the mean curvature of F^2 vanishes.*

As for isotropic (light-like) surfaces in M^n, this statement needs the refinement. The metric induced on an isotropic surface F^2 is degenerate, hence we cannot define objects of exterior geometry of F^2, e.g., the mean curvature, in the standard way. The conjecture that *any* isotropic surface F^2 in M^n admits continuous nontrivial conformal G-deformations fails for $n \geq 5$, too. Namely, we prove the following classification statement.

Theorem. *Let F^2 be an isotropic surface in M^n with radius vector $x = \rho(u^1, u^2)$ such that the metric induced on F^2 is $ds^2 = g_{22}(du^2)^2$, i.e., the coordinate curves $u^2 = \mathrm{const}$ on F^2 are isotropic curves of M^n. The surface F^2 admits continuous nontrivial conformal G-deformations if and only if either*

1) *the vectors $\partial_{u^1}\rho$, $\partial_{u^2}\rho$, $\partial_{u^1u^1}\rho$ are linearly dependent, or*
2) *the vectors $\partial_{u^1}\rho$, $\partial_{u^2}\rho$, $\partial_{u^1u^1}\rho$ are linearly independent and $\partial_{u^1u^2}\rho$ depends on them.*

In the first case, F^2 is a null-ruled surface, this means that the isotropic curves $u^2 = \mathrm{const}$ on F^2 are isotropic geodesics of the ambient Minkowski space. This simple and quite natural class of isotropic surfaces has been well studied (see [2]–[4] and the references given there).

In the second case we get isotropic surfaces in M^n which are not null-ruled. In the present paper we call them strongly isotropic. These surfaces also have been considered, e.g., in K.V. Ilyenko's papers [4], [5]. By dimension arguments, one can easily prove that in M^3 any isotropic surface is null-ruled; in M^4 any isotropic surface is either null-ruled, or strongly isotropic, and the second class is more general; in M^n, $n \geq 5$, the null-ruled surfaces and the strongly isotropic surfaces form