Russian Mathematics (Iz. VUZ) *Izvestiya VUZ. Matematika*
Vol. 49, No. 4, pp.27–34, 2005 *UDC 517.546*

REMARKS ON THE BOUNDARY BEHAVIOR OF BLOCH FUNCTIONS

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1. Introduction and main results

The class of Bloch functions $\mathcal{B}$ consists of all analytic in the unit disk $\Delta = \{z : |z| < 1\}$ functions f such that

$$\|f\|_{\mathcal{B}} = \sup_{z\in\Delta}(1 - |z|^2)|f'(z)| < \infty.$$

It is a linear space. The value $|f(0)| + \|f\|_{\mathcal{B}}$ defines a norm in $\mathcal{B}$.

The space $\mathcal{B}_0$ consists of all functions $f \in \mathcal{B}$ such that $(1 - |z|^2)|f'(z)| \to 0$ for $|z| \to 1$, and the subset $\mathcal{B}_1 \subset \mathcal{B}$ does of all functions $f \in \mathcal{B}$ which satisfy the following restriction: if $\{z_n\}$ is a sequence in Δ such that $|f(z_n)| \to \infty$, then $(1 - |z_n|^2)|f'(z_n)| \to 0$. Thus, $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}$.

Let S be the well known class of analytic and univalent in Δ functions f normalized by the condition $f(0) = f'(0) - 1 = 0$. Denote by $\operatorname{Aut}\Delta$ the set of all conformal automorphisms

$$\varphi(z) = e^{i\theta}\frac{a + z}{1 + \overline{a}z}, \quad a \in \Delta, \quad \theta \in \mathbb{R},$$

of the disk Δ. A *linear-invariant family* is defined in [1] as the set $\mathfrak{M}$ of all locally univalent in Δ functions $h(z) = z + \cdots$ satisfying the condition: *for any functions* $h \in \mathfrak{M}$ *and* $\varphi \in \operatorname{Aut}\Delta$ *the function*

$$F_\varphi(z) = \frac{h(\varphi(z)) - h(\varphi(0))}{h'(\varphi(0))\varphi'(0)} = z + \cdots$$

also belongs to $\mathfrak{M}$.

The number

$$\operatorname{ord} h = \sup\{|F''_\varphi(0)|/2 : \varphi \in \operatorname{Aut}\Delta\}$$

is called the *order of function* h. Many well known classes of conformal mappings are linear-invariant families [1].

The totality of all locally univalent in Δ functions $h(z) = z + \cdots$ with $\operatorname{ord} h \le \alpha$ is called the universal linear-invariant family of functions and is denoted by $\mathcal{U}_\alpha$. It is proved in [1] that $\mathcal{U}_\alpha = \emptyset$ for $\alpha < 1$, $\mathcal{U}_1$ is the well known class of convex functions. Note also that the linear-invariant family $S \subset \mathcal{U}_2$.

In [2], the liner-invariant families $\mathcal{U}'_\alpha \subset \mathcal{U}_\alpha$ are introduced which are important for the solution of extremal problems in $\mathcal{U}_\alpha$ (see, e. g., [3]).

A function f *belongs to* $\mathcal{U}'_\alpha$ *if and only if*

$$f'(z) = \exp\left[-2\int_0^{2\pi}\log(1 - ze^{it})\,d\mu(t)\right], \quad z \in \Delta, \quad \log 1 = 0,$$