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Double $SO(2,1)$-Integrals and Formulas for Whittaker Functions

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Abstract—With the help of some double integral bilinear functionals with homogeneous kernels defined on a pair of representation spaces of the group $SO(2,1)$ we obtain some functional relations for Whittaker functions and calculate the sum of one series of Gauss hypergeometric functions converging to a Whittaker function.

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1. INTRODUCTION

Let T_1 and T_2 be presentations of a group G in complex spaces L_1 and L_2, respectively. Invariant bilinear functionals (i. b. f.) $f : L_1 \times L_2 \longrightarrow \mathbb{C}$ satisfying the conditions

$$f(\alpha u + \beta v, \widetilde{\alpha}\widetilde{u} + \widetilde{\beta}\widetilde{v}) = \alpha\widetilde{\alpha} f(u, \widetilde{u}) + \alpha\widetilde{\beta} f(u, \widetilde{v}) + \beta\widetilde{\alpha} f(v, \widetilde{u}) + \beta\widetilde{\beta} f(v, \widetilde{v}),$$
$$f([T_1(g)](u), [T_2(g)](v)) = f(u, v)$$

play an important role in studying presentations T_1 and T_2. For example, in [1] (Chap. 3, § 4) one demonstrates the application of such functionals to the study of properties of presentations of the group $SL(2,\mathbb{C})$. On the other hand, such functionals can be used for obtaining formulas which connect special functions. Thus, in [2, 3] with the help of such a functional one obtains formulas for the Legendre functions. In this paper we study the connection between some i. b. f. with respect to the group $SO(2,1)$ and the Whittaker functions $M_{\lambda,\mu}$ and $W_{\lambda,\mu}$.

In what follows, q is the bilinear form $x_0\widetilde{x}_0 - x_1\widetilde{x}_1 - x_2\widetilde{x}_2$, σ is a complex number, and D_σ is the space of infinitely differentiable functions f on the cone $C : q(x,x) = 0$ which correspond to the σ-homogeneity condition $f(\alpha x) = \alpha^\sigma f(x)$. The presentation of the group $SO(2,1)$ in the space D_σ is defined by the formula $[T_\sigma(g)](u(x)) = u(g^{-1}x)$.

2. MAIN THEOREMS

Theorem A. *With $-1 < \operatorname{Re}\sigma < -\frac{1}{2}$ we have*

$$\Gamma(2\sigma+2)\Bigg[C_{2,+}\cdot 2^{-\sigma}\pi^{-\frac{1}{2}}\operatorname{ch}(\pi\rho)\Gamma\left(-\sigma-\frac{1}{2}\right)\Gamma^{-1}\left(\sigma+\frac{3}{2}\right)$$
$$\times \mathrm{B}(-\sigma-\mathbf{i}\,\rho, -\sigma+\mathbf{i}\,\rho)M_{-\mathbf{i}\,\rho,-\sigma-\frac{1}{2}}(2\,\mathbf{i}\,\lambda) + C_{2,-}\cdot 2^{\sigma+1}(\mathbf{i}\,\lambda)^\sigma\Gamma(2\sigma+2)$$
$$\times\left[\Gamma(-\sigma-\mathbf{i}\,\rho)W_{-\mathbf{i}\,\rho,\sigma+\frac{1}{2}}(2\,\mathbf{i}\,\lambda) + (-1)^\sigma\Gamma(-\sigma+\mathbf{i}\,\rho)W_{\mathbf{i}\,\rho,\sigma+\frac{1}{2}}(-2\,\mathbf{i}\,\lambda)\right]\Bigg]$$
$$= C_3\cdot 2^{-\sigma-1}\pi^{\frac{1}{2}}\,\mathbf{i}^{-\sigma-1}\,\lambda^\sigma\Gamma\left(-\sigma-\frac{1}{2}\right)\Gamma^{-1}(\sigma+1)M_{-\mathbf{i}\,\rho,\sigma+\frac{1}{2}}(2\,\mathbf{i}\,\lambda).$$

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