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WELL-POSEDNESS OF BOUNDARY VALUE PROBLEM ON STRAIGHT LINE FOR THREE ANALYTIC FUNCTIONS

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Introduction. Let Π_+ and Π_- be the upper and lower halfplanes, respectively,

$$\Pi_\pm = \{z = x + iy : x \in R,\ \pm y > 0\}.$$

The boundary value problem for three analytic functions on the real straight line R will be posed as follows: Find three vanishing on infinity functions $F^\pm(z)$, $F_0^-(z)$, holomorphic, respectively, in the halfplanes $\Pi_\pm$ and Π_-, whose limit values on the real axis R satisfy the boundary value condition

$$F^+(x) = G(x)F^-(x) + G_0(x)F_0^-(x) + g(x), \quad x \in R, \tag{1}$$

where $G(x)$, $G_0(x)$, and $g(x)$ are given functions.

It is assumed that

$$G \in L_\infty(R), \quad G_0 \in C(R), \quad g \in L_2(R). \tag{2}$$

In addition, the coefficients are assumed to satisfy the conditions

$$G_0(x) \neq 0, \quad x \in R, \quad \lim_{x\to\pm\infty} G_0(x) = 1, \tag{3}$$

$$G(x) = O(e^{-bx}) \ \text{ as } \ x \to \infty, \quad b > 0. \tag{4}$$

A solution of the boundary value problem (1)–(4) will be sought in the Hardy classes H^2 (see [1], p. 79)

$$F^\pm \in H^2(\Pi_\pm), \quad F_0^-(z), e^{idz}F_0^-(z) \in H^2(\Pi_-), \quad d > 0. \tag{5}$$

Being posed in that way, the problem seems to be new in the theory of boundary value problems (see [2]–[7]). In what follows, without loss of generality, we assume that the support of the function $F_0^-(x)$ in not empty.

The objective of this article is to obtain in an explicit form the necessary and sufficient conditions of the solvability (and the uniqueness) of the solution of problem (1)–(5), determine this solution in an explicit form.

Under the condition $d = G = 0$ the boundary value problem (1)–(5) passes into the well-studied Riemann problem on straight line (see [2]–[5]). Under the condition $G_0 = 0$, the boundary value problem (1)–(5) turns into the Riemann problem with the minus infinite index (see [2], [7]). In [8], in the form of explicit formulas of the Carleman type, the necessary and sufficient conditions of solvability and the unique solution of this Riemann problem with minus infinite index were obtained. In that paper the fulfillment of the condition $G(x) = O(e^{-bx})$ as $x \to \infty$, $b > 0$ upon the coefficient of the problem made it possible to avoid the infinite number of conditions for the solvability of the Riemann problem with minus infinite index (see [7]).

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From conditions (2)–(3) imposed upon the coefficient G_0 in accordance with [2]–[3] we obtain the following

Proposition. *Let $\mu = \operatorname{Ind} G_0(x)$, $x \in R$. If $\mu > 0$ ($\mu < 0$), then the unique correct factorization*

$$G_0(x) = X^-(x)X^+(x), \quad x \in R, \tag{6}$$

will always exist in which the factor $X^+(z)$ ($X^-(z)$) will have inside the halfplane $\operatorname{Im} z > 0$ ($\operatorname{Im} z < 0$) the unique zero of the multiplicity $|\mu|$ at the point $z = i$ ($z = -i$) and will not possess other zeros.

If $\mu = 0$, then the factors $X^\pm(z)$ have no zeros in the halfplanes $\pm \operatorname{Im} z > 0$, respectively.

For $p \in [-d, 0]$, we set

$$\begin{aligned} w_0(p) &:= -\mathcal{F}^{-1}\Big\{\frac{g(t)}{X^+(t)}\Big\}(p) \quad \text{for} \quad \mu \le 0, \\ w_0(p) &:= -\mathcal{F}^{-1}\Big\{\frac{g(t)}{X^+(t)}\Big\}(p) - e^p \sum_{k=1}^{\mu} \frac{a_{\mu-k}}{(k-1)!i^k} p^{k-1} \quad \text{for} \quad \mu > 0, \end{aligned} \tag{7}$$

where $\mathcal{F}^{-1}$ in the inverse Fourier transform, a_k, $k = 0, \dots, \mu - 1$, are constants.

By the Cauchy problem for the function

$$w \in H^2(\Pi_0^b), \quad \text{where} \quad \Pi_0^b = \{p = x + iy : x \in R, \ 0 < y < b\}, \tag{8}$$

we will mean the problem of restoration of the function w in a closed strip Π_0^b by the condition

$$w(p) = w_0(p), \quad p \in [-d, 0], \quad w_0 \in L_2(-d, 0). \tag{9}$$

The Cauchy problem (8), (9) was completely investigated. In [9], in the form of explicit formulas of the Carleman type, the necessary and sufficient conditions of solvability and the unique solution of problem (8), (9) were given.

In this article we will also consider intrinsic applications of the boundary value problem (1)–(5) to the theory of integral equations in convolution and singular integral equations (applications of the Riemann problem to integral equations can be found in [2]–[4]). An interesting peculiarity of equations considered in this article is the fact, that from one integral equation one can correctly determine two (independent) functions.

We will study the following equations.

a) Equation in convolution with respect to the pair of functions u, u_0

$$u(x)(1 - \chi(x)) - \int_0^\infty k_1(x-t)u(t)\,dt - \int_{-\infty}^{-d} k_2(x-t)u(t)\,dt - \int_{-\infty}^{0} k_0(x-t)u_0(t)\,dt = f(x), \tag{10}$$

where $x \in R$, $\chi(x)$ is the characteristic function of the interval $(-d, 0)$,

$$\begin{gathered} k_j \in L_1(R), \quad j = 0, 1, 2, \quad f \in L_2(R) \ \text{ are the known functions}, \\ u \in L_2(R \setminus [-d, 0]), \quad u_0 \in L_2(-\infty, -d), \quad d > 0, \quad \text{are the desired functions}; \end{gathered} \tag{11}$$

in addition, it is assumed that

$$\mathcal{F}k_0(t) = O(e^{-bt}) \quad \text{as} \quad t \to \infty, \quad b > 0, \quad 1 - \mathcal{F}k_1(t) \ne 0, \quad t \in R. \tag{12}$$

b) Singular integral equation with respect to the pair of functions ψ, ψ_0

$$(1 + A(t))\psi(t) + B(t)(S\psi)(t) + B_0(t)(S^-\psi_0)(t) = f(t), \quad t \in R, \tag{13}$$

where

$$(S\psi)(x) = \frac{1}{\pi i}\mathcal{P}\int_{-\infty}^{\infty} \frac{\psi(t)}{t - x}\,dt, \quad (S^\pm\psi)(x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\psi(t)}{t - (x \pm i0)}\,dt,$$

$A, B \in C(R)$, $B_0 \in L_\infty(R)$, $f \in L_2(R)$ are coefficients of the equation,

$$\begin{gathered}\psi,\ \psi_0 \in L_2(R) \quad \text{are the desired functions,}\\ \text{the function } \psi \text{ is such that} \quad \mathcal{F}^{-1}\psi(x) = 0, \quad x \in (-d, 0), \quad d > 0.\end{gathered} \tag{14}$$

In addition, it is assumed that

$$1 + A(t) + B(t) \neq 0, \quad t \in R, \quad B_0(t) = O(e^{-bt}) \quad \text{as} \quad t \to \infty, \quad b > 0. \tag{15}$$

1. *Basic results.*

Theorem. *For the solvability of the boundary value problem* (1)–(5) *it is necessary and sufficient that the following two conditions be valid.*

(i) *For $\mu \leq 0$, a solution $w \in H^2(\Pi_0^b)$ of the Cauchy problem* (7)–(9) *exists and the relation is fulfilled*

$$\frac{X^+W}{G} \in H^2(\Pi_-), \quad \textit{where} \quad W = \mathcal{F}w. \tag{16}$$

For $\mu > 0$, constants a_k, $k = 0, \dots, \mu - 1$, exist such that the Cauchy problem (7)–(9) *has a solution $w \in H^2(\Pi_0^b)$ and relation* (16) *is fulfilled.*

(ii) *A solution of the Riemann boundary value problem*

$$F^+(t) = G_0(t)F_0^-(t) + g(t) + W(t)X^+(t), \quad t \in R, \tag{17}$$

exists in the Hardy classes

$$F^+ \in H^2(\Pi_+), \quad F_0^-(z), e^{idz}F_0^-(z) \in H^2(\Pi_-), \quad d > 0. \tag{18}$$

Under the fulfillment of conditions (i), (ii) *the solution of problem* (1)–(5) *is unique and can be found in the explicit form*

$$F^-(t) = \frac{X^+(t)W(t)}{G(t)}, \quad t \in R, \tag{19}$$

the functions F^+, F_0^- are solution of the Riemann boundary value problem (17), (18).

Proof. Necessity. Suppose that the solution of problem (1)–(5) exists. We have $F^\pm, F_0^- \in L_2(R)$ by virtue of the known property of the Hardy classes (see [1]).

We put

$$W(t) := \frac{F^+(t)}{X^+(t)} - X^-(t)F_0^-(t) - \frac{g(t)}{X^+(t)}, \quad t \in R. \tag{20}$$

Then from (1) with regard for (6), (20) we obtain

$$W(t) = \frac{F^-(t)G(t)}{X^+(t)}. \tag{21}$$

From (20) and (2), (3) it follows that $W \in L_2(R)$. Then from (4) we have

$$e^{-ipt}W(t) \in L_1(R) \cap L_2(R) \quad \text{for} \quad 0 < \operatorname{Im} p < b. \tag{22}$$

Consequently,

$$w(p) := \mathcal{F}^{-1}W(p) \in H^2(\Pi_0^b). \tag{23}$$

Applying to equality in (20) the inverse Fourier transform, with regard for (23) we obtain

$$w(p) = \mathcal{F}^{-1}\Big\{\frac{F^+(t)}{X^+(t)}\Big\}(p) - \mathcal{F}^{-1}\{X^-(t)F_0^-(t)\}(p) - \mathcal{F}^{-1}\Big\{\frac{g(t)}{X^+(t)}\Big\}(p), \quad p \in R. \tag{24}$$

Let $\mu \le 0$. Then from Proposition (see (6)) we have

$$\frac{F^+(z)}{X^+(z)} \in H^2(\Pi_+), \quad e^{idz}X^-(z)F_0^-(z) \in H^2(\Pi_-). \tag{25}$$

From (25) in accordance with [1], [2] we obtain

$$\mathcal{F}^{-1}\Big\{\frac{F^+(t)}{X^+(t)}\Big\}(p) = 0, \quad p < 0, \quad \mathcal{F}^{-1}\{X^-(t)F_0^-(t)\}(p) = 0, \quad p > -d. \tag{26}$$

From (24) and (26) we have

$$w(p) = -\mathcal{F}^{-1}\Big\{\frac{g(t)}{X^+(t)}\Big\}(p), \quad p \in [-d, 0]. \tag{27}$$

Relations (23), (27) represent the Cauchy problem (7)–(9). From (21) and (5) relation (16) follows. Thus, condition (i) is fulfilled. By multiplying equality (20) by $X^+(t)$, we obtain the Riemann problem (17), (18), whose solution for $\mu \le 0$ will be unique. We have proved the necessity for $\mu \le 0$.

Sufficiency. Let $\mu \le 0$ and assume that the existence conditions (i), (ii) are fulfilled. Let us show that the functions $F^\pm$, F_0^-, defined by formulas (17)–(19), supply a solution to the initial problem (1)–(5).

From (19) equality (21) immediately follows. Substituting it into the boundary condition (17), we obtain (1). From (18), (19), and (16) the validity of relations in (5) follows. The theorem has been proved for $\mu \le 0$.

The proof for the case $\mu > 0$ is similar to that in case $\mu \le 0$ with the unique difference: In argumentation one should take into account the presence of zero (of multiplicity μ) for the factor $X^+(z)$ in the halfplane of regularity $\operatorname{Im} z > 0$. Let us first prove the uniqueness of the solution of problem (1)–(5). Let $F^\pm, F_0^- \in L_2(R)$ be a solution of the initial problem (1)–(5). If the Riemann boundary value problem (17), (18) has two solutions, then the corresponding homogeneous problem ($W = g = 0$) will possess a nontrivial solution of the form

$$F^+(z) = X^+(z)z^{l-1}(z-i)^{-\mu}, \quad F^-(z) = \frac{1}{X^-(z)}z^{l-1}(z-i)^{-\mu}, \quad l \in \{1, \dots, \mu\},$$

which contradicts the condition $e^{izb}F^-(z) \in H^2(\Pi_-)$. Consequently, the homogeneous problem (17), (18) has only the trivial solution, and the solution of problem (1)–(5) is unique.

The formulas and relations (20)–(24) are valid also for $\mu > 0$. The first relation in (25) is not fulfilled (in view of presence of zero for the function $X^+(z)$ at the point $z = i$). Let us obtain an analog of this relation in this case.

We put

$$X_1^+(t) := (t-i)^{-\mu}X^+(t), \quad a_k := \frac{1}{k!}\left(\frac{\partial^k}{\partial t^k}\frac{F^+(t)}{X_1^+(t)}\right)_{t=i}, \quad k = 0, \dots, \mu - 1. \tag{28}$$

From (28) the required relation follows

$$\frac{F^+(z)}{X^+(z)} - \sum_{k=1}^{\mu}(z-i)^{-k}a_{\mu-k} \in H^2(\Pi_+). \tag{25$_1$}$$

From (25_1) an analog of equality (27) follows

$$w(p) = -\mathcal{F}^{-1}\Big\{\frac{g(t)}{X^+(t)}\Big\}(p) - e^p\sum_{k=1}^{\mu}\frac{a_{\mu-k}}{(k-1)!i^k}p^{k-1}, \quad p \in [-d, 0]. \tag{27$_1$}$$

Relations (23), (27_1) is the Cauchy problem (7)–(9). Relation (16) is fulfilled due to the same reasons that in the case $\mu \le 0$. Thus, condition (i) is valid.

The solution of the Cauchy problem (23), (27_1) is known (see [9]) and is representable in the form

$$w(p) = v_0(p) + \sum_{k=1}^{\mu} a_{\mu-k} v_k(p), \quad p \in R,$$

where the functions v_j, $j = 0, \dots, \mu$, are calculated by the respective Carleman formulas. Then

$$W(t) = \mathcal{F}v_0(t) + \sum_{k=1}^{\mu} a_{\mu-k} \mathcal{F}v_k(t), \quad t \in R. \tag{29}$$

Multiplying equality (20) by X^+, we obtain that the functions $F^\pm$, F_0^-, subordinated to condition (5), satisfy the boundary value problem (17), (18). Consequently, condition (ii) is also fulfilled.

The solution of the boundary value problem (17), (18) has the form (see [2]–[4])

$$F^+(t) = X^+(t)(S^+g_1)(t) + X^+(t)P_\mu(t)(t-i)^{-\mu}, \tag{30}$$

$$F^-(t) = \frac{1}{X^-(t)}(S^-g_1)(t) + P_\mu(t)(t-i)^{-\mu}, \tag{31}$$

where P_μ is a certain polynomial of degree $\mu - 1$, $g_1 = g/X^+ + W$.

Substituting the expression for F^+ from (30) into (28), with regard for (29) we obtain the relations for the constants a_k, $k = 0, \dots, \mu - 1$, and the coefficients of the polynomial P_μ. We have

$$a_k = \frac{\partial^k}{k!\partial t^k}\big((t-i)^\mu((S^+g_1)(t) + (S+W)(t)) + P_\mu(t)\big)_{t=i}, \quad k = 0, \dots, \mu-1,$$

or

$$a_k = \frac{\partial^k}{k!\partial t^k}\big(P_\mu(t)\big)_{t=i}, \quad k = 0, \dots, \mu-1. \tag{32}$$

From (32) the coefficients of the polynomial P_μ can be determined in the unique way.

The sufficiency of conditions (i), (ii) for the existence of the solution of problem (1)–(5) for the case where $\mu > 0$ is obvious. □

2. *Integral equations.* Having applied to equation (10) the Fourier transform and dividing the resulting equation by $1 - \mathcal{F}k_1(p)$, we have

$$\mathcal{F}u_+(p) - \mathcal{F}u_-(p)G_0(p) - \mathcal{F}u_0(p)G(p) = g(p), \quad p \in R, \tag{33}$$

where

$$u_\pm(t) = \pm u(t)\theta(\pm t), \quad \theta \text{ is the Heaviside function}, \quad u_-(t) = 0, \quad t \in (-d, 0),$$

$$G(p) = \frac{\mathcal{F}k_0(p)}{1 - \mathcal{F}k_1(p)}, \quad G_0(p) = \frac{1 - \mathcal{F}k_2(p)}{1 - \mathcal{F}k_1(p)}, \quad g(p) = \frac{\mathcal{F}f(p)}{1 - \mathcal{F}k_1(p)}.$$

By putting in equation (33) $F^+ := \mathcal{F}u_+$, $F_0^- := \mathcal{F}u_-$, $F^- := \mathcal{F}u_0$, we obtain the boundary value problem (1)–(5). One can easily show that from (1)–(5) equation (10) follows with conditions (11)–(12). Thus, equation (10) with conditions (11)–(12) is equivalent to problem (1)–(5).

Let us consider the singular equation (13). By the Sokhotskiĭ formulas (see [2]–[4]) we have

$$\psi(t) = (S^+\psi)(t) - (S^-\psi)(t), \quad (S\psi)(t) = (S^+\psi)(t) + (S^-\psi)(t). \tag{34}$$

From (13) and (34) we obtain

$$(S^+\psi)(t)(1 + A(t) + B(t)) - (S^-\psi)(t)(1 + A(t) - B(t)) + B_0(t)(S^-\psi_0)(t) = f(t). \tag{35}$$

In equation (35), having put

$$F^+(t) := (S^+\psi)(t), \quad F^-(t) := (S^-\psi_0)(t), \quad F_0^-(t) := (S^-\psi)(t), \quad t \in R,$$
$$G := -\frac{B_0}{1+A+B}, \quad G_0 = \frac{1+A-B}{1+A+B}, \quad g := \frac{f}{1+A+B},$$

we obtain the boundary value problem (1)–(5). Thus, equation (13) with conditions (14), (15) is equivalent to problem (1)–(5).

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