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EXPLICIT FORMULAS FOR CHROMATIC POLYNOMIALS OF SOME SERIES-PARALLEL GRAPHS

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Abstract

The main goal of our paper is to present explicit formulas for chromatic polynomials of some planar series-parallel graphs (sp-graphs). The necklace-graph considered in this paper is the simplest non-trivial sp-graph. We have provided the explicit formula for calculating the chromatic polynomial of common sp-graphs. In addition, we have presented the explicit formulas for calculating chromatic polynomials of the ring of the necklace graph and the necklace of the necklace graph. Chromatic polynomials of the necklace graph and the ring of the necklace graph have been initially obtained by transition to the dual graph and the subsequent using of the flow polynomial. We have also used the technique of finite Fourier transformations. The use of the partition function of the Potts model is a more general way to evaluate chromatic polynomials. In this method, we have used the parallel- and series-reduction identities that were introduced by A. Sokal. We have developed this idea and introduced the transformation of the necklace-graph reduction. Using this transformation makes it easier to calculate chromatic polynomials for the necklace-graph, the ring of the necklace graph, as well as allows to calculate the chromatic polynomial of the necklace of the necklace graph.

Keywords: chromatical polynomial, partition function of Potts model, Tutte polynomial, Fourier transform, series-parallel graph, necklace graph

Introduction

As is well known, the notion of the chromatic polynomial for a graph was introduced by George Birkhoff in 1912 in connection with the 4 color problem. Namely, the chromatic polynomial $P_G(q)$ for the given graph G specifies the number of distinct colorings of vertices of G with q colors (colorings are said to be distinct if they differ only in the permutation of colors). For example, for any triangular graph, $P_{K_3}(q) = q(q-1)(q-2)$. Formulas for chromatic polynomials of complete graph K_n, cycle graph C_n, wheel graph W_n, and for some other classes of graphs are well-known. In this paper, we omit basic formulas for chromatic polynomials and their properties (see, e.g., [1–4]). Let us only mention one remarkable technique for calculating chromatic polynomials of graphs with a complex crystal structure, which was proposed by N. Biggs. It is based on the transfer matrix method that was initially used in statistical physics.

The goal of our paper is to present explicit formulas of chromatic polynomials of some planar series-parallel graphs (sp-graphs). The graph is called an sp-graph if one can reduce it to K_2 by several consecutive replacements, i.e., by replacing a pair of edges that are incident to a vertex of degree 2 with a single edge and/or a pair of parallel edges with a single edge that connects their common endpoints. The SP-graph is a necklace graph, if one can reduce it to K_2 by several operations of the first type followed by operations of only the second type (see Fig. 1). We present explicit formulas

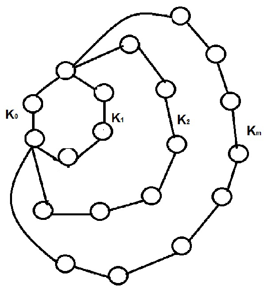

Fig. 1. An m-level necklace graph. For this graph, $m = 3$, $k_0 = 2$, $k_1 = 4$, $k_2 = 7$, and $k_3 = 9$

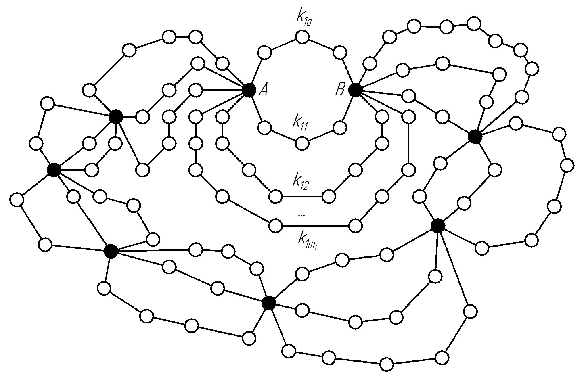

Fig. 2. A ring of t "necklaces". For this graph, $t = 8$ and $m_i = 2$ for $i = 2, \dots, 8$. Sources and a sinks are bolded

for necklace graphs, for the ring of necklace graphs, and for the necklace of necklace graphs. The main method uses the Potts model partition function for calculating chromatic polynomials. Note that the partition function in the Potts model coincides with the Feynman amplitude for the vacuum case with an arbitrary propagator over a finite field [5]. The chromatic polynomial is a special case of the Potts model partition function, where all weights of edges are equal to -1. We reduce the complicated graphs to simpler ones by using Sokal's parallel- and series-reduction identities of edges (see [6]) for the multidimensional case. However, the weights in the resulting simple graphs differ from -1.

Therefore, we perform these calculations for arbitrary weights.

One can also calculate chromatic polynomials for necklace graphs and for the rings of necklace graphs without the use of partition functions. The approach described in section 1 is based on the transition to dual graphs and the use of the finite Fourier transform. In section 2, we describe a method which is based on the use of Potts partition functions. In Conclusions, we compare the applicability of these two methods for calculating chromatic polynomials of more general graphs.

For the necklace graph, we number its "levels" from 0 to m and denote the number of edges at each level by k_i, $i = 0, \dots, m$ (see Fig. 1).

We understand the ring of the necklace graph as the following graph (see Fig. 2). Each necklace graph has key vertices A and B. In sp-graphs, these key vertices are usually called a source and a sink.

We consider the graph, in which all necklaces (t pieces) are attached to each other in a circle by means of these vertices. We denote by m_i, $i = 1, \dots, t$, the number of nonzero levels in the i-th necklace (the total number of levels in it equals $m_i + 1$). Correspondingly, we denote by k_{ij}, $j = 0, \dots, m_i$, the number of edges on the j-th level in the i-th necklace.

1. Using "classical" methods for calculating chromatical polynomials of sp-graphs

1.1. Evaluation of chromatic polynomials for the necklace graph. The flow polynomial of a directed graph G is the number of nonzero values of variables $x_e \in \mathbb{Z}_q$ (where $e \in E(G)$ and $\mathbb{Z}_q$ is the residue group modulo q) such that for each vertex $v \in V(G)$ the sum of x_e over all edges e that "enter" v equals the sum of x_e over all edges e that "emanate" from v (see [2] for more detail). As is known, $F_G(q)$ is

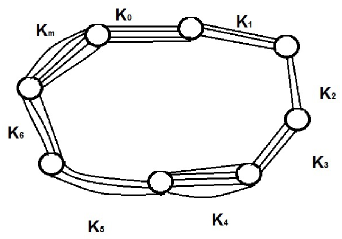

Fig. 3. The multigraph dual to the necklace graph

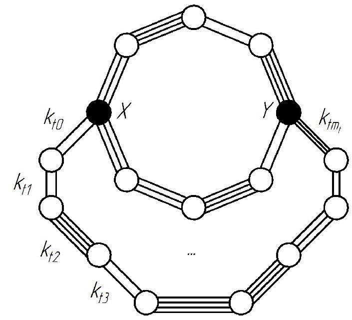

Fig. 4. The multigraph dual to the ring of necklaces graph with $t = 3$

independent of the orientation of edges of the graph G, and for any planar graph,

$$P_G(q) = q\, F_{\widetilde{G}}(q), \tag{1}$$

where $\widetilde{G}$ is the graph dual to G.

As a simple illustration of the mentioned terms and methods, let us consider the case of a necklace graph with $m = 1$, which is reducible to a cycle graph C_r with $r = k_0 + k_1$ vertices. The chromatic polynomial for this graph is well known. The dual (multi) graph for the graph C_r consists of two vertices and r edges connecting them. Thus, in order to find the flow polynomial of the graph C_r, one has to find the number of nonzero (in each variable) solutions to the equation $\sum_{j=1}^{r} x_j = 0$ in the group $\mathbb{Z}_q$.

Lemma 1. *Let variables x_j take on nonzero values in a group of order q, $q \in \mathbb{N}$, and let c be an arbitrary fixed nonzero element of this group. Then, the numbers of nonzero (in each variable) solutions to the homogeneous equation $\sum_{j=1}^{r} x_j = 0$ and to the inhomogeneous one $\sum_{j=1}^{r} x_j = c$ equal, correspondingly,*

$$((q-1)^r + (-1)^r (q-1))/q, \tag{2}$$

$$((q-1)^r - (-1)^r)/q. \tag{3}$$

One can easily prove lemma 1 by induction.

Using lemma 1 and formula (1), we prove that $P_{C_r}(q) = (q-1)^r + (-1)^r (q-1)$, which coincides with the well-known result the chromatic polynomial of the graph C_r.

In what follows we use lemma 1 for calculating the value of the chromatic polynomial of the necklace graph. We treat the set of multigraph edges that connect the same vertices as a *"thick"* edge.

Theorem 1. *The chromatic polynomial for a necklace graph obeys the formula*

$$q \left\{ \prod_{i=0}^{m} \frac{(q-1)^{k_i} + (-1)^{k_i}(q-1)}{q} + (q-1) \prod_{i=0}^{m} \frac{(q-1)^{k_i} - (-1)^{k_i}}{q} \right\}. \tag{4}$$

Proof. In order to prove this theorem, it suffices to make sure that the expression in curly brackets specifies the flow polynomial for the graph shown in Fig. 3 (it is dual to the necklace graph).

Let us direct the edges of the graph in Fig. 3 clockwise. Denote the variables that correspond to the "thick" edge k_i, $i = 0, \dots, m$ by x_{ij}, $j = 1, \dots, k_i$. Then, the flow polynomial coincides with the number of nonzero (in each variable) solutions to the following system of equations:

$$\sum_{j=1}^{k_0} x_{0j} = \sum_{j=1}^{k_1} x_{1j} = \ldots = \sum_{j=1}^{k_m} x_{mj} = y$$

for all $y \in \mathbb{Z}_q$.

Considering all possible values of y and using explicit formulas given in lemma 1, we obtain formula (4). □

1.2. Evaluation of the chromatic polynomial for the ring of the necklace graph. The number ν_r of nonzero solutions to the homogeneous equation $\sum\limits_{j=1}^{r} x_j = 0$ in the group of order q obeys formula (2), while the number ν'_r of nonzero solutions to the corresponding inhomogeneous equation does formula (3). Choose an arbitrary set of natural numbers $(k_0, \dots, k_m)$. For convenience of further considerations, let us introduce denotations f and g as follows:

$$f(k_0, \dots, k_m) = \prod_{j=0}^{m} \nu_{k_j}, \quad g(k_0, \dots, k_m) = \prod_{j=0}^{m} {\nu'}_{k_j}. \tag{5}$$

In accordance with theorem 1, the value of the chromatic polynomial for the graph shown in Fig. 1 is $q\{f(k_0, \dots, k_m) + (q-1)g(k_0, \dots, k_m)\}$.

In order to evaluate the chromatic polynomial of the ring of necklace graphs, we need to evaluate the flow polynomial for the dual multigraph. One can easily draw this graph by placing some vertex of the dual graph (let us denote it by X) inside the ring of necklaces and doing some other one (let us denote it by Y) outside the ring. The resulting graph is shown in Fig. 4. It is a single necklace with "thick" edges, its vertices X and Y are, respectively, the source and the sink of this necklace. In this case, the number of "thick" paths (paths of "thick" edges) that connect X and Y is equal to t, i.e., the number of different necklaces connected in a ring. The length of the i-th "thick" path equals m_i, i.e., the number of levels in the i-th necklace. The "thickness" of edges in this path coincides with values of parameters $k_{i0}, k_{i1}, \dots, k_{im_i}$. For fixed (for the i-th necklace) values of these parameters, we define the function $h_i(y)$, $y \in \mathbb{Z}_q$, as follows:

$$h_i(y) = \begin{cases} f(k_{i0}, k_{i1}, \dots, k_{im_i}), & y = 0, \\ g(k_{i0}, k_{i1}, \dots, k_{im_i}), & y \neq 0. \end{cases} \tag{6}$$

In other words, $h_i(y)$ is the number of nonzero solutions to the system

$$\sum_{j=1}^{k_{i0}} x_{0j} = \sum_{j=1}^{k_{i1}} x_{1j} = \cdots = \sum_{j=1}^{k_{im_i}} x_{m_i j} = y.$$

Theorem 2. *For the given ring of necklace graphs, the chromatic polynomial obeys the formula*

$$(q-1)\prod_{i=1}^{t}(f(k_{i0}, \dots, k_{im_i}) - g(k_{i0}, \dots, k_{im_i})) + \\ + \prod_{i=1}^{t}((q-1)g(k_{i0}, \dots, k_{im_i}) + f(k_{i0}, k_{i1}, \dots, k_{im_i})), \tag{7}$$

where f and g are defined by formulas (5).

Proof. Let us direct all edges of the graph in Fig. 4 from X to Y. We denote by y_E the total flow along the "thick" edge e, i.e., the sum of values of variables associated with each of multiple edges. Evidently, the value of y_e is the same for all "thick" edges. Let y_i, $i = 1, \ldots, t$, be the flow along the i-th "thick" path. Since the total flow from the vertex X to the vertex Y equals 0, it holds that $\sum_{i=1}^{t} y_i = 0$. We only need to impose constraints on the variables associated with edges of the graph shown in Fig. 4 in order to equate the total "incoming" and "outcoming" flows for each vertex. As a result, we conclude that the flow polynomial of the graph shown in Fig. 4 equals

$$\sum{}' \, h_1(y_1) \times \ldots \times h_t(y_t). \tag{8}$$

Here, the symbol $\sum'$ means that the summation is performed over all vectors $\mathbf{y} = (y_1, \ldots, y_t)$, $y_i \in \mathbb{Z}_q$, such that $\sum_{i=1}^{t} y_i = 0$. One can evaluate the convolution of functions $h_1, \ldots, h_t$ at the point 0 given by formula (8) by the standard technique based on the use of the Fourier transform.

Proposition 1. *Let* $\widehat{h}_j(z) = \sum_{y \in \mathbb{Z}_q} h_j(y) e^{2\pi i z y/q}$ *– the Fourier transform of some complex-valued function* $h_j(y)$, $y \in \mathbb{Z}_q$, $j = 1, \ldots, t$. *Then*

$$\sum{}' \, h_1(y_1) \times \ldots \times h_t(y_t) = \frac{1}{q} \sum_{z \in \mathbb{Z}_q} \prod_{j=1}^{t} \widehat{h_j}(z).$$

Proposition 1 easily follows from the basic Fourier transformation formula in the group $\mathbb{Z}_q$, i.e.,

$$\sum_{z \in \mathbb{Z}_q} e^{2\pi i z y/q} = q\, \delta(y), \tag{9}$$

where

$$\delta(y) = \begin{cases} 1, & y = 0, \\ 0, & y \neq 0. \end{cases}$$

In accordance with the proved proposition 1, for evaluating the chromatic polynomial given in theorem 2, it suffices to calculate the Fourier transform of the function h_i which is given by formula (6). Correlation (10) given below is a simple corollary of formula (9).

Proposition 2. *The Fourier transform of a certain complex-valued function* $h(y)$, $y \in \mathbb{Z}_q$, *representable as* $a + b\,\delta(z)$, $a, b \in \mathbb{C}$, *obeys the formula*

$$\widehat{h}(z) = b + aq\, \delta(z). \tag{10}$$

The end of the proof of theorem 2 is based on using of propositions 1 and 2. □

2. Evaluation of the chromatic polynomial with the help of the partition function of the Potts model

2.1. Definition of the partition function and its connection with the chro-matic polynomial. The Potts model describes a crystal magnetic structure as a graph. In this model, atoms are represented as vertices of the graph connected by edges with some "neighbor" vertices. Each atom can be in any of q different states. Atoms of the crystal interact with each other. Each edge of the graph is associated with some potential energy. According to the Boltzmann postulate adopted in the classical statistical physics, the probability that a system is in a certain state obeys a certain equality. In our case, for each edge e of the graph this probability is proportional to $(1+v_e)$ if states of atoms at vertices of the edge e are different (or proportional to unity, otherwise). The normalization factor in this equation is called the partition function.

Thus, the partition function of the Potts model (which is known to mathematicians as the multivariate Tutte polynomial [6]) can be defined on an arbitrary finite graph. Let $G = (V, E)$ be a finite undirected graph with the vertex set V and the edge set E. The partition function equals

$$Z_G(q, \mathbf{v}) = \sum_{A \subseteq E} q^{K(A)} \prod_{e \in A} v_e, \tag{11}$$

where q and $\mathbf{v} = \{v_e, e \in E\}$ are commuting variables, while $K(A)$ is the number of connected components in the subgraph (V, A). It is important to note that if $v_e = -1$ for all edges e, then the set of states gives the weight 1 for each proper coloring and the weight 0 for an improper one. Thus, $P_G(q) = Z_G(q, -1)$.

2.2. Graph transformations based on parallel-, series-, and necklace-reduction identities of edges.

Proposition 3. *Let us assume that the graph G contains edges $e_1, \ldots, e_n$, that connect the same pairs of vertices x and y (the parallel connection of x and y). Then, the value $Z_G(q, \mathbf{v})$ remains constant, even if we replace these edges with a single edge e with the following weight:*

$$v_e = \prod_{i=1}^{n} (1 + v_{e_i}) - 1. \tag{12}$$

We obtain formula (12) by converting the parallel-reduction identity for $n = 2$ (see [6, pp. 16–17]).

Let us assume that edges $e_1, \ldots, e_n$ are sequentially connected, i.e., there exist distinct vertices $x_1, x_2, \ldots, x_{n+1}$, such that they are sequentially adjacent to each other (this means that there exists an edge that connects x_i and x_{i+1}, $i = 1, \ldots, n$), while vertices $x_2 \ldots x_n$ have the degree of 2. We also consider the case when $x_1 = x_{n+1}$.

Proposition 4. *In the case of a sequential connection, the following two transformations do not affect the partition function:*
1) *Replace edges $e_1, \ldots, e_n$ with a single one $e = x_1 x_{n+1}$ with the weight*

$$v_e = \frac{q \prod_{i=1}^{n} v_i}{\prod_{i=1}^{n} (v_i + q) - \prod_{i=1}^{n} v_i}. \tag{13}$$

2) *Multiply the partition function $Z_G(q, \mathbf{v})$ for the obtained graph by the prefactor*

$$\frac{\prod_{i=1}^{n} (v_i + q) - \prod_{i=1}^{n} v_i}{q}. \tag{14}$$

We obtain formula (13) by converting the series-reduction identity for the case when $n = 2$ (see [6, p. 17]).

When using the partition function for evaluating the chromatic polynomial, we set all weights to -1, i.e., $v_e = -1$. Then, formula (13) and prefactor (14) turn into

$$v_e = \frac{q(-1)^n}{(q-1)^n - (-1)^n}, \quad \text{pref} = \frac{(q-1)^n - (-1)^n}{q}.$$

By simple arithmetic transformations we obtain the following corollary of propositions 3 and 4.

Proposition 5. *Let G contain a necklace subgraph H with arbitrary weights v_{ij}, $i = 0, \dots, m$, $j = 1, \dots, k_i$ (see Fig. 1) associated with vertices of the i-th level of this necklace. Let us assume that all vertices of H, except the source and the sink, are not incident to the rest of vertices of the graph G. Then, the replacement of the subgraph H with one edge e (connecting the source and the sink) with the weight*

$$v_e = \prod_{i=0}^{m} \left(\frac{\prod_{j=1}^{k_i}(v_{ij}+q) + (q-1)\prod_{j=1}^{k_i} v_{ij}}{\prod_{j=1}^{k_i}(v_{ij}+q) - \prod_{j=1}^{k_i} v_{ij}} \right) - 1. \tag{15}$$

and the further multiplication of the resulting partition function by the prefactor

$$\prod_{i=0}^{m} \frac{\prod_{j=1}^{k_i}(v_{ij}+q) - \prod_{j=1}^{k_i} v_{ij}}{q}.$$

do not affect the partition function Z_G.

2.3. The application of formulas established in section 2.2 for evaluation of chromatic polynomials of the necklace graph, ring of the necklaces graph, and necklace of the necklaces graph. As was mentioned above, the partition function coincides with the chromatic polynomial for $v_{ij} = -1$. Therefore, we can apply proposition 5 for evaluating the chromatic polynomial of the necklace graph. To this end, it suffices to evaluate the partition function for the graph consisting of one edge that connects distinct vertices. We denote the weight of this edge by w. Using formula (11), we obtain the partition function for this graph, i.e.,

$$Z(G) = q^2 \times 1 + q \times w. \tag{16}$$

Replacing w in the latter formula with v_e (see formula (15)) and multiplying the result by the corresponding prefactor for $v_{ij} = -1$, we evaluate the chromatic polynomial for the necklace graph

$$Z(G) = \prod_{i=0}^{m} \frac{(q-1)^{k_i} - (-1)^{k_i}}{q} \left(q^2 \times 1 + q \times \left(\prod_{i=0}^{m} \frac{(q-1)^{k_i} + (-1)^{k_i}(q-1)}{(q-1)^{k_i} - (-1)^{k_i}} - 1 \right) \right) =$$
$$= q \left[(q-1) \prod_{i=0}^{m} \frac{(q-1)^{k_i} - (-1)^{k_i}}{q} + \prod_{i=0}^{m} \frac{(q-1)^{k_i} + (-1)^{k_i}(q-1)}{q} \right].$$

This formula coincides with formula (4) in theorem 1, which was derived in another way (see section 1.1).

Let us now apply formulas established in section 2.2 for evaluating the chromatic polynomial of the ring of the necklace graphs. We use denotations introduced at the end of Introduction.

Converting each necklace into an edge by using proposition 5, we get a graph C_t with weights v_i, $i = 1, \ldots, t$, which are equal to

$$v_i = \prod_{j=0}^{m_i} \frac{(q-1)^{k_{ij}} + (-1)^{k_{ij}}(q-1)}{(q-1)^{k_{ij}} - (-1)^{k_{ij}}} - 1.$$

We also need to take into account the prefactor

$$\prod_{j=0}^{m_i} \frac{(q-1)^{k_{ij}} - (-1)^{k_{ij}}}{q}$$

for each necklace. Now, by applying formula (13), we get a "loop" graph with the weight

$$w = \frac{q \prod_{i=1}^{t} v_i}{\prod_{i=1}^{t} (v_i + q) - \prod_{i=1}^{t} v_i}.$$

We also have to take into account the prefactor that appears under the contraction of edges of the ring, i.e.,

$$\text{pref} = \frac{\prod_{i=1}^{t} (v_i + q) - \prod_{i=1}^{t} v_i}{q}.$$

As a particular case of equality (11), we get the following partition function formula for the "loop" graph with the weight w:

$$Z(G) = q + q \times w = q(1 + w).$$

With the help of algebraic transformations, we obtain a formula for the chromatic polynomial of the ring of necklace graphs, which coincides with formula (7) in theorem 2.

Finally, let us consider a necklace sp-graph, in which each edge is replaced with an arbitrary necklace graph. We treat the outer necklace as a large one and do the inner necklaces as small.

Let us introduce denotations for the large necklace. Let M be the quantity of levels in the large necklace, denote the level number in the large necklace by r, $r = 0, \ldots, M$. Let t_r be the quantity of small necklaces on the rth level of the large necklace.

Let us now introduce denotations for small necklaces. Instead of the variable i used in the Introduction, we introduce the double index $r\ell$ for denoting the ℓth small necklace on the rth level, $r = 0, \ldots, M$, $\ell = 1, \ldots, t_r$. For convenience of further considerations, let $B_{r\ell}$ stand for the ℓth small necklace on the rth level.

It remains to introduce the denotation $m_{r\ell}$ for the number of levels in the small necklace $B_{r\ell}$ and the denotation $k_{r\ell_j}$ for the number of edges on the j-th level in the necklace $B_{r\ell}$, $j = 0 \ldots m_{r\ell}$.

Theorem 3. *Let G be a necklace of necklace graphs with parameters introduced above. We put*

$$x_{r\ell} = \prod_{j=0}^{m_{r\ell}} \left((q-1)^{k_{r\ell_j}} + (-1)^{k_{r\ell_j}}(q-1) \right),$$

$$y_{r\ell} = \prod_{j=0}^{m_{r\ell}} \left((q-1)^{k_{r\ell_j}} - (-1)^{k_{r\ell_j}} \right),$$

$$s = (M+1) + \sum_{r=0}^{M} \sum_{\ell=1}^{t_r} (m_{r\ell} + 1).$$

Then, the chromatic polynomial of the graph G obeys the formula

$$P_G(q) = q^{1-s} \left[(q-1) \prod_{r=0}^{M} \left\{ \prod_{\ell=1}^{t_r} ((q-1)y_{r\ell} + x_{r\ell}) - \prod_{\ell=1}^{t_r} (x_{r\ell} - y_{r\ell}) \right\} + \prod_{r=0}^{M} \left\{ \prod_{\ell=1}^{t_r} ((q-1)y_{r\ell} + x_{r\ell}) + (q-1) \prod_{\ell=1}^{t_r} (x_{r\ell} - y_{r\ell}) \right\} \right].$$

For the proof of theorem 3, we need to make calculations, applying proposition 5 for small necklaces and for a large one and then making use of formula (16).

Conclusions

In this paper, we have derived explicit formulas for chromatic polynomials of some sp-graphs. Chromatic polynomials of necklace graphs and the ring of necklace graphs were initially obtained by transition to the dual graph and the further use of the flow polynomial. We have also used the technique of finite Fourier transformations. Fourier transforms were used for evaluating the Tutte polynomials in [5, 7].

The use of the partition function of the Potts model is a more general way to evaluate chromatic polynomials. The parallel-reduction and series-reduction identities (introduced by A. Sokal) play an important role in this method. The multidimensional case of these identities is presented in propositions 3 and 4. The case of the necklace reduction is considered in proposition 5. The use of these transformations simplifies the evaluation of chromatic polynomials of the necklace, ring of necklaces, and necklace of necklace graphs. In general, this technique allows us to evaluate the chromatic polynomial of any sp-graph. Note that the same technique was used in [8] for estimating the radius of the circle that contains all the roots of the chromatic polynomial of an sp-graph.

In our further works, we are going to use the necklace reduction technique for evaluating the partition function of graphs with a hierarchical structure, namely, the necklace-type sp-graphs, where all edges are replaced with the same necklaces for many times.

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Нахождение хроматических многочленов некоторых параллельно-последовательных графов

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Аннотация

Основная цель работы – представить явные формулы для вычисления хроматических многочленов некоторых параллельно-последовательных графов (sp-графов). Граф колье, рассматриваемый в данной работе, это простейший нетривиальный sp-граф. Мы дадим явную формулу для вычисления графа-колье общего вида. Кроме того, будут даны явные формулы для вычисления хроматических многочленов графов кольцо из колье и колье из колье произвольного вида.

Хроматические многочлены графов колье и кольцо из колье изначально получены нами при помощи перехода к двойственному графу и дальнейшему использованию потокового многочлена. Кроме того, мы использовали технику конечного преобразования Фурье.

Более универсальным способом подсчета хроматических многочленов является использование статистической суммы модели Поттса. Идея используемых здесь тождеств параллельного и последовательного сокращения была предложена Аланом Сокалом. Мы разовьем эту идею и введем преобразование сокращения графа колье. Использование этого преобразования позволяет упростить вычисление хроматических многочленов графов колье, кольцо из колье и вычислить хроматический многочлен графа колье из колье.

Ключевые слова: хроматические многочлены, статистическая сумма модели Поттса, полином Татта, преобразование Фурье, параллельно-последовательный граф, граф колье

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