Russian Mathematics (Iz. VUZ) *Izvestiya VUZ. Matematika*
Vol. 49, No. 4, pp. 6–14, 2005 *UDC 517.983*

AN UNSTABLE DIFFERENTIAL TURNING POINT IN THE THEORY OF SINGULAR PERTURBANCES

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1. Problem definition

Consider the singularly perturbed differential equation (SPDE)

$$\mathbf{L}_\varepsilon y(x,\varepsilon) \equiv \varepsilon^3 y'''(x,\varepsilon) + x\widetilde{a}(x)y'(x,\varepsilon) + b(x)y(x,\varepsilon) = h(x) \tag{1.1}$$

with $\varepsilon \to +0$, $x \in I = [0,1]$.

SPDE (1.1) is investigated in [1] and [2], correspondingly, subject to $\widetilde{a}(x) > 0$, $b(x) < 0$, and $\widetilde{a}(x) > 0$, $b(x) > 0$. In the cited works, it is essential that in both cases the turning point is stable. The stability of the turning point is ensured by the application of the apparatus of the Airy–Dorodnitsyn functions for constructing the asymptotic of a solution of this equation.

In this paper, we investigate equation (1.1) under the following conditions.

Condition 1. $a(x), b(x), h(x) \in \mathbf{C}^\infty[0,1]$.

Condition 2. $a(x) = x\widetilde{a}(x)$, moreover, $\widetilde{a}(x) < 0$ and $b(x) < 0$ with $x \in I \equiv [0,1]$.

Unfortunately, up to now the apparatus of the Airy–Dorodnitsyn functions ([3], section 1; [4]; [5], p. 198) is not applicable if the turning point is instable. Therefore, in this paper, for constructing a uniformly suitable asymptotic of a solution of SPDE (1.1) on the entire segment $[0,1]$, instead of the Airy–Dorodnitsyn functions we use other modifications of the Airy function, namely, the Airy–Langer ones. The properties of the latter are described in ([3], section 1; [6], p. 264–270; [7], Chap. 11). Using the Airy–Langer functions, we have developed a method for constructing a uniformly suitable asymptotic of a solution of the Liouville equation and systems of SPDE with stable and unstable inner algebraic turning points ([3], sections 5–7). Therefore, the main goal of our investigation is to generalize the results which were obtained and described in [3] for an unstable algebraic turning point to the case of an unstable differential turning point.

In this paper, we consider the case, when $\rho = \frac{b(0)}{\widetilde{a}(0)} \in N$. Since $\rho > 0$, the solution of the confluent equation

$$\mathbf{L}_0\omega(x) \equiv x\widetilde{a}(x)\omega'(x) + b(x)\omega(x) = h(x) \tag{1.2}$$

and its derivatives, similarly to [2], are not smooth in a neighborhood of the turning point $x = 0$. Hence we cannot use the solution of equation (1.2) in the explicit form for constructing a uniformly suitable asymptotic of a solution of SPDE (1.1).