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BEHAVIOR OF THE CONFORMAL RADIUS IN SUBCLASSES OF UNIVALENT DOMAINS

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In this article we continue investigation of the behavior of conformal (interior) radius of a simply connected domain (see [1]–[4]). In the first Section we give a formula for representation of the conformal radius in a small neighborhood of a fixed point with special formulation of the representation for the case where the point is critical. This formula allows us to give a new proof of the known condition (see [5]) of uniqueness of the critical point in the form $|\{f(\zeta),\zeta\}| \le \frac{2}{(1-|\zeta|^2)^2}$ with Schwarzian in the left side of the inequality. In the second Section we write out two families of spiral-like functions with violation of the uniqueness of critical point of the corresponding conformal radius and formulate two theorems on the uniqueness of the critical point of conformal radius for the subclasses of spiral-like domains. The third Section contains a proposition which complements the results in [4], related to the behavior of the conformal radius of almost convex domains.

1.

The conformal radius of a domain D at the point z is calculated by means of the function $z = f(\zeta)$, which maps conformally the disk $E = \{\zeta : |\zeta| < 1\}$ onto the domain D,

$$R(D, z) = R(f(E), f(\zeta)) = |f'(\zeta)|(1 - |\zeta|^2) \equiv R(\zeta).$$

We will obtain a decomposition of this real positive (because $f'(\zeta) \neq 0$, $\zeta \in E$) function in a neighborhood of the point a by rewriting the last equality in the previous formula in the form

$$R(a + \rho e^{i\theta}) = |f'(a + \rho e^{i\theta})|(1 - |a + \rho e^{i\theta}|^2). \tag{1}$$

For the second factor in (1) we have

$$1 - (a + \rho e^{i\theta})(\overline{a} + \rho e^{-i\theta}) = 1 - |a|^2 - 2\,\mathrm{Re}(\overline{a}e^{i\theta})\rho - \rho^2. \tag{2}$$

We write the first factor in the form $|f'(a + \rho e^{i\theta})| = \exp(\ln |f'(a + \rho e^{i\theta})|)$ and first decompose the function

$$\ln f'(a + \rho e^{i\theta}) = \ln f'(a) + \frac{f''(a)}{f'(a)}e^{i\theta}\rho + \frac{1}{2}\Big(\frac{f'''(a)}{f'(a)} - \Big(\frac{f''(a)}{f'(a)}\Big)^2\Big)e^{i2\theta}\rho^2 + O(\rho^3).$$

This decomposition leads us to the representation

$$\ln |f'(a + \rho e^{i\theta})| = \mathrm{Re} \ln f'(a + \rho e^{i\theta}) = \ln |f'(a)| + a_1\rho + a_2\rho^2 + O(\rho^3),$$

which implies

$$|f'(a + \rho e^{i\theta})| = |f'(a)|e^{a_1\rho + a_2\rho^2 + O(\rho^3)} = |f'(a)|\Big(1 + a_1\rho + \Big(a_2 + \frac{a_1^2}{2}\Big)\rho^2 + O(\rho^3)\Big),$$