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Sequential Differentiation in Nonsmooth Infinite-Dimensional Extremal Problems

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Qualitative and quantitative optimization methods imply the differentiation of the objective functional (e.g., [1—4]). In nonlinear optimal control problems, calculation of a derivative is rather difficult. The reason is that one has to substantiate the differentiability of the state function of a system with respect to the control. In an infinite-dimensional case one can verify this property with the help of the inverse function theorem ([5], P. 40) or implicit function one ([6], P. 660). These theorems assume that the derivative of the state operator is invertible. Otherwise, the state function is not necessarily differentiable with respect to the control [7]. Nevertheless, one can obtain the desired result with the help of the notion of an extended operator derivative and the corresponding generalizations of inverse and implicit functions theorems [7, 8]. The case is much more difficult, if the state operator is nonsmooth. If nonsmooth terms enter in the optimality criterion, then one can obtain the desired result with the help of subdifferential calculus [9], the Clarke derivative [10], and other generalizations of derivatives of functionals. The use of these notions in nonsmooth infinite-dimensional optimal control problems leads to major difficulties because of the absence of effective nonsmooth analogs of inverse and implicit functions theorems.

Papers [11—13] are dedicated to optimal control problems for elliptic equations with nonsmooth nonlinearities. In these papers the state operators are approximated by smooth ones; this fact suggests the existence of certain operator derivatives. In [14] one introduces a sequential operator derivative which is analogous to the derivative of a generalized function in the sequential distribution theory [15]. The corresponding constructions used for this purpose are similar to those described in [11—13]. However, this definition does not allow one to deduce the results of the mentioned papers from theorems of the general extremum theory (e.g., [1—4]), replacing the classical operator derivatives with their sequential analogs. In this paper, we propose a more general definition of the sequential operator derivative and use it, studying abstract extremal problems. As an example we consider the optimal control problem for a parabolic equation with a nonsmooth nonlinearity. This problem does not admit the application of the known solution methods used for similar problems (e.g., [16—24]).

1. SEQUENTIAL DIFFERENTIATION OF OPERATORS

The adduced below definition of the sequential differentiation follows the scheme described in [14]. The main distinction here is the fact that points of differentiability of approximating operators are not identified with the point, where one differentiates the initial operator. Due to this fact the mentioned results are practically applicable in the theory of extremum. We consider linear normalized spaces V, Y and a point $v_0 \in V$. Let Σ stand for the family of sequences $\{A_k\}$ of operators with the following properties. These operators act from V into Y; one can find a neighborhood O^A of the point v_0 and a sequence $\{v_k^A\}$ of elements of the space V such that with any number k the operator A_k is Gâteaux differentiable at the point v_k^A; the sequence of values $\{A_k v\}$ converges in Y uniformly with respect to $v \in O^A$, and limits of sequences $\{A_k v_0\}$ and $\{A_k v_k^A\}$ coincide. On the set Σ we define the equivalence relation σ, assuming that the condition $\{A_k\}\sigma\{B_k\}$ is fulfilled, if in the space Y the sequence $\{A_k v - B_k v\}$ tends to zero uniformly with respect to $v \in (O^A \cap O^B)$. Then sequences $\{A_k v_k^A\}$ and

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