Finite groups and their arithmetic characteristics

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Arithmetic Characteristics of a Finite Group

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- $\omega(G)$, the spectrum of G that is the set of its element orders

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- $\omega(G)$, the spectrum of G that is the set of its element orders
- $GK(G)$ is the prime (or Gruenberg – Kegel) graph of G : the graph with the vertex set equal to $\pi(G)$ and $p, q \in \pi(G)$ are adjacent iff $pq \in \omega(G)$ and $p \neq q$.

Finite Simple Groups

According to their classification, the finite simple groups are

1. the groups of prime order;
2. the alternating groups of degree at least 5;
3. the simple classical groups;
4. the simple exceptional groups of Lie type;
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A finite group G is almost simple if $S \leq G \leq \mathrm{Aut}(S)$ for some nonabelian simple group S.

Two Conjectures of 1987

Conjecture (Shi Wujie, 1987)

If L is a finite simple group and G is a finite group with $\omega(G) = \omega(L)$ and $|G| = |L|$, then $G \simeq L$.

Conjecture (John Thompson, 1987)

If L is a finite simple group and G is a finite group with $N(G) = N(L)$ and $Z(G) = 1$, then $G \simeq L$.

Shi's Conjecture

W. Shi, J. Bi, H. Cao, M. Xu, 1987,...,2003:
Shi's conjecture is valid for all simple groups except symplectic and orthogonal groups (more precisely, except the simple groups $P\Omega^+_{2n}(q)$ with n even, $P\Omega_{2n+1}(q)$ and $Sp_{2n}(q)$).

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Shi's conjecture is true for the remaining groups. It follows

Theorem A

If L is a finite simple group, and G is a finite group with $\omega(G) = \omega(L)$ and $|G| = |L|$, then $G \simeq L$.

Thompson's Conjecture

G. Chen, A. Vasil'ev, W. Shi, M. Xu, N. Ahanjideh, 1996,...,2019: Thomson's conjecture is valid for all simple groups except alternating groups and orthogonal groups of dimensions 8 and 16 (more precisely, except the simple groups $P\Omega_8^+(q)$ and $P\Omega_{16}^+(q)$).

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Theorem B

If L is a finite simple group and G is a finite group with $N(G) = N(L)$ and $Z(G) = 1$, then $G \simeq L$.

Mazurov's Conjecture

- Groups G and H are isospectral if $\omega(G) = \omega(H)$
- G is recognizable (by spectrum) if $G \simeq H$ for every H isospectral to G
- G is almost recognizable if there are only finitely many (up to isomorphism) groups H isospectral to G.

Theorem (Shi, Mazurov, 1998)

If a finite group G includes a nontrivial normal abelian subgroup, then there exist infinitely many finite groups H with $\omega(H) = \omega(G)$.

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Conjecture (Mazurov, 2007)

Generally, if L is a finite nonabelian simple group and G is a finite group with $\omega(G) = \omega(L)$, then $L \leqslant G \leqslant \mathrm{Aut}(L)$.

- A nonabelian simple group L is almost recognizable if $L \leqslant G \leqslant \mathrm{Aut}(L)$ for all groups G isospectral to L.

Mazurov's conjecture is valid in the following sense:

Theorem C (current version)

Let L be one of the following nonabelian simple groups:

- Sporadic groups other than J_2
- Alternating groups other than A_6 and A_{10}
- Exceptional groups of Lie type other than ${}^3D_4(2)$
- Classical groups of dimension at least 38.

If $\omega(G) = \omega(L)$, then $L \leq G \leq \operatorname{Aut} L$; all such groups G are known.

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Open Problem 1

Obtain the complete answer for classical groups of dimension n, where $5 \leq n \leq 38$.

Mazurov's Conjecture: Sporadic and Alternating Groups

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Theorem 1 (... Mazurov-Shi, 1998)

Let L be a sporadic simple group and $\omega(G) = \omega(L)$.

- If $L \neq J_2$, then $G \simeq L$.
- If $L = J_2$, then $\omega(L) = \omega(V \leftthreetimes A_8)$, where $V \simeq 2^6$.

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Theorem 2 (... Gorshkov, 2012)

Let L be an alternating group A_n, $n \geqslant 5$, and $\omega(G) = \omega(L)$.

- If $n \notin \{6, 10\}$, then $G \simeq L$.
- If $n = 6$, then $\omega(L) = \omega(V \leftthreetimes A_5)$, where $V \simeq 2^4$.
- If $n = 10$, then $\omega(L) = \omega(V \leftthreetimes H)$, where V is abelian and H contains a section isomorphic to A_5.

Mazurov's Conjecture: Groups of Lie Type

Theorem 3 (...Staroletov-Vasil'ev, 2013)

Let L be a simple exceptional group of Lie type, and $\omega(G) = \omega(L)$.

- If $L \neq {}^3D_4(2)$, then $L \leqslant G \leqslant \mathrm{Aut}(L)$.
- If $L = {}^3D_4(2)$, then $\omega(L) = \omega(V \leftthreetimes L)$, where $V \simeq 2^{24}$.

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Theorem 4 (... Grechkoseeva-Vasil'ev, 2015)

If L is a classical group of sufficiently large dimension, and $\omega(G) = \omega(L)$, then $L \leqslant G \leqslant \mathrm{Aut}(L)$.

General Structure of a Group Isospectral to Simple

Let L be a finite nonabelian simple group.
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Williams (+ Gruenberg-Kegel's theorem), 1981, Kondrat′ev, 1989, Vasil′ev, Vasil′ev-Vdovin, 2005, Gorshkov, 2012:

Proposition

If L differs from $PSL_3(3)$, $PSU_3(3)$, $PSp_4(3)$, then G has exactly one nonabelian composition factor. Thus,

$$1 \leqslant K < H \leqslant G,$$

where K is the solvable radical of G,

$H/K \simeq S$, where S is a nonabelian simple group,

and $G/H \leq \mathrm{Out}(S)$ is a solvable group.

L is a nonabelian simple group, G is a group with $\omega(G) = \omega(L)$

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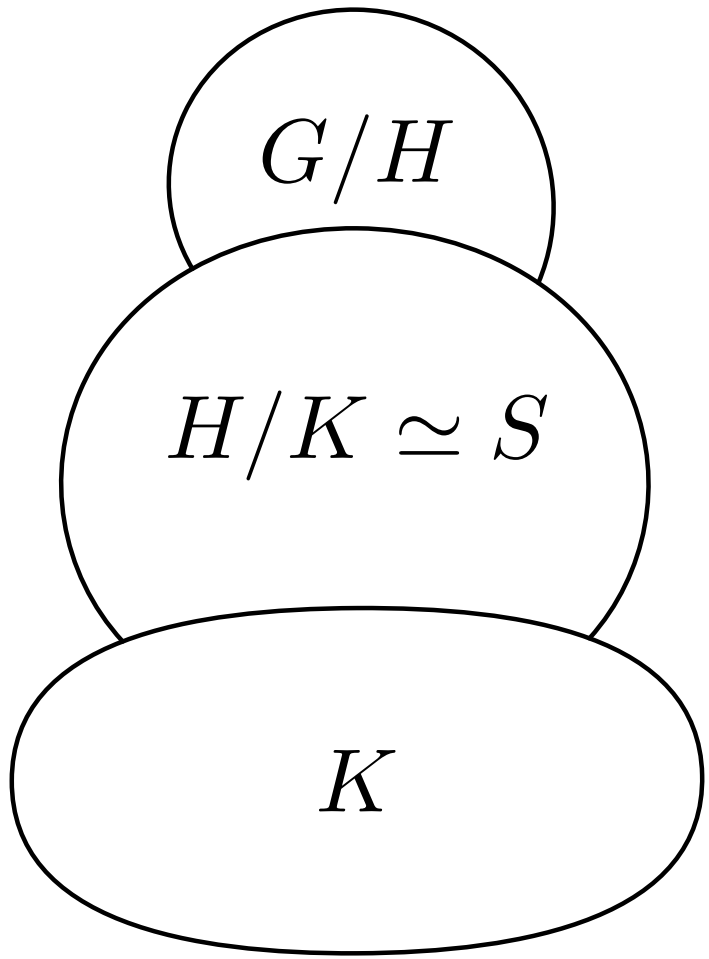

Theorem (N. Yang, M. Grechkoseeva, A. Vasil′ev, 2020)

If G is an nonsolvable finite group isospectral to some finite simple group L, then

$$1 \leqslant K < H \leqslant G,$$

where H/K is a nonabelian simple group, and K being the solvable radical of G is nilpotent provided $L \neq A_{10}$.

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The quotient $G/H \leq \mathrm{Out}(S)$, so it is a solvable group by the validity of the Schreier conjecture.

Open Problem 2

Is it true that G/H is cyclic?

Remark. This is true for all known almost recognizable simple groups (see Theorem C).

Recognition of non-simple groups

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Theorem (Mazurov, 1997)

Let $L = Sz(2^7) = {}^2B_2(2^7)$. The following groups are recognizable:

- $L \times L$;
- $L \wr F$, the permutation wreath product of L and the Frobenius group $F = 23 : 11$ of degree 23.

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Theorem (Gorshkov, Maslova, 2020)

The group $J_4 \times J_4$, where J_4 is the sporadic Janko group, is recognizable.

Despite the examples from the previous slide, it is easy to see that

- $\omega(A_5 \times A_5) = \omega(F_1 \times F_2)$, where F_1 and F_2 are Frobenius groups with the kernels isomorphic to Z_5^2 and complements of order 2 and 3, respectively.
- $\omega(A_5 \times A_5 \times A_5) = \omega(Z_2^6 \times Z_3^3 \times Z_5^3)$ and these two groups are of the same order!

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The infinite series of recognizable non-simple groups is given by

Theorem (Gorshkov, 2021)

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Open Problem 3

Does there exist a recognizable group which is a direct product of

1. non-isomorphic simple groups,
2. m copies of a simple group for arbitrarily large m?

Spectra of Finite Simple Groups

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Set $\mu(G)$ for the set of maximal w.r.t divisibility elements of the spectrum $\omega(G)$ of a group G. Clearly, $\omega(G)$ can be recovered from $\mu(G)$ and even from any set $\nu(G)$ satisfying

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The spectrum of G is known if some $\nu(G)$ is described.

Theorem D

If G is a finite simple group, then $\omega(G)$ is known.

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Open Problem 4

Describe the spectra of the finite almost simple groups.

Constructive Recognition of Simple Groups

$\mathcal{M}$ is a finite set of positive integers
$\omega(\mathcal{M})$ is the set of all divisors of elements of $\mathcal{M}$
$\mu(\mathcal{M}) \subseteq \mathcal{M}$ consists of the elements maximal w.r.t divisibility.

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General problem

Given a finite set $\mathcal{M}$ of positive integers, determine whether there exists a finite group G such that

$$\omega(G) = \mathcal{M},$$

and if it exists, describe all finite groups with this property.

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Since we have the description of spectra of finite simple groups (Theorem D), we can solve the problem (at least theoretically).

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More precisely, can we do this in time polynomial in the length m of a set $\mathcal{M}$, that is the sum of $\log x$ where x runs over $\mathcal{M}$.

Theorem (Buturlakin – Vasil′ev, 2019)

There is an algorithm that, given a set $\mathcal{M}$ of positive integers, outputs either an empty set, in which case there is no finite simple group H with $\omega(H) = \omega(\mathcal{M})$, or a unique simple group G satisfying the properties: (1) $\omega(\mathcal{M}) \subseteq \omega(G)$; (2) $\omega(H) \neq \omega(\mathcal{M})$ for every finite simple group H whose spectrum differs from $\omega(G)$. The running time of the algorithm is polynomial in the length of the input.

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Thus, to complete the task, one need to verify that

$$\omega(G) \subseteq \omega(\mathcal{M})$$

for the unique possible simple group G (if it exists).

Generating the Spectra of Simple Groups

Theorem (Buturlakin, 2008)

Let $G = PSL_n(q)$, where $n \geqslant 2$ and q is a power of a prime p. Put $d = (n, q-1)$. Let $\nu(G)$ consists of the following numbers:

(1) $\frac{q^n-1}{d(q-1)}$;

(2) $\frac{[q^{n_1}-1, q^{n_2}-1]}{(n/(n_1,n_2),q-1)}$ for $n_1, n_2 > 0$ such that $n_1 + n_2 = n$;

(3) $[q^{n_1}-1, q^{n_2}-1, \ldots, q^{n_s}-1]$ for $s \geqslant 3$ and $n_1, n_2, \ldots, n_s > 0$ such that $n_1 + n_2 + \ldots + n_s = n$;

(4) $p^k \frac{q^{n_1}-1}{d}$ for $k, n_1 > 0$ such that $p^{k-1} + 1 + n_1 = n$;

(5) $p^k[q^{n_1}-1, q^{n_2}-1, \ldots, q^{n_s}-1]$ for $s \geqslant 2$ and $k, n_1, n_2 \ldots, n_s > 0$ such that $p^{k-1} + 1 + n_1 + n_2 + \ldots + n_s = n$;

(6) p^k, if $p^{k-1} + 1 = n$ for $k > 0$.

Then $\mu(G) \subseteq \nu(G) \subseteq \omega(G)$.

The analysis of the efficiency of the most straightforward algorithms provided by the known description of spectra of groups of Lie type implies that $\mu(G)$ can be found

- in time polynomial in the length m of $\mu(G)$ if G is an alternating group or exceptional group of Lie type;
- in time $m^{O(\sqrt{\log \log m})}$ if G is a classical group.

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Theorem E

There is an algorithm that given a set $\mathcal{M}$ of positive integers outputs a finite simple group G with $\omega(G) = \omega(\mathcal{M})$, or says that there is no such a group. The running time of the algorithm is $m^{O(\sqrt{\log\log m})}$, where m is the length of $\mathcal{M}$.

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Open Problem 5

Does there exist a polynomial-time algorithm?

$N(G) = \{|x^G| : x \in G\}$, the set of element indices of G.

Theorem B

If L is a finite simple group and G is a finite group with $N(G) = N(L)$ and $Z(G) = 1$, then $G \simeq L$.

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If L is a finite simple group and G is a finite group with $N(G) = N(L)$ and $Z(G) = 1$, then $G \simeq L$.

Open Problem 6

Given a finite set $\mathcal{M}$ of positive integers, determine whether there exists a finite simple group G such that $N(G) = \mathcal{M}$.

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Open Problem 6

Given a finite set $\mathcal{M}$ of positive integers, determine whether there exists a finite simple group G such that $N(G) = \mathcal{M}$.

Open Problem 7

Describe the sets $N(G)$ for all finite (almost) simple groups G.

Theorem A (Shi's Conjecture)

If L is a finite simple group and G is a finite group with $\omega(G) = \omega(L)$ and $|G| = |L|$, then $G \simeq L$.

Theorem B (Thompson's Conjecture)

If L is a finite simple group and G is a finite group with $N(G) = N(L)$ and $Z(G) = 1$, then $G \simeq L$.

Theorem C (Mazurov's Conjecture)

If L is a finite simple group and G is a finite group with $\omega(G) = \omega(L)$, then (in most cases) $L \leq G \leq \mathrm{Aut}(L)$.

Theorem D

If G is a finite simple group, then $\omega(G)$ is known.

Theorem E

There is a quasi-polynomial algorithm that given $\mathcal{M} \subseteq \mathbb{N}$ decides if there is a finite simple group G with $\omega(G) = \omega(\mathcal{M})$.