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Invariants of the Action of a Semisimple Finite-Dimensional Hopf Algebra on Special Algebras

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Abstract—In this paper we extend classical results of the invariant theory of finite groups to the action of a finite-dimensional semisimple Hopf algebra H on a special algebra A, which is homomorphically mapped onto a commutative integral domain, and the kernel of this map contains no nonzero H-stable ideals. We prove that the algebra A is finitely generated as a module over a subalgebra of invariants, and the latter is finitely generated as a $\mathbf{k}$-algebra. We give a counterexample to the finite generation of a non-semisimple Hopf algebra.

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1. INTRODUCTION

In this paper we extend some classical results of the theory of invariants of finite groups to the case of the action of a semisimple finite-dimensional Hopf algebra on an algebra of a special kind homomorphically mapped to a commutative domain of integrity.

All considered algebras, coalgebras, and Hopf algebras are defined over a field $\mathbf{k}$. Algebras are assumed to be associative with unit, while coalgebras are assumed to be coassociative with counit. We denote by $\Delta : C \to C \otimes C$ the comultiplication and we do by $\varepsilon : C \to \mathbf{k}$ the counit of the coalgebra C. Hereinafter H is a Hopf algebra, and A is an associative $\mathbf{k}$-algebra. All necessary information about the Hopf algebras can be found in [1]. All tensor products, unless the contrary is specified, are calculated over $\mathbf{k}$, and Hom is $\mathrm{Hom}_{\mathbf{k}}$.

Definition 1.1. An algebra H is said to act on A, if A has the structure of a left H-module, and for any $h \in H$ and $a, b \in A$ it holds

$$h \cdot (ab) = \sum_h (h_{(1)} \cdot a)(h_{(2)} \cdot b), \quad h \cdot 1_A = \varepsilon(h) 1_A.$$

An algebra A with a given action on H is said to be an *H-module algebra*.

Definition 1.2. An algebra H is said to coact on A, if A has the structure of a right H-comodule and $\rho : A \to A \otimes H$ is a homomorphism of algebras.

In this case, A is also called an *H-comodule algebra*.

If H is finite-dimensional, then H^* is a Hopf algebra, and one can treat any left H-module U as a right H^*-comodule with the help of the isomorphism $\mathrm{Hom}(H \otimes U, U) \cong \mathrm{Hom}(U, U \otimes H^*)$. Moreover, any H-module algebra turns into an H^*-comodule one with respect to the structural mapping

$$\rho : A \to A \otimes H^* \cong \mathrm{Hom}(H, A),$$
$$\rho(a)(h) = h \cdot a, \quad h \in H, \quad a \in A.$$

Therefore all assertions stated for an H-module algebra (an H-comodule algebra) remain valid for an H-comodule algebra (an H-module algebra).

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