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DETERMINATION OF CLASSES OF DATA FOR CORRECT STATEMENT OF THE INVERSE BOUNDARY VALUE PROBLEM BY MEANS OF RE-PARAMETERIZATION

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In the solution of the inverse boundary value problem with respect to the parameter s (see [1]) the known contour $L_w = \{w = u(s) + iv(s),\ s \in [0, l]\}$ — a simple closed curve — plays an important role. In the process of solving the problem either a mapping of the domain D_w bounded by the contour L_w to the canonical domain, i.e., the unit disk is used (see [1]), or the respective Schwartz problem is solved directly in the domain D_w, which leads us to the solution of the Fredholm integral equation (see [2]). In this article the boundary data $u(s) + iv(s)$ of the inverse boundary value problem are given as the superposition of the complex-valued function $\widetilde{w}(\sigma) = \widetilde{u}(\sigma) + i\widetilde{v}(\sigma)$, $0 \le \sigma \le \sigma_k$, which gives a simple closed contour L_w with the use of the intrinsic parameter σ, and the re-parameterization function $\sigma = \sigma(s)$, $s \in [0, l]$. The contour L_w is assumed to be a Lyapunov's curve, i.e., $|\widetilde{w}'(\sigma)| = 1$, $\arg \widetilde{w}'(\sigma) \in H_\alpha[0, \sigma_k]$, while $\sigma(s)$ is a monotone function. The problem is reduced to process of solving an integral equation. The integral equation is solvable in the case where $\sigma'(s) \in L_{1+\varepsilon}[0, l]$, $[\sigma'(s)]^{-1} \in L_\delta[0, l]$, $\varepsilon, \delta > 0$. We obtain a sufficient condition for the simplicity of the desired contour L_z, which is not related to the auxiliary mapping of the given domain onto the unit disk, therefore this condition is a result within the frameworks of the strong problem of univalence (see [3]). On the other hand, the re-parameterization of the known contour can be treated as a univalent change of a univalent solution of the inverse boundary value problem (see [4]). In this article we obtain a sufficient condition of almost convexity (see [5]) of the desired contour L_z in the case where the known contour L_w is convex. The sufficient conditions of almost convexity of solution of problem were earlier obtained within the frameworks of the weak problem of univalence (see [6]–[11]).

1. Let us consider the simplest case of re-parameterization $\sigma(s) = s$. The integral equation for the function $q(\sigma) = \operatorname{Im} \ln z'(w)|_{w \in L_w}$ has the form (see [2]):

$$q = T[q] - R[p], \tag{1}$$

where

$$T[q] \equiv \frac{1}{\pi} \int_0^{\sigma_k} q(\tau)[\arg[\widetilde{w}(\tau) - \widetilde{w}(\sigma)]]'_\tau d\tau, \tag{2}$$

$$R[p] \equiv \frac{1}{\pi} \int_0^{\sigma_k} p(\tau)[\ln |\widetilde{w}(\tau) - \widetilde{w}(\sigma)|]'_\tau d\tau, \tag{3}$$

$$p(\sigma) = \ln s'(\sigma).$$

In the present case, we have $p(\sigma) \equiv 0$, therefore in equation (1) the free term $R[p]$ will be absent. A set of solutions of equation (1) satisfies the relation: $q(\sigma) \equiv \alpha$, $\alpha \in \mathbf{R}$ (see [2]). Consequently, $z(w) = e^{i\alpha} w + C$, $C \in \mathbf{C}$, $g(z) = (z - C)e^{-i\alpha}$. Clearly, the desired contour L_z, which results from the simple contour L_w after shift and rotation, will also be a simple curve.

2. If $\sigma'(s) \in H_\gamma[0,l]$, then the problem can be reduced to solving integral equation (1) in the space H_δ, $\delta = \min\{\alpha,\gamma\}$. The solution in this class was investigated in [2], where also a sufficient condition of the simplicity — the almost convexity — of the desired contour was given.

3. Let us consider in detail the case of weaker constraints upon $\sigma'(s)$. Note that one can obtain a sufficient condition of the almost convexity of the desired contour L_z for the convex contour L_w only under the condition that

$$q(\sigma) = \arg \frac{dz}{dw}\Big|_{w\in L_w}$$

is a bounded value. Following [2], in the case where

$$\sigma'(s) \in L_{1+\varepsilon}[0,\sigma_k], \quad [\sigma'(s)]^{-1} \in L_\delta[0,\sigma_k], \quad \varepsilon,\delta > 0,$$

equation (1) has a solution $q(\sigma) \in L_\nu[0,\sigma_k]$ $\forall \nu > 1$. In order to obtain a bounded function $q(\sigma)$ it suffices to ensure the boundedness of the function $f(\sigma) = R[p]$ (3).

Indeed, if

$$\arg \widetilde{w}'(\sigma) \in H_\alpha[0,\sigma_k], \quad \alpha \in (0,1];$$
$$\left|\frac{\widetilde{w}(\sigma_1) - \widetilde{w}(\sigma_2)}{\sigma_1 - \sigma_2}\right| \ge m > 0, \quad |\sigma_1 - \sigma_2| \le \frac{\sigma_k}{2},$$

then in accordance with [2] the kernel of the operator T (2), i. e., the function $K(\tau,\sigma) \equiv \{\arg[\widetilde{w}(\tau) - \widetilde{w}(\sigma)]\}'_\tau$, has the estimate $|K(\tau,\sigma)| \le D|\tau-\sigma|^{\alpha-1}$, where $D \le [\|\widetilde{v}'\|_C \|\widetilde{u}'\|_{H_\alpha} + \|\widetilde{u}'\|_C \|v'\|_{H_\alpha}]m^{-2}$. Taking into account that for the given conditions equation (1) has a solution in the space L_ν $\forall\nu > 1$, we obtain

$$\sup|q(\sigma)| \le \frac{1}{\pi}\|q\|_{L_{r/(r-1)}} \left[\max_\sigma \int_0^{\sigma_k} |K(\tau,\sigma)|^r d\tau\right]^{1/r} + \sup|f(\sigma)|,$$

where $1 < r < \frac{1}{1-\alpha}$. Consequently, in this case, the function $q(\sigma)$ is bounded.

One can ensure the boundedness of the function $f(\sigma)$ only if the density $p(\sigma)$ of the operator R (3) will be a continuous function; in the contrary case $f(\sigma)$ will possess a logarithmic singularity as the imaginary part of the Cauchy integral with a discontinuous real density.

Thus, let us suppose that

$$\arg[\widetilde{w}(\tau) - \widetilde{w}(\sigma)] = \arg[e^{i2\pi\tau/\sigma_k} - e^{i2\pi\sigma/\sigma_k}] + \sum_{j=0}^{\infty}\sum_{n=0}^{j} a_{nj}\left[\cos\frac{2\pi}{\sigma_k}(\tau j + \sigma n) + \right.$$
$$\left. + \cos\frac{2\pi}{\sigma_k}(\tau n + \sigma j)\right] + b_{nj}\left[\sin\frac{2\pi}{\sigma_k}(\tau j + \sigma n) + \sin\frac{2\pi}{\sigma_k}(\tau n + \sigma j)\right]; \quad (4)$$

moreover,

$$\sum_{j=0}^{\infty}\sum_{n=0}^{j}(|a_{nj}| + |b_{nj}|)(j+n) \le b < \frac{1}{2}, \tag{5}$$

$$\sum_{j=0}^{\infty}\sum_{n=0}^{j}(|a_{nj}| + |b_{nj}|) \le c < \frac{1}{\pi}. \tag{6}$$

Assume that a function $\sigma(s)$, $s \in [0,l]$, such that $0 < m_1 \le \sigma'(s) \le M_1$, satisfies the condition

$$|\sigma'(s_1) - \sigma'(s_2)| \le \frac{C}{|\ln\frac{|s_1-s_2|}{l}|^p}, \quad p > 1, \quad |s_1 - s_2| \le l/2. \tag{7}$$

In accordance with [2], the corresponding equation (1) in this case is solvable in the space $L_\nu[0,\sigma_k]$ $\forall\nu > 1$. Let us determine restrictions upon the introduced parameters, which are sufficient for the desired contour L_z to be almost convex. We divide the derivation of the condition of the almost convexity into steps.

a) Let us show that condition (5) ensures the convexity of the contour L_w. In correspondence with (4) we have

$$\arg \widetilde{w}'(\sigma) = \frac{\pi}{2} + \frac{2\pi}{\sigma_k}\sigma + 2\sum_{j=0}^{\infty}\sum_{n=0}^{j} a_{nj}\cos\frac{2\pi}{\sigma_k}\sigma(j+n) + b_{nj}\sin\frac{2\pi}{\sigma_k}\sigma(j+n).$$

Consequently,

$$(\arg \widetilde{w}'(\sigma))' = \frac{2\pi}{\sigma_k} + \frac{4\pi}{\sigma_k}\sum_{j=0}^{\infty}\sum_{n=0}^{j}(j+n)\Big[-a_{nj}\sin\frac{2\pi}{\sigma_k}\sigma(j+n) + b_{nj}\cos\frac{2\pi}{\sigma_k}\sigma(j+n)\Big] \geq$$

$$\geq \frac{2\pi}{\sigma_k} - \frac{4\pi}{\sigma_k}\sum_{j=0}^{\infty}\sum_{n=0}^{j}(j+n)(|a_{nj}| + |b_{nj}|) \geq \frac{2\pi}{\sigma_k}(1-2b) > 0$$

in accordance with (5).

At the same time, let us note that $\arg \widetilde{w}'(\sigma) \in H_1[0,\sigma_k]$, i. e., $\alpha = 1$.

b) Now let us find a number $m > 0$ such that

$$\left|\frac{\widetilde{w}(\sigma_1) - \widetilde{w}(\sigma_2)}{\sigma_1 - \sigma_2}\right| \geq m, \quad |\sigma_1 - \sigma_2| \leq \frac{\sigma_k}{2}.$$

Let us note that the above inequality which seems to be intrinsic for simple curves was earlier used in M.A. Lavrent'yev's works (see, e. g., [12]).

We have

$$\widetilde{w}(\sigma_1) - \widetilde{w}(\sigma_2) = \int_{\sigma_2}^{\sigma_1} e^{i\arg \widetilde{w}'(t)}dt = \int_{\sigma_2}^{\sigma_1} \exp\Big[i\Big(\frac{\pi}{2} + \frac{2\pi}{\sigma_k}t + \phi_0(t)\Big)\Big]dt,$$

where

$$\phi_0(\sigma) = 2\sum_{j=0}^{\infty}\sum_{n=0}^{j} a_{nj}\cos\frac{2\pi}{\sigma_k}\sigma(j+n) + b_{nj}\sin\frac{2\pi}{\sigma_k}\sigma(j+n).$$

Consequently,

$$\widetilde{w}(\sigma_1) - \widetilde{w}(\sigma_2) = i\int_{\sigma_2}^{\sigma_1}\exp(i2\pi t/\sigma_k)dt + i\int_{\sigma_2}^{\sigma_1}\exp(i2\pi t/\sigma_k)[\exp(i\phi_0(t)) - 1]dt =$$

$$= \frac{i\sigma_k}{\pi}\exp[i\pi(\sigma_1+\sigma_2)/\sigma_k]\sin\frac{\pi(\sigma_1-\sigma_2)}{\sigma_k} + i\int_{\sigma_2}^{\sigma_1}\exp(i2\pi t/\sigma_k)[\exp(i\phi_0(t)) - 1]dt.$$

From the geometrical standpoint it is evident that $|\exp(ia) - 1| \leq |a|$, and therefore

$$|\widetilde{w}(\sigma_1) - \widetilde{w}(\sigma_2)| \geq \left|\frac{\sigma_k}{\pi}\right|\left|\sin\frac{\pi}{\sigma_k}(\sigma_1-\sigma_2)\right| - \max_t|\phi_0(t)|\,|\sigma_1-\sigma_2| \geq \Big(\frac{2}{\pi} - 2c\Big)|\sigma_1-\sigma_2|,$$

where c is from (6). Thus, $m = \frac{2}{\pi} - 2c$.

c) Let us show that for $|\sigma_1 - \sigma_2| \leq lm_1/2$ the inequality is fulfilled:

$$\left|\ln\frac{ds}{d\sigma}\Big|_{\sigma_1} - \ln\frac{ds}{d\sigma}\Big|_{\sigma_2}\right| \leq \frac{C_1}{|\ln\frac{|\sigma_1-\sigma_2|}{\sigma_k}|^p},$$

where

$$C_1 = \frac{C|\ln lm_1/(2\sigma_k)|^p}{m_1(\ln 2)^p}. \tag{8}$$

Let $|\sigma_1 - \sigma_2| \leq lm_1/2$; then we have

$$|s(\sigma_1) - s(\sigma_2)| \leq \frac{|\sigma_1-\sigma_2|}{\min\sigma'(s)} \leq l/2.$$

Consequently,

$$\left|\ln\frac{ds}{d\sigma}\Big|_{\sigma_1}-\ln\frac{ds}{d\sigma}\Big|_{\sigma_2}\right|=\left|\ln\frac{d\sigma}{ds}\Big|_{s(\sigma_1)}-\ln\frac{d\sigma}{ds}\Big|_{s(\sigma_2)}\right|\le\frac{1}{\min\sigma'(s)}\left|\frac{d\sigma}{ds}\Big|_{s(\sigma_1)}-\frac{d\sigma}{ds}\Big|_{s(\sigma_2)}\right|\le$$
$$\le\frac{C}{|\ln(|s(\sigma_1)-s(\sigma_2)|l^{-1})|^p m_1}=$$
$$=\frac{C}{m_1|\ln[s'(\sigma_0)|\sigma_1-\sigma_2|/l]|^p}\le\frac{C}{m_1}|\ln|\sigma_1-\sigma_2|/\sigma_k+\ln\sigma_k/l-\ln\min_s\sigma'(s)|^{-p}=$$
$$=\frac{C}{m_1}\left|1+\frac{\ln\sigma_k/l-\ln\min\sigma'(s)}{\ln|\sigma_1-\sigma_2|/\sigma_k}\right|^{-p}\left|\ln\frac{|\sigma_1-\sigma_2|}{\sigma_k}\right|^{-p}\le$$
$$\le\frac{C}{m_1}\left|1+\frac{\ln\sigma_k/l-\ln\min\sigma'(s)}{\ln[lm_1(2\sigma_k)^{-1}]}\right|^{-p}\left|\ln\frac{|\sigma_1-\sigma_2|}{\sigma_k}\right|^{-p}=\frac{C_1}{|\ln\frac{|\sigma_1-\sigma_2|}{\sigma_k}|^p},$$

where C_1 is from (8).

d) Let us estimate $f(\sigma)=R[p]$ (3). Note that

$$f(\sigma)=\ln\frac{ds}{d\sigma}\operatorname{Re}\frac{1}{\pi}\int_{L_w}\frac{d\widetilde{w}(\tau)}{\widetilde{w}(\tau)-\widetilde{w}(\sigma)}+\operatorname{Re}\frac{1}{\pi}\int_{L_w}\frac{\ln\frac{ds}{d\sigma}|_\tau-\ln\frac{ds}{d\sigma}|_\sigma}{\widetilde{w}(\tau)-\widetilde{w}(\sigma)}d\widetilde{w}(\tau)=$$
$$=\operatorname{Re}\frac{1}{\pi}\int_{L_w}\frac{\ln\frac{ds}{d\sigma}|_\tau-\ln\frac{ds}{d\sigma}|_\sigma}{\widetilde{w}(\tau)-\widetilde{w}(\sigma)}d\widetilde{w}(\tau).$$

Consequently,

$$|f(\sigma)|\le\frac{1}{\pi}\Bigg[\int\limits_{|\tau-\sigma|\le lm_1/2}\left|\frac{\ln\frac{ds}{d\sigma}|_\tau-\ln\frac{ds}{d\sigma}|_\sigma}{\widetilde{w}(\tau)-\widetilde{w}(\sigma)}\right|d\tau+\int\limits_{lm_1/2<|\tau-\sigma|\le\sigma_k/2}\left|\frac{\ln\frac{ds}{d\sigma}|_\tau-\ln\frac{ds}{d\sigma}|_\sigma}{\widetilde{w}(\tau)-\widetilde{w}(\sigma)}\right|d\tau\Bigg]\le$$
$$\le\frac{1}{\pi}\Bigg[\frac{C_1}{m}\int\limits_{|\tau-\sigma|\le lm_1/2}\frac{d\tau}{|\tau-\sigma|\,|\ln\frac{|\tau-\sigma|}{\sigma_k}|^p}+\frac{4\max|\ln\frac{d\sigma}{ds}|}{m}(\ln\sigma_k/2-\ln lm_1/2)\Bigg]\le M,$$

where

$$M=\frac{1}{1-\pi c}\left[\frac{C_1}{p-1}\left|\ln\frac{lm_1}{2\sigma_k}\right|^{1-p}+2\ln M_1\ln\frac{\sigma_k}{lm_1}\right].\qquad(9)$$

e) Let us estimate $|q(\sigma)|$, where $q(\sigma)=\arg\frac{dz}{dw}|_{w=\tilde{w}(\sigma)}$ is a solution of equation (1). Since the solution of this equation can be found up to a constant addend, we choose this addend so that the relation be valid

$$\int_0^{\sigma_k}q(\sigma)d\sigma=0.$$

Then

$$q(\sigma)=\frac{1}{\pi}\int_0^{\sigma_k}q(\tau)\Big[\{\arg[\widetilde{w}(\tau)-\widetilde{w}(\sigma)]\}'_\tau-\frac{\pi}{\sigma_k}\Big]d\tau-f(\sigma),$$

and therefore

$$\sup|q(\sigma)|\le\frac{1}{\pi}\sup|q(\sigma)|\sup\int_0^{\sigma_k}\left|\{\arg[\widetilde{w}(\tau)-\widetilde{w}(\sigma)]\}'_\tau-\frac{\pi}{\sigma_k}\right|d\tau+\sup|f(\sigma)|.\qquad(10)$$

By condition (5),

$$\Big|\{\arg[\widetilde{w}(\tau)-\widetilde{w}(\sigma)]\}'_\tau-\frac{\pi}{\sigma_k}\Big|\le\frac{2\pi}{\sigma_k}\Big|\sum_{j=0}^{\infty}\sum_{n=0}^{j}a_{nj}\Big(-\sin\frac{2\pi}{\sigma_k}(\tau j+\sigma n)j-$$
$$-\sin\frac{2\pi}{\sigma_k}(\tau n+\sigma j)n\Big)+b_{nj}\Big(\cos\frac{2\pi}{\sigma_k}(\tau j+\sigma n)j+\cos\frac{2\pi}{\sigma_k}(\tau n+\sigma j)n\Big)\Big|\le$$
$$\le\frac{2\pi}{\sigma_k}\sum_{j=0}^{\infty}\sum_{n=0}^{j}(|a_{nj}|+|b_{nj}|)(j+n)\le\frac{2\pi b}{\sigma_k}.$$

Consequently,

$$\frac{1}{\pi}\sup\int_0^{\sigma_k}\Big|\{\arg[\widetilde{w}(\tau)-\widetilde{w}(\sigma)]\}'_\tau-\frac{\pi}{\sigma_k}\Big|d\tau\le 2b<1.$$

Thus, following (10), we have $\sup|q(\sigma)|(1-2b)\le\sup|f(\sigma)|$, i.e., $|q(\sigma)|\le\frac{M}{1-2b}$, where M is from (9).

Consequently, if M and b are small so that $M/(1-2b)<\pi/2$, then due to the convexity of the contour L_w we obtain

$$\Delta\arg\frac{dz}{d\sigma}\Big|_{L_w}=\Delta\arg\frac{dz}{dw}\Big|_{L_w}+\Delta\arg\frac{dw}{d\sigma}\Big|_{L_w}>-\pi,$$

which ensures the almost convexity and therefore the simplicity of the contour L_z. Thus we have proved the

Theorem. *Let the initial function in the statement of the inverse boundary value problem with respect to the parameter s be given in the form $w(s)=\widetilde{w}(\sigma(s))$, where $|\widetilde{w}'(\sigma)|\equiv 1$, constraints* (4)–(6) *are assumed to be fulfilled*; *$\sigma(s)$ is a monotone function satisfying both inequality* (7) *and the constraints $0<m_1\le\sigma'(s)\le M_1<\infty$. If, in addition, $M<\pi(1-2b)/2$, where M is from* (9), *C_1 is from* (8), *b and c are from* (5) *and* (6), *respectively, then the statement of this problem is correct, i.e., the desired contour will be simple* (*almost convex*).

The obtained sufficient condition of the almost convexity of the solution of the interior inverse boundary value problem with respect to the parameter s could be compared with the sufficient condition of the univalence of the solution to the same problem, which was given in [13] (p. 163), where a constraint was found relating the ratio $\max\sigma'(s)/\min\sigma'(s)$ and the two geometric characteristics of the known domain D_w. Unfortunately, the determination of the constant (B) for the second characteristic in the case of an arbitrary convex domain is rather complex task by itself. Though an undoubtful advantage of the sufficient condition given in [13] is the sole requirement of continuity of $\sigma'(s)$ and absence of a constraint of the type (7).

4. Condition (7) in the previous Item can be replaced with a similar inequality with another module of continuity. Thus, assume that under the fulfillment of conditions (4)–(6) upon the function $\sigma(s)$ the constraints are imposed $0<m_1\le\sigma'(s)\le M_1<\infty$ and the inequality is valid $|\sigma'(s_1)-\sigma'(s_2)|\le\omega(s_1-s_2)$, where a continuous function $\omega(t)$ is such that $\omega(0)=0$, $\int\limits_0^\delta\frac{\omega(t)}{t}dt<\infty$, $\delta>0$. In this case, we also can ensure the almost convexity of the desired contour L_z for some modules of continuity $\omega(t)$. In addition, only the estimate for M will change for $\sup|f(\sigma)|$. Since

$$|f(\sigma)|\le\frac{1}{\pi}\Big[\int\limits_{\sigma(-\delta)\le\tau-\sigma\le\sigma(\delta)}\Big|\frac{\ln\frac{ds}{d\sigma}|_\tau-\ln\frac{ds}{d\sigma}|_\sigma}{\widetilde{w}(\tau)-\widetilde{w}(\sigma)}\Big|d\tau+$$
$$+\int\limits_{\sigma(\delta)<\tau-\sigma\le\sigma(l/2),\ \sigma(-l/2)\le\tau-\sigma<\sigma(-\delta)}\Big|\frac{\ln\frac{ds}{d\sigma}|_\tau-\ln\frac{ds}{d\sigma}|_\sigma}{\widetilde{w}(\tau)-\widetilde{w}(\sigma)}\Big|d\tau\Big]=$$

$$= \frac{1}{\pi}\left[2\int_0^\delta \frac{|\ln\sigma'(\xi) - \ln\sigma'(s)|}{mm_1|\xi - s|}M_1 d\xi + 4\frac{M_1|\ln M_1|}{mm_1}\ln\frac{l}{2\delta}\right] \le$$

$$\le \frac{2M_1}{\pi m m_1}\left[\frac{1}{m_1}\int_0^\delta \frac{\omega(t)}{t}dt + 2|\ln M_1|\,|\ln(l/(2\delta))|\right],$$

in the capacity of M we can take the value

$$\frac{M_1}{(1-\pi c)m_1}\inf_{\delta>0}\left[\frac{1}{m_1}\int_0^\delta \frac{\omega(t)}{t}dt + 2|\ln M_1|\,|\ln(l/(2\delta))|\right].$$

In the case where $M < \frac{\pi(1-2b)}{2}$, we again obtain a simple — almost convex — contour.

In the capacity of an example which is more convenient for application of the sufficient condition of univalence we can cite the following

Corollary (see [14]). Let $\widetilde{w}(\sigma) = \widetilde{u}(\sigma) + i\widetilde{v}(\sigma)$, $0 \le \sigma \le \sigma_k$, be the equation of a closed curve, σ the intrinsic parameter, and

$$\max_{0\le\tau,\sigma\le\sigma_k} |\Phi^{(5)}_{\tau\tau\tau\sigma\sigma}|\frac{\sigma_k^5}{\pi 144} \equiv d < \frac{1}{\pi},$$

where $\Phi(\tau,\sigma) \equiv \arg[\widetilde{w}(\tau) - \widetilde{w}(\sigma)] - \arg[\exp(2\pi i\tau/\sigma_k) - \exp(2\pi i\sigma/\sigma_k)]$.

If $\sigma = \sigma(s)$, $0 \le s \le l$, is a monotone function satisfying the conditions $0 < m_1 \le \sigma'(s) \le M_1 < \infty$, $|\sigma'(s_1) - \sigma'(s_2)| \le \omega(s_1 - s_2)$, where

$$\omega(0) = 0, \int_0^\delta \frac{\omega(t)}{t}dt < \infty, \quad \delta > 0,$$

and

$$\inf_{0<\delta\le\sigma_k/2}\left[\frac{1}{m_1}\int_0^\delta \frac{\omega(t)}{t}dt + 2|\ln M_1|\,|\ln(l/(2\delta))|\right] < \frac{\pi(1-2d)(1-\pi d)m_1}{2M_1},$$

then the dependence $w(s) \equiv \widetilde{w}(\sigma(s))$ represents the initial data under which the solution of the corresponding interior inverse boundary value problem with respect to the parameter s will be univalent (almost convex).

Proof. If

$$\Phi(\tau,\sigma) = \sum_{j=0}^{\infty}\sum_{n=0}^{j} a_{nj}\left[\cos\frac{2\pi}{\sigma_k}(\tau j + \sigma n) + \cos\frac{2\pi}{\sigma_k}(\tau n + \sigma j)\right] +$$

$$+ b_{nj}\left[\sin\frac{2\pi}{\sigma_k}(\tau j + \sigma n) + \sin\frac{2\pi}{\sigma_k}(\tau n + \sigma j)\right],$$

as in (4), and the inequality is fulfilled

$$\sum_{j=0}^{\infty}\sum_{n=0}^{j}(|a_{nj}| + |b_{nj}|)(j+n) \le b < \frac{1}{\pi}, \tag{11}$$

then conditions (5), (6) of Theorem are valid. Let us ensure the fulfillment of inequality (11) by estimating the coefficients of the decomposition $\Phi'_\tau(\tau,\sigma)$. We have

$$a_{nj}(j+n) = -\frac{1}{\pi\sigma_k}\int_0^{\sigma_k}\int_0^{\sigma_k} \Phi'_\tau(\tau,\sigma)\left[\sin\frac{2\pi}{\sigma_k}(\tau j + \sigma n) + \sin\frac{2\pi}{\sigma_k}(\tau n + \sigma j)\right]d\tau\, d\sigma,$$

$$b_{nj}(j+n) = \frac{1}{\pi\sigma_k}\int_0^{\sigma_k}\int_0^{\sigma_k} \Phi'_\tau(\tau,\sigma)\left[\cos\frac{2\pi}{\sigma_k}(\tau j + \sigma n) + \cos\frac{2\pi}{\sigma_k}(\tau n + \sigma j)\right]d\tau\, d\sigma.$$

If in the expressions for $a_{nj}(j+n)$, $b_{nj}(j+n)$ we integrate by parts twice with respect to each of the variables, then we obtain

$$|a_{nj}(j+n)| = \left|\frac{\sigma_k^3}{2^4\pi^5 j^2 n^2}\int_0^{\sigma_k}\int_0^{\sigma_k}\Phi^{(5)}_{\tau\tau\tau\sigma\sigma}(\tau,\sigma)\left[\sin\frac{2\pi}{\sigma_k}(\tau j+\sigma n)+\sin\frac{2\pi}{\sigma_k}(\tau n+\sigma j)\right]d\tau\, d\sigma\right|,$$

$$|b_{nj}(j+n)| = \left|\frac{\sigma_k^3}{2^4\pi^5 j^2 n^2}\int_0^{\sigma_k}\int_0^{\sigma_k}\Phi^{(5)}_{\tau\tau\tau\sigma\sigma}(\tau,\sigma)\left[\cos\frac{2\pi}{\sigma_k}(\tau j+\sigma n)+\cos\frac{2\pi}{\sigma_k}(\tau n+\sigma j)\right]d\tau d\sigma\right|.$$

Consequently,

$$\sum_{j=0}^{\infty}\sum_{n=0}^{j}(|a_{nj}|+|b_{nj}|)(j+n) \le \frac{4\sigma_k^5}{2^4\pi^5}\max_{0\le\tau,\sigma\le\sigma_k}|\Phi^{(5)}_{\tau\tau\tau\sigma\sigma}(\tau,\sigma)|\sum_{j=1}^{\infty}j^{-2}\sum_{n=1}^{\infty}n^{-2}=d,$$

where d is from Theorem's condition. Thus, inequality (11) is fulfilled; consequently, the almost convexity of the solution of the corresponding inverse boundary value problem is ensured by virtue of the estimate in Item 4, where both constants b and c coincide with d.

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